Weakly de Rham complexes

David Hansen

December 9, 2021

1 Introduction and main theorem

This note is a contribution to the six functor formalism in étale cohomology.

Fix a prime p, and let K be a p-adic field, i.e. a complete discretely valued extension of $\mathbf{Q}_p$ with perfect residue field. Let X be any finite type K-scheme. Recall from [Sch13] that there is a well-behaved category of *de Rham* $\mathbf{Q}_p$*-local systems* on X,[1] which specializes to Fontaine's category of de Rham G_K-representations when $X = \mathrm{Spec} K$. This category includes many lisse $\mathbf{Q}_p$-sheaves of geometric origin. In particular, if $f : Y \to X$ is any smooth and proper morphism, then each $R^n f_* \mathbf{Q}_p$ is a de Rham $\mathbf{Q}_p$-local system.

Now, on the categories $D^b_c(X, \mathbf{Q}_p)$ for X a finite type K-scheme, we have Grothendieck's six operations $\otimes$, $R\mathscr{H}\mathrm{om}$, f^*, Rf_*, $Rf_!$, and $Rf^!$ (see [Eke90], or [BS15] for a modern treatment). It is then natural to hope for a reasonable triangulated subcategory $D^b_?(X, \mathbf{Q}_p) \subset D^b_c(X, \mathbf{Q}_p)$ containing all de Rham local systems and stable under the six operations, and minimal with respect to these two properties. In fact, there is only one reasonable candidate for this category.

Definition 1.1. An object $A \in D^b_c(X, \mathbf{Q}_p)$ is *weakly de Rham* if it lies in the thick triangulated subcategory generated by objects of the form $i_! \mathbf{L}$, where $i : V \to X$ is a locally closed immersion and $\mathbf{L}$ is a de Rham $\mathbf{Q}_p$-local system on V. We write $D^b_{\mathrm{wdR}}(X, \mathbf{Q}_p) \subset D^b_c(X, \mathbf{Q}_p)$ for the thick triangulated subcategory spanned by weakly de Rham objects. Similarly, we write $\mathrm{Perv}_{\mathrm{wdR}}(X, \mathbf{Q}_p) \subset \mathrm{Perv}(X, \mathbf{Q}_p)$ for the category of weakly de Rham perverse sheaves.

The main result of this note is the following theorem.

Theorem 1.2. i. *On finite type K-schemes, the subcategories $D^b_{\mathrm{wdR}}(-, \mathbf{Q}_p) \subset D^b_c(-, \mathbf{Q}_p)$ are stable under the six operations and the perverse truncation functors.*

ii. *Intermediate extensions of weakly de Rham perverse sheaves are weakly de Rham.*

iii. *Any perverse subquotient of a weakly de Rham perverse sheaf is weakly de Rham. In particular,* $\mathrm{Perv}_{\mathrm{wdR}}(X, \mathbf{Q}_p)$ *is an abelian Serre subcategory of* $\mathrm{Perv}(X, \mathbf{Q}_p)$.

In short, weakly de Rham complexes on varieties over p-adic fields enjoy the same stabilities as mixed ℓ-adic complexes on varieties over finite fields.

Since the actual proof of this result is not very long, we content ourselves with a very brief sketch here. The stabilities of D^b_{wdR} under f^*, $\otimes$, and $j_!$ for j an open immersion are all trivial. The two essential non-formal results are stability under $Rf_!$, and stability under Verdier duality. For

[1] In [Sch13] this notion is only defined for smooth varieties. For a general variety X, we say a lisse $\mathbf{Q}_p$-sheaf is de Rham if its pullback to some resolution of singularities $\tilde{X}$ is de Rham in the sense of [Sch13]. By [LZ17, Theorem 1.5.(iii)], this condition is independent of the choice of resolution.

stability under proper pushfoward, the key input is a remarkable theorem of Liu-Zhu [LZ17], which says that a $\mathbf{Q}_p$-local system on a connected variety is de Rham if its stalk at a single closed point x is a de Rham G_{K_x}-representation. For stability under Verdier duality, we reduce to showing that if X is smooth, $j : U \to X$ is the complement of a strict normal crossings divisor, and $\mathbf{L}$ is a de Rham $\mathbf{Z}_p$-local system on U with $\mathbf{L}/p$ trivial, then $Rj_*\mathbf{L}[\frac{1}{p}]$ is weakly de Rham. This can be reduced by induction to the situation where $X - U$ is smooth, and in this case it follows from a result of Diao-Lan-Liu-Zhu on preservation of the de Rham property under unipotent nearby cycles. The rest of i. and ii. now follow by formal arguments. For iii., we argue by induction on the dimension of support. The key observation here is that the functors of unipotent nearby and vanishing cycles preserve weakly de Rham perverse sheaves, which follows easily from i. and ii. using Beilinson's construction of these functors.

Conventions. Unless noted otherwise, all schemes are finite type K-schemes. By assumption, a $\mathbf{Q}_p$-local system always contains a $\mathbf{Z}_p$-lattice.

Acknowledgments. It's a pleasure to thank Bhargav Bhatt for many helpful and inspiring conversations.

2 Proofs

2.1 Reduction

In this section, we prove Theorem 1.2. Propositions 2.1 and 2.3 will be proved in the next section.

Proposition 2.1. *Suppose that X is smooth, $j : U \to X$ is the complement of a strict normal crossings divisor D, and $\mathbf{L}$ is a de Rham $\mathbf{Z}_p$-local system on U such that $\mathbf{L}/p\mathbf{L}$ is trivial. Then $Rj_*\mathbf{L}[\frac{1}{p}]$ is weakly de Rham.*

Corollary 2.2. *If $j : U \to X$ is an open immersion and $\mathbf{L}$ is a de Rham $\mathbf{Z}_p$-local system on U, then $Rj_*\mathbf{L}[\frac{1}{p}]$ is weakly de Rham.*

More generally, if $j : U \to X$ is an open immersion and $A \in D^b_{\mathrm{wdR}}(U, \mathbf{Q}_p)$ is lisse, then Rj_*A is weakly de Rham.

Proof. The second part of the corollary reduces immediately to the first part.

For the first part, we can assume X is reduced. By a standard argument, we can find a proper hypercover $\epsilon_\bullet : X_\bullet \to X$ such that X_n is smooth, $U_n = U \times_X X_n$ is the complement of an snc divisor, and $\mathbf{L}|_{U_n}$ is trivial mod p. Then $Rj_*\mathbf{L}[\frac{1}{p}] = R\epsilon_{\bullet *}Rj_{\bullet *}\epsilon^*_{U\bullet}\mathbf{L}[\frac{1}{p}]$ where $\epsilon_{U\bullet} : U_\bullet \to U$ is the evident base change of $\epsilon_\bullet$. Then each $Rj_{n*}\epsilon^*_{U,n}\mathbf{L}[\frac{1}{p}]$ is weakly de Rham by Proposition 2.1, so each $R\epsilon_{n*}Rj_{n*}\epsilon^*_{U,n}\mathbf{L}[\frac{1}{p}]$ is weakly de Rham by Proposition 2.3 and the properness of ϵ_n. ☐

Proposition 2.3. *If $f : X \to Y$ is any finite type map, then $Rf_!$ preserves D^b_{wdR}.*

Proof of Theorem 1.2. By Proposition 2.3 and the trivial stabilities mentioned in the introduction, we already have stability under the operations $f^*, \otimes, Rf_!$.

Next we establish stability under Verdier duality. Let X and $A \in D^b_{\mathrm{wdR}}(X, \mathbf{Q}_p)$ be given, and let $\mathbf{D}_X = R\mathscr{H}\mathrm{om}(-, \omega_X)$ be the Verdier duality functor. The result is clear if A is lisse and X is smooth. For the general case, we can assume X is reduced. Pick a dense open $j : U \to X$ with U smooth and $A|_U$ lisse, and let $i : Z \to X$ be the closed complement. Dualizing the triangle $j_!A|_U \to A \to i_*A|_Z \to$, we get a triangle

$$i_*\mathbf{D}_Z(A|_Z) \to \mathbf{D}_X(A) \to Rj_*\mathbf{D}_U(A|_U) \to .$$

By induction on dimension, we can assume $i_*\mathbf{D}_Z(A|_Z)$ is weakly de Rham. Now observe that $\mathbf{D}_U(A|_U)$ is weakly de Rham and lisse, so then $Rj_*\mathbf{D}_U(A|_U)$ is weakly de Rham by Corollary 2.2. Stability under Verdier duality now follows.

The stabilities under the remaining six operations now follow from the results proved so far, by the habitual formulas $Rf_* = \mathbf{D} \circ Rf_! \circ \mathbf{D}$, $Rf^! = \mathbf{D} \circ f^* \circ \mathbf{D}$, and $R\mathscr{H}\mathrm{om}(A, B) = \mathbf{D}(A \otimes \mathbf{D}(B))$ on D^b_c.

For stability under the perverse truncation functors, we argue by induction on the dimension of X. The result is clear if $\dim X = 0$. In general, pick some $A \in D^b_{\mathrm{wdR}}(X, \mathbf{Q}_p)$. It suffices to prove that ${}^{\mathfrak{p}}\tau^{\leq 0}A \in D^b_{\mathrm{wdR}}(X, \mathbf{Q}_p)$. We can assume X is reduced. Pick a smooth dense open $j : U \to X$ on which A is lisse, with closed complement $i : Z \to X$. Then ${}^{\mathfrak{p}}\tau^{\leq 0}A$ is defined iteratively by consideration of the triangles

$$B \to A \to Rj_*{}^{\mathfrak{p}}\tau_U^{>0}j^*A \to$$

and

$${}^{\mathfrak{p}}\tau^{\leq 0}A \to B \to i_*{}^{\mathfrak{p}}\tau_Z^{>0}i^*B \to .$$

Since j^*A is lisse and U is smooth, ${}^{\mathfrak{p}}\tau_U^{>0}j^*A$ agrees up to shift with a standard truncation of j^*A, and hence is weakly de Rham. Therefore $Rj_*{}^{\mathfrak{p}}\tau_U^{>0}j^*A$ is weakly de Rham, so B is weakly de Rham. Then also ${}^{\mathfrak{p}}\tau_Z^{>0}i^*B$ is weakly de Rham by the induction hypothesis, so ${}^{\mathfrak{p}}\tau^{\leq 0}A$ is weakly de Rham as desired.

For stability under intermediate extensions, let $j : U \to X$ be an open immersion with closed complement $i : Z \to X$, and let $A \in \mathrm{Perv}(U, \mathbf{Q}_p)$ be any weakly de Rham perverse sheaf. The claim now follows from the exact triangle

$$j_{!*}A \to Rj_*A \to i_*{}^{\mathfrak{p}}\tau_Z^{\geq 0}i^*Rj_*A \to$$

and the stabilities proved so far.

Finally, it remains to prove stability under perverse subquotients. For this we argue by induction on $\dim X$. Let A be a weakly de Rham perverse sheaf on X. Since the weak de Rham property can be checked locally, we can assume X is affine. By induction on the length of A, it suffices to prove that any simple subobject $B \subset A$ is weakly de Rham. We now divide into two cases.

Case 1. $B_{\overline{\eta}} \neq 0$ for some generic point $\eta \in X$.

Case 2. $B_{\overline{\eta}} = 0$ for all generic points $\eta \in X$.

In case 1, we can find a smooth irreducible open subscheme $j : U \to X$ such that $B|_U$ and $A|_U$ are (shifted) lisse $\mathbf{Q}_p$-sheaves and $B \simeq j_{!*}(B|_U)$. Since $A|_U$ is lisse and weakly de Rham and $B|_U$ is a lisse subsheaf of $A|_U$, $B|_U$ is also weakly de Rham, so then B is weakly de Rham by part ii.

In case 2, we can find a nowhere-dense closed subscheme $i : Z \to X$ with open complement $j : U \to X$ such that $B|_U = 0$, so i^*B is perverse and $B = i_*i^*B$. We can assume that Z is the vanishing locus of a non-zero-divisor $f \in \mathcal{O}(X)$. Let $\Psi_f^u : \mathrm{Perv}(U, \mathbf{Q}_p) \to \mathrm{Perv}(Z, \mathbf{Q}_p)$ and $\Phi_f^u : \mathrm{Perv}(X, \mathbf{Q}_p) \to \mathrm{Perv}(Z, \mathbf{Q}_p)$ be the associated unipotent nearby cycle and vanishing cycle functors. These are exact functors, and for any $C \in \mathrm{Perv}(X, \mathbf{Q}_p)$ there is a natural exact sequence

$$0 \to {}^{\mathfrak{p}}\mathcal{H}^{-1}(i^*C) \to \Psi_f^u(j^*C) \to \Phi_f^u(C) \to {}^{\mathfrak{p}}\mathcal{H}^0(i^*C) \to 0$$

(see [Mor18] for these fundamental facts). We next observe that Ψ_f^u and Φ_f^u preserve weakly de Rham perverse sheaves. For Ψ_f^u this follows from [Mor18, Corollary 3.2] and the stabilities already proved in i.-ii. For Φ_f^u this follows from the result for Ψ_f^u, the stabilities proved in i.-ii., and the previous exact sequence. Returning to the situation at hand, we see that $B = i_*i^*B \simeq i_*\Phi_f^u(B)$. On the other hand, $\Phi_f^u(B)$ is a subobject of $\Phi_f^u(A)$ by the exactness of Φ_f^u, and $\Phi_f^u(A)$ is weakly de Rham, so $\Phi_f^u(B)$ is weakly de Rham by the induction hypothesis. □

2.2 Proofs of key propositions

Proof of Proposition 2.3. We argue by induction on $\dim Y$. When $\dim Y = 0$, the claim reduces easily to the following theorem of Lan-Liu-Zhu [LLZ19, Theorem 1.1] : if U/K is a smooth variety and $\mathbf{L}$ is a de Rham $\mathbf{Q}_p$-local system on U, then $H^n_c(U_{\overline{K}}, \mathbf{L})$ is a de Rham G_K-representation.

For the induction step, let $f : X \to Y$ and $A \in D^b_{\mathrm{wdR}}(X, \mathbf{Q}_p)$ be given. We can assume X and Y are reduced. Fix a finite set of pairs $\{(X_i, \mathbf{L}_i)\}_{i \in I}$ where $X_i \subset X$ is a smooth locally closed subvariety and $\mathbf{L}_i$ is a de Rham $\mathbf{Q}_p$-local system on X_i, such that A is built from the objects $j_{i!}\mathbf{L}_i$ by finitely many shifts and cones. Here $j_i : X_i \to X$ is the evident immersion. Write $f_i = f \circ j_i$, so $Rf_!A$ is built from the objects $Rf_{i!}\mathbf{L}_i$ by finitely many shifts and cones. It thus suffices to show that each $Rf_{i!}\mathbf{L}_i$ is weakly de Rham. Since X_i is smooth and Y is reduced, there is a dense open $U_i \subset Y$ over which f_i is smooth. Let $Z_i \subset Y$ be the closed complement. By the induction hypothesis, $(Rf_{i!}\mathbf{L}_i)|_{Z_i}$ is weakly de Rham, so it suffices to show that $(Rf_{i!}\mathbf{L}_i)|_{U_i}$ is weakly de Rham.

In other words, we've reduced to showing that if $f : X \to Y$ is smooth and $\mathbf{L}$ is a de Rham $\mathbf{Q}_p$-local system on X, then $Rf_!\mathbf{L}$ is weakly de Rham. By proper base change and the theorem of Lan-Liu-Zhu quoted above, we see that for any n, the stalk of $R^nf_!\mathbf{L}$ at any closed point x is a de Rham G_{K_x}-representation. By [LZ17, Theorem 1.5.(iii)], this implies that $(R^nf_!\mathbf{L})|_{Y_i}$ is a de Rham $\mathbf{Q}_p$-local system for some constructible stratification $Y = \coprod Y_i$. Therefore $Rf_!\mathbf{L}$ is weakly de Rham. □

Proof of Proposition 2.1. Let $D = X - U$ be the strict normal crossings divisor in question, with irreducible components $\{D_i\}_{i \in I}$. Let $c(X, U)$ be the largest integer n such that $\cap_{i \in J} D_i \neq \emptyset$ for some subset $J \subseteq I$ with $|J| = n$. In other words, $c(X, U)$ is the maximal codimension of any nonempty intersection of irreducible components of D. Note that $c(X, U) = 1$ exactly when D is smooth. We will argue by induction on $c(X, U)$.

To facilitate the argument, it will actually be convenient to prove a slightly different statement. Let us say $\mathbf{L} \in D^b_c(X, \mathbf{Z}_p)$ is *special* if

a) the cohomology sheaves of $\mathbf{L}[\frac{1}{p}] \in D^b_c(X, \mathbf{Q}_p)$ are weakly de Rham $\mathbf{Q}_p$-local systems, and

b) after pullback to $X_{\overline{K}}$, the cohomology sheaves of $\mathbf{L} \otimes^L \mathbf{F}_p \in D^b_c(X, \mathbf{F}_p)$ are iterated extensions of the constant sheaf $\underline{\mathbf{F}_p}$.

We are going to prove by induction on $c(X, U)$ that if $\mathbf{L} \in D^b_c(U, \mathbf{Z}_p)$ is special, then $Rj_*\mathbf{L}[\frac{1}{p}]$ is weakly de Rham. If $c(X, U) = 1$, then D is smooth, and the claim follows from [DLLZ18, Theorem 3.7.11] (and note that we are only using condition a) here).

In general, it clearly suffices to show that $(Rj_*\mathbf{L}[\frac{1}{p}])|_{D_i}$ is weakly de Rham for each $i \in I$. Set $U_{(i)} = X - \cup_{j \in I \smallsetminus \{i\}} D_j$, so $j : U \to X$ factors as $U \overset{j^{(i)}}{\to} U_{(i)} \overset{j_{(i)}}{\to} X$, and these maps fit into a diagram

$$\begin{tikzcd}
U \arrow[r, "j^{(i)}"] & U_{(i)} \arrow[r, "j_{(i)}"] & X \\
& V_i \arrow[u, "b_i"] \arrow[r, "j'_{(i)}"] & D_i \arrow[u, "a_i"']
\end{tikzcd}$$

where the horizontal maps are open immersions and the square is cartesian. Note that the complement of U in $U_{(i)}$ is a smooth snc divisor, and that the complement of V_i in D_i is an snc divisor with $c(D_i, V_i) < c(X, U)$. Then

$$\begin{aligned}(Rj_*\mathbf{L})|_{D_i} &= a_i^* Rj_{(i)*} Rj_*^{(i)}\mathbf{L} \\ &\cong Rj'_{(i)*} b_i^* Rj_*^{(i)}\mathbf{L}\end{aligned}$$

where the second isomorphism follows from condition b), [Zhe08, Lemma 3.7] and the compatibility of the base change map with reduction mod p. Moreover, the proof of [Zhe08, Lemma 3.7] shows that $b_i^* Rj_*^{(i)}\mathbf{L}$ satisfies condition b) above, while [DLLZ18, Theorem 3.7.11] shows that it satisfies condition a). Therefore $b_i^* Rj_*^{(i)}\mathbf{L}$ is special, so $Rj'_{(i)*} b_i^* Rj_*^{(i)}\mathbf{L}[\frac{1}{p}]$ is weakly de Rham by the induction hypothesis. This concludes the proof. □

References

[BS15] Bhargav Bhatt and Peter Scholze, *The pro-étale topology for schemes*, Astérisque (2015), no. 369, 99–201. MR 3379634

[DLLZ18] Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu, *Logarithmic Riemann-Hilbert correspondences for rigid varieties*, preprint.

[Eke90] Torsten Ekedahl, *On the adic formalism*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 197–218. MR 1106899

[LLZ19] Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu, *de Rham comparison and Poincaré duality for rigid varieties*, preprint.

[LZ17] Ruochuan Liu and Xinwen Zhu, *Rigidity and a Riemann-Hilbert correspondence for p-adic local systems*, Invent. Math. **207** (2017), no. 1, 291–343. MR 3592758

[Mor18] Sophie Morel, *Beilinson's construction of nearby cycles and gluing*, available at http://perso.ens-lyon.fr/sophie.morel/gluing.pdf.

[Sch13] Peter Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77. MR 3090230

[Zhe08] Weizhe Zheng, *Sur la cohomologie des faisceaux l-adiques entiers sur les corps locaux*, Bull. Soc. Math. France **136** (2008), no. 3, 465–503. MR 2415350