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OVERCONVERGENT COHOMOLOGY: THEORY AND APPLICATIONS

a dissertation

by

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Abstract

We analyze the overconvergent cohomology modules introduced by Ash and Stevens, applying them to construct eigenvarieties for fairly general reductive groups. We then establish several instances of p-adic Langlands functoriality for these eigenvarieties, and we use these functorialities to give evidence for a precise conjecture relating trianguline Galois representations to overconvergent cohomology classes. Our main technical innovations are a family of universal coefficients spectral sequences for overconvergent cohomology and a generalization of Chenevier's interpolation theorem.

Contents

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For my parents.

Chapter 1

Introduction

1.1 A curious prime

The prime number 691 appears in four unusual places.

1. The rational number $\zeta(-11)$ is divisible by 691, where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function; more precisely,

$$\zeta(-11) = \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}.$$

2. Let K denote the number field $\mathbf{Q}(\zeta_{691})$. Then there is a Galois extension L/K which is cyclic of degree 691 and unramified everywhere, and such that $a \in \mathrm{Gal}(K/\mathbf{Q}) \simeq (\mathbf{Z}/691\mathbf{Z})^{\times}$ acts on $x \in \mathrm{Gal}(L/K) \simeq \mathbf{Z}/691\mathbf{Z}$ through the automorphism $x \mapsto a^{-11}x$.

3. If $\tau(n)$ is defined by the formal equality $\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}$, then the congruence $\tau(p) \equiv p^{11} + 1 \bmod 691$ hold for all primes p.

4. Setting $L(s,\Delta) = \sum_{n=1}^{\infty} \tau(n) n^{-s}$, we have

$$\frac{L(11,\Delta)}{L(9,\Delta)} = \frac{2^3 \cdot 3^2}{691} \pi^2.$$

As the reader may have guessed, there is no coincidence here. In fact, *any one of these results implies the other three.* The equivalence 1. $\Leftrightarrow$ 3. was observed by Ramanujan as a consequence of the basic theory of modular forms, while 2. $\Rightarrow$ 1. is a classic result of Herbrand. The implication 1. $\Rightarrow$ 2. lies much deeper and was discovered by Ribet in the late '70s. Aside from appealing to a great deal of 20th century algebra and algebraic geometry, Ribet's proof crucially relies on the fact that the function $\sum \tau(n)q^n$ is a *modular form.* Recall that a modular form of weight $k \geq 1$ and level N is a holomorphic function on the open upper half-plane $\mathfrak{h} = \{z = x + iy,\, y > 0\}$ which satisfies the transformation law

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$$

and doesn't grow "too quickly" near the boundary of $\mathfrak{h}$; here $\Gamma_1(N)$ consists of those matrices in $\mathrm{SL}_2(\mathbf{Z})$ with lower row $\equiv (0,1) \bmod N$. Let $M_k(N)$ denote the finite-dimensional $\mathbf{C}$-vector space of modular forms of weight k and level N. Note that since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1(N)$, any modular form has a Fourier expansion

$$f(z) = \sum_{n \geq 0} a_n(f) q^n,\ q = e^{2\pi i z}.$$

(The absence of negative terms in this sum follows from the growth condition on f.) Hecke, following an idea of Mordell, constructed a family of commuting operators acting on $M_k(N)$ and indexed by primes; these operators are traditionally denoted T_l for $l \nmid N$ and U_l for $l|N$. An element $f \in M_k(N)$ is a *normalized eigenform* if $a_1(f) = 1$ and if f is a common eigenvector of all the Hecke operators, say with $T_l f = \lambda_l(f) f$. If f is a normalized eigenform, a pleasant calculation with the definition of the Hecke operators shows that in fact $\lambda_l(f) = a_l(f)$; furthermore, a fundamental theorem of Shimura asserts that the coefficients $a_n(f)$ are algebraic integers which simultaneously lie in a single finite-degree extension of $\mathbf{Q}$. This remarkable property is the tip of the iceberg concerning the arithmetic properties of normalized eigenforms and more general automorphic representations.

1.2 Families of modular forms

Our story really begins with an amazing theorem of Hida and Coleman (Col97, Hid86).

Theorem. *Fix a prime p with $p|N$, and let $f \in M_{k_0}(N)$ be a normalized eigenform. If*

$v_p(a_p(f)) < k_0 - 1$, *there is a* p*-adic disk* $U_f \subset \mathbf{Z}_p$ *containing* k_0 *together with a collection* $\mathscr{C}_f = \{a_n(x)\}_{n\geq 0}$ *of analytic functions on* U_f *valued in a finite extension of* $\mathbf{Q}_p$ *such that for any integer* $k \in U_f \cap \mathbf{Z}_{\geq 2}$, *the specialization of the formal* q*-series*

$$f_x(q) = \sum_{n\geq 0} a_n(x)q^n$$

at $x = k$ *is a normalized eigenform of weight* k *and level* N, *with* $f_{k_0} = f$. *Furthermore the data of* U_f *and* $\mathscr{C}_f$ *is* unique, *possibly up to shrinking* U_f.

We refer to the formal series $f_x(q)$ as the *Coleman family* of f; the notation is meant to suggest that f_x is a "p-adic deformation" of f. This theorem raises several immediate questions:

Q1. What do we mean by "analytic functions" on U_f?

Q2. Does the formal series $f_x(q)$ have any intrinsic meaning for *noninteger* points $x \in U_f$?

Q3. There is a great deal of redundancy among the Coleman families of distinct modular forms: for example, if $k \in U_f \cap \mathbf{Z}_{\geq 2}$, then $f' = f_k$ is a normalized eigenform, and the Coleman families of f and f' tautologically overlap. Is there any order to be found in the disorder of these overlaps?

A1. The only possible thing: a function represented by a convergent power series.

A2. Yes: it is the q-expansion of an *overconvergent modular eigenform* of weight x.

A3. Yes. Before explaining this, we need to shift slightly our perspective on one aspect of the picture. Rather than recording the weight as an integer, it is better for a number of reasons to record the weight *and* the p-part of the character of f simultaneously as a point in the *weight space*

$$\mathscr{W} = \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{G}_m).$$

More precisely, given a modular form f of weight k, level N, and character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, we may factor χ as $\chi_p\chi^p$, and we map f to the point w_f in $\mathscr{W}$ corresponding to the character $w_f(x) = x^{k-1}\chi_p(x)$. The object $\mathscr{W}$ is our first example of a *rigid analytic space.* Rigid analytic geometry, as conceived by Tate and developed by a number of mathematicians (with the most significant contributions due to Bartenwerfer, Bosch, Conrad, Gerritzen, Grauert, Guntzer, Kiehl, Lutkebohmert, Raynaud, Remmert, and Temkin), is an amazing nonarchimedean analogue of complex manifold theory. When $N = 1$, Coleman and Mazur proved the following theorem, with Buzzard treating the case of arbitrary level (Buz07, CM98).

Theorem (Coleman-Mazur, Buzzard). *There is a reduced, equidimensional rigid analytic curve $\mathscr{C} = \mathscr{C}(N)$ together with a morphism $w : \mathscr{C}(N) \to \mathscr{W}$ and global sections $\mathrm{U}_p \in \mathscr{O}(\mathscr{C})$, $\mathrm{T}_l \in \mathscr{O}(\mathscr{C})$ for all $l \nmid Np$, such that:*

i. *The morphism w has discrete fibers.*

ii. *The points $x \in w^{-1}(\lambda)$ in the fiber over a fixed weight $\lambda \in \mathscr{W}$ are in bijection with overconvergent modular eigenforms of weight λ.*

iii. *If f_x is an overconvergent modular eigenform corresponding to a point $x \in \mathscr{C}$, the eigenvalue of T_l (resp. U_p) acting on f_x equals the image $\mathrm{T}_l(x)$ (resp. $\mathrm{U}_p(x)$) of T_l (resp. U_p) in the residue field of the stalk $\mathscr{O}_{\mathscr{C},x}$.*

iv. *If f is a classical normalized eigenform, with $x_f \in \mathscr{C}$ the associated point, then $w(x_f) = w_f$. If furthermore $v_p(a_f(p)) < k-1$, then there is an open neighborhood U of $x_f \in \mathscr{C}$ such that $w|_U : U \to \mathscr{W}$ is a homeomorphism onto its image and $\mathrm{T}_l|_U = a_l \circ w$.*

This construction and the ideas driving it have been vastly generalized by a number of authors, resulting in a whole universe of **eigenvarieties**.

1.3 This thesis

In this thesis we accomplish three tasks:

- Using overconvergent cohomology, we give a construction of eigenvarieties associated with an arbitrary connected, reductive $\mathbf{Q}$-group $\mathbf{G}$ such that $\mathbf{G}/\mathbf{Q}_p$ is split, and we prove some basic structural results on their geometry (Theorems 4.3.3 and 4.5.1).

- We establish a flexible, axiomatic framework for comparing different constructions on eigenvarieties on the same group, and for interpolating Langlands functoriality maps into morphisms of eigenvarieties (Theorems 5.1.2 and 5.1.5), applying it to construct p-adic analogues of the symmetric square lift (Theorem 5.4.1) and the Rankin-Selberg lift (Theorem 5.5.1).

- We formulate a generalization of the Fontaine-Mazur-Langlands conjecture which encompasses trianguline Galois representations (Conjecture 6.1.2) and give some significant evidence for it (Theorems 6.1.3-6.1.5).

For groups such that $\mathbf{G}^{\mathrm{der}}(\mathbf{R})$ is either compact or has a discrete series, our constructions mostly reduce to those of previous authors, and our driving goal is the development of techniques for treating groups satisfying neither of these hypotheses. This was our initial stimulus for studying overconvergent cohomology, a beautifully simple construction introduced by Stevens (Ste94) and developed by Ash-Stevens (AS08). Our motivations for working in this generality are both philosophical and practical. Philosophically speaking, p-adic automorphic forms seem to play as fundamental a role in number theory as classical automorphic forms, and it's imperative to avoid any restrictice hypotheses on the underlying groups. Practically speaking, it's natural to construct p-adic L-functions by pairing overconvergent cohomology classes with cycles in homology, in analogy with integral representations of classical L-functions; it's essential to work with $\mathbf{R}$-split and even $\mathbf{Q}$-split groups in this context.

Our main technical result is a pair of spectral sequences which play the role of a universal coefficients theorem for overconvergent cohomology (Theorem 3.3.1). We also introduce an axiomatic notion of "eigenvariety datum" (Definition 4.2.1); in contrast with Buzzard's "eigenvariety machine", we shift the emphasis from systems of Banach modules over weight space to coherent sheaves over Fredholm hypersurfaces. This shift was very important psychologically in the development of our ideas: our most interesting results (e.g. Theorems 4.4.1, 5.1.5, 5.5.1, 6.1.5) all make use of the technique of "analytic continuation along a Fredholm hypersurface".

1.4 Notation and terminology

Our notation and terminology is mostly standard. In nonarchimedian functional analysis and rigid analytic geometry we follow (BGR84). If M and N are $\mathbf{Q}_p$-Banach spaces, we write $\mathcal{L}(M,N)$ for the space of continuous $\mathbf{Q}_p$-linear maps between M and N; the operator norm

$$|f| = \sup_{m\in M,\, |m|_M\leq 1} |f(m)|_N$$

makes $\mathcal{L}(M,N)$ into a Banach space. If $(A, |\bullet|_A)$ is a Banach space which furthermore is a commutative Noetherian $\mathbf{Q}_p$-algebra whose multiplication map is (jointly) continuous, we say A is a $\mathbf{Q}_p$-*Banach algebra*. An A-module M which is also a Banach space is a *Banach A-module* if the structure map $A\times M\to M$ extends to a continuous map $A\widehat{\otimes}_{\mathbf{Q}_p} M \to M$, or equivalently if the norm on M satisfies $|am|_M \leq C|a|_A|m|_M$ for all $a \in A$ and $m \in M$ with some fixed constant C. For a topological ring R and topological R-modules M, N, we write $\mathcal{L}_R(M, N)$ for the R-module of continuous R-linear maps $f : M \to N$. When A is a Banach algebra and M, N are Banach A-modules, we topologize $\mathcal{L}_A(M, N)$ via its natural Banach A-module structure. We write Ban_A for the category whose objects are Banach A-modules and whose morphisms are elements of $\mathcal{L}_A(-,-)$. If I is any set and A is a Banach algebra, we write $c_I(A)$ for the module of sequences $\mathbf{a} = (a_i)_{i \in I}$ with $|a_i|_A \to 0$; the norm $|\mathbf{a}| = \sup_{i \in I} |a_i|_A$ gives $c_I(A)$ the structure of a Banach A-module. If M is any Banach A-module, we say M is *orthonormalizable* if M is *isomorphic* to $c_I(A)$ for some I (such modules are called "potentially orthonormalizable" in (Buz07)).

If A is an affinoid algebra, then $\mathrm{Sp}A$, the *affinoid space* associated with A, denotes the locally G-ringed space $(\mathrm{Max}A, \mathcal{O}_A)$ where $\mathrm{Max}A$ is the set of maximal ideals of A endowed with the Tate topology and $\mathcal{O}_A$ is the extension of the assignment $U \mapsto A_U$, for affinoid subdomains $U \subset \mathrm{Max}A$ with representing algebras A_U, to a structure sheaf on $\mathrm{Max}A$. If X is an affinoid space, we write $\mathscr{O}(X)$ for the coordinate ring of X, so $A \simeq \mathscr{O}(\mathrm{Sp}A)$. If A is reduced we equip A with the canonical supremum norm. If X is a rigid analytic space, we write $\mathscr{O}_X$ for the structure sheaf and $\mathscr{O}(X)$ for the ring of global sections of $\mathscr{O}_X$. Given a point $x \in X$, we write $\mathfrak{m}_x$ for the corresponding maximal ideal in $\mathscr{O}_X(U)$ for any admissible affinoid open $U \subset X$ containing x, and $k(x)$ for the residue field $\mathscr{O}_X(U)/\mathfrak{m}_x$; $\mathscr{O}_{X,x}$ denotes the local ring of $\mathcal{O}_X$ at x in the Tate topology, and $\widehat{\mathcal{O}_{X,x}}$ denotes the $\mathfrak{m}_x$-adic completion of $\mathcal{O}_{X,x}$. A Zariski-dense subset S of a rigid analytic space X is *very Zariski-dense* if for any connected affinoid open $U \subset X$, either $U \cap S = \emptyset$ or $U \cap S$ is Zariski-dense in U.

In homological algebra our conventions follow (Wei94). If R is a ring, we write $\mathbf{K}^?(R)$, $? \in \{+, -, b, \emptyset\}$ for the homotopy category of ?-bounded R-module complexes and $\mathbf{D}^?(R)$ for its derived category.

We normalize the reciprocity maps of local class field theory so uniformizers map to geometric Frobenii. If π is an irreducible admissible representation of $\mathrm{GL}_n(F)$ for a nonarchimedian local field F, we write $\mathrm{rec}(\pi)$ for the Frobenius-semisimple Weil-Deligne representation associated with π via the local Langlands correspondence, normalized as in Harris and Taylor's book. For p the prime with respect to which things are -adic, we fix an algebraic closure $\overline{\mathbf{Q}_p}$ and an isomorphism $\iota : \overline{\mathbf{Q}_p} \xrightarrow{\sim} \mathbf{C}$. If $f = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(\Gamma_1(N))$ is a classical newform, we write $V_{f,\iota}$ (or sometimes V_f) for the two-dimensional semisimple $\overline{\mathbf{Q}_p}$-linear representation of $G_{\mathbf{Q}}$ characterized by the equality $\iota \mathrm{trFrob}_\ell | V_{f,\iota} = a_f(\ell)$ for all

$\ell \nmid Np$.

Chapter 2

Background material

Throughout the remainder of this thesis, unless otherwise mentioned, $\mathbf{G}$ denotes a connected, reductive $\mathbf{Q}$-group and p denotes a prime such that $G = \mathbf{G} \times_{\mathrm{Spec}\mathbf{Q}} \mathrm{Spec}\mathbf{Q}_p$ is split and admits a split Borel pair (B, T). The group G spreads out to a group scheme (also denoted G) over $\mathrm{Spec}\mathbf{Z}_p$ with reductive special fiber, and we may assume that B and T are defined over $\mathbf{Z}_p$ as well. Set $X^* = \mathrm{Hom}(T, \mathbf{G}_m)$ and $X_* = \mathrm{Hom}(\mathbf{G}_m, T)$, and let Φ, Φ^+ and Φ^- be the sets of roots, positive roots, and negative roots respectively, for the Borel B. We write X^*_+ for the cone of B-dominant weights; $\rho \in X^* \otimes_{\mathbf{Z}} \frac{1}{2}\mathbf{Z}$ denotes half the sum of the positive roots.

We write $\overline{B}$ for the opposite Borel, N and $\overline{N}$ for the unipotent radicals of B and $\overline{B}$, and I for the Iwahori subgroup

$$I = \{g \in G(\mathbf{Z}_p) \,\mathrm{with}\, g \bmod p \in B(\mathbf{Z}/p\mathbf{Z})\}\,.$$

For any integer $s \geq 1$, set $\overline{B}^s = \left\{b \in \overline{B}(\mathbf{Z}_p),\, b \equiv 1 \,\mathrm{in}\, G(\mathbf{Z}/p^s\mathbf{Z})\right\}$, $\overline{N}^s = \overline{N}(\mathbf{Z}_p) \cap \overline{B}^s$ and $T^s = T(\mathbf{Z}_p) \cap \overline{B}^s$, so the Iwahori decomposition reads $I = \overline{N}^1 \cdot T(\mathbf{Z}_p) \cdot N(\mathbf{Z}_p)$. We also set

$$I_0^s = \left\{g \in I,\, g \bmod p^s \in \overline{B}(\mathbf{Z}/p^s\mathbf{Z})\right\}$$

and

$$I_1^s = \left\{g \in I,\, g \bmod p^s \in \overline{N}(\mathbf{Z}/p^s\mathbf{Z})\right\}.$$

Note that I_1^s is normal in I_0^s, with quotient $T(\mathbf{Z}/p^s\mathbf{Z})$. Finally, we set $I^s = I \cap \ker\{G(\mathbf{Z}_p) \to G(\mathbf{Z}/p^s\mathbf{Z})\}$.

We define semigroups in $T(\mathbf{Q}_p)$ by

$$T^+ = \left\{ t \in T(\mathbf{Q}_p),\, t\overline{N}^1 t^{-1} \subseteq \overline{N}^1 \right\}$$

and

$$T^{++} = \left\{ t \in T(\mathbf{Q}_p),\, \bigcap_{i=1}^{\infty} t^i \overline{N}^1 t^{-i} = \{1\} \right\}.$$

A simple calculation shows that $t \in T(\mathbf{Q}_p)$ is contained in T^+ (resp. T^{++}) if and only if $v_p(\alpha(t)) \leq 0$ (resp. $v_p(\alpha(t)) < 0$) for all $\alpha \in \Phi^+$. Using these semigroups, we define a semigroup of $G(\mathbf{Q}_p)$ by $\Delta = IT^+I$. The Iwahori decompsition extends to Δ: any element $g \in \Delta$ has a unique decomposition $g = \mathrm{n}^\circ(g)\mathrm{t}(g)\mathrm{n}(g)$ with $\mathrm{n}^\circ \in \overline{N}^1$, $\mathrm{t} \in T^+$, $\mathrm{n} \in N(\mathbf{Z}_p)$. There is a canonical group homomorphism $\sigma : T(\mathbf{Q}_p) \to T(\mathbf{Z}_p)$ which splits the inclusion $T(\mathbf{Z}_p) \subset T(\mathbf{Q}_p)$, and we set $\Lambda = T^+ \cap \ker \sigma$ and $\Lambda^+ = T^{++} \cap \ker \sigma$.

2.1 Symmetric spaces and Hecke operators

In this section we set up our conventions for the homology and cohomology of local systems on locally symmetric spaces. Following (AS08), we compute homology and cohomology using two different families of resolutions: some extremely large "adelic" resolutions which have the advantage of making the Hecke action transparent, and resolutions with good finiteness properties constructed from simplicial decompositions of the Borel-Serre compactifications of locally symmetric spaces.

Resolutions and complexes

Let $\mathbf{G}/\mathbf{Q}$ be a connected reductive group with center $Z_{\mathbf{G}}$. Let $\mathbf{G}(\mathbf{R})^\circ$ denote the connected component of $\mathbf{G}(\mathbf{R})$ containing the identity element, with $\mathbf{G}(\mathbf{Q})^\circ = \mathbf{G}(\mathbf{Q}) \cap \mathbf{G}(\mathbf{R})^\circ$. Fix a maximal compact subgroup $K_\infty \subset \mathbf{G}(\mathbf{R})$ with K_∞° the connected component containing the identity, and let $Z_\infty = Z_{\mathbf{G}}(\mathbf{R})$. Given an open compact subgroup $K_f \subset \mathbf{G}(\mathbf{A}_f)$, we define the *locally symmetric space of level* K_f by

$$Y(K_f) \quad = \quad \mathbf{G}(\mathbf{Q})\backslash\mathbf{G}(\mathbf{A})/K_f K_\infty^\circ Z_\infty.$$

This is a possibly disconnected Riemannian orbifold. By strong approximation there is a finite set of elements $\gamma(K_f) = \{x_i, x_i \in \mathbf{G}(\mathbf{A}_f)\}$ with

$$\mathbf{G}(\mathbf{A}) = \coprod_{x_i \in \gamma(K_f)} \mathbf{G}(\mathbf{Q})^\circ \mathbf{G}(\mathbf{R})^\circ x_i K_f.$$

Defining $Z(K_f) = Z_{\mathbf{G}}(\mathbf{Q}) \cap K_f$ and $\Gamma(x_i) = \mathbf{G}(\mathbf{Q})^\circ \cap x_i K_f x_i^{-1}$, we have a decomposition

$$Y(K_f) = \mathbf{G}(\mathbf{Q})\backslash \mathbf{G}(\mathbf{A})/K_f K_\infty^\circ Z_\infty \simeq \coprod_{x_i \in \gamma(K_f)} \Gamma^{\mathrm{ad}}(x_i)\backslash D_\infty,$$

where $D_\infty = \mathbf{G}(\mathbf{R})^\circ/K_\infty^\circ Z_\infty$ is the symmetric space associated with $\mathbf{G}$ and $\Gamma^{\mathrm{ad}}(x_i) \cong \Gamma(x_i)/Z(K_f)$ denotes the image of $\Gamma(x_i)$ in the adjoint group. If N is any left K_f-module, the double quotient

$$\widetilde{N} = \mathbf{G}(\mathbf{Q})\backslash \left(D_\infty \times \mathbf{G}(\mathbf{A}_f) \times N\right)/K_f$$

naturally gives rise to a local system on $Y(K_f)$, which is trivial unless $Z(K_f)$ acts trivially on N. Set $D_{\mathbf{A}} = D_\infty \times \mathbf{G}(\mathbf{A}_f)$, and let $C_\bullet(D_{\mathbf{A}})$ denote the complex of singular chains on $D_{\mathbf{A}}$ endowed with the natural action of $\mathbf{G}(\mathbf{Q}) \times \mathbf{G}(\mathbf{A}_f)$. If M and N are right and left K_f-modules, respectively, we define the complexes of *adelic chains* and *adelic cochains* by

$$C_\bullet^{ad}(K_f, M) = C_\bullet(D_{\mathbf{A}}) \otimes_{\mathbf{Z}[\mathbf{G}(\mathbf{Q})\times K_f]} M$$

and

$$C^\bullet_{ad}(K_f, N) = \mathrm{Hom}_{\mathbf{Z}[\mathbf{G}(\mathbf{Q})\times K_f]}(C_\bullet(D_{\mathbf{A}}), N),$$

and we define functors $H_*(K_f, -)$ and $H^*(K_f, -)$ as their cohomology.

Proposition 2.1.1. *There is a canonical isomorphism*

$$H^*(Y(K_f), \widetilde{N}) \simeq H^*(K_f, N) = H^*(C^\bullet_{ad}(K_f, N)).$$

Proof. Let $C_\bullet(D_\infty)(x_i)$ denote the complex of singular chains on D_∞, endowed with the natural left action of $\Gamma(x_i)$ induced from the left action of $G(\mathbf{Q})^\circ$ on D_∞; since D_∞ is contractible, this is a free resolution of $\mathbf{Z}$ in the category of $\mathbf{Z}[\Gamma(x_i)]$-modules. Let $N(x_i)$ denote the left $\Gamma(x_i)$-module whose underlying module is N but with the action $\gamma \cdot_{x_i} n =$

$x_i^{-1}\gamma x_i|n$. Note that the local system $\widetilde{N}(x_i)$ obtained by restricting $\widetilde{N}$ to the connected component $\Gamma(x_i)\backslash D_\infty$ of $Y(K_f)$ is simply the quotient $\Gamma(x_i)\backslash(D_\infty \times N(x_i))$. Setting

$$C^\bullet_{sing}(K_f, N) = \oplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma(x_i)]}(C_\bullet(D_\infty)(x_i), N(x_i)),$$

the map $D_\infty \to (D_\infty, x_i) \subset D_\mathbf{A}$ induces a morphism $x_i^* = \mathrm{Hom}(C_\bullet(D_\mathbf{A}), N) \to \mathrm{Hom}(C_\bullet(D_\infty), N)$, which in turn induces an isomorphism

$$\oplus_i x_i^* : C^\bullet_{ad}(K_f, N) \xrightarrow{\sim} \oplus_i \mathrm{Hom}_{\Gamma(x_i)}(C_\bullet(D_\infty)(x_i), N(x_i)),$$

and passing to cohomology we have

$$\begin{aligned} H^*(C^\bullet_{ad}(K_f, N)) &\simeq \oplus_i H^*(\Gamma(x_i)\backslash D_\infty, \widetilde{N}(x_i)) \\ &\simeq H^*(Y(K_f), \widetilde{N}) \end{aligned}$$

as desired. □

When $\Gamma(x_i)$ is torsion-free for each $x_i \in \gamma(K_f)$, we choose a finite resolution $F_\bullet(x_i) \to \mathbf{Z} \to 0$ of $\mathbf{Z}$ by free left $\mathbf{Z}[\Gamma(x_i)]$-modules of finite rank as well as a homotopy equivalence $F_\bullet(x_i) \underset{g_i}{\overset{f_i}{\rightleftarrows}} C_\bullet(D_\infty)(x_i)$. We shall refer to the resolution $F_\bullet(x_i)$ as a *Borel-Serre resolution*; the existence of such resolutions follows from taking a finite simplicial decomposition of the Borel-Serre compactification of $\Gamma(x_i)\backslash D_\infty$. (BS73) Setting

$$C_\bullet(K_f, N) = \oplus_i F_\bullet(x_i) \otimes_{\mathbf{Z}[\Gamma(x_i)]} M(x_i)$$

and

$$C^\bullet(K_f, N) = \oplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma(x_i)]}(F_\bullet(x_i), N(x_i)),$$

the maps f_i, g_i induce homotopy equivalences

$$C_\bullet(K_f, M) \underset{g_*}{\overset{f_*}{\rightleftarrows}} C^{ad}_\bullet(K_f, M)$$

and

$$C^\bullet(K_f, N) \underset{f^*}{\overset{g^*}{\rightleftarrows}} C^\bullet_{ad}(K_f, M).$$

We refer to the complexes $C_\bullet(K_f,-)$ and $C^\bullet(K_f,-)$ as *Borel-Serre complexes*, and we refer to these complexes together with a *fixed* set of homotopy equivalences $\{f_i, g_i\}$ as *augmented Borel-Serre complexes.* When the $\Gamma(x_i)$'s are not torsion-free but M is uniquely divisible as a $\mathbf{Z}$-module, we may still define $C_\bullet(K_f, M)$ in an ad hoc manner by taking the K_f/K'_f-coinvariants of $C_\bullet(K'_f, M)$ for some sufficiently small normal subgroup $K'_f \subset K_f$.

Hecke operators

A *Hecke pair* consists of a monoid $\Delta \subset \mathbf{G}(\mathbf{A}_f)$ and a subgroup $K_f \subset \Delta$ such that K_f and $\delta K_f \delta^{-1}$ are commensurable for all $\delta \in \Delta$. Given a Hecke pair and a commutative ring R, we write $\mathbf{T}(\Delta, K_f)_R$ for the R-algebra generated by the double coset operators $T_\delta = [K_f \delta K_f]$ under convolution.[1]

Suppose M is a right $R[\Delta]$-module. The complex $C_\bullet(D_{\mathbf{A}}) \otimes_{\mathbf{Z}[\mathbf{G}(\mathbf{Q})]} M$ receives a right Δ-action via $(\sigma \otimes m)|\delta = \sigma\delta \otimes m\delta$, and $C^{ad}_\bullet(K_f, M)$ is naturally identified with the K_f-coinvariants of this action. Given any double coset $K_f \delta K_f = \coprod_j \delta_j K_f$, the action defined on pure tensors by the formula

$$(\sigma \otimes m) \cdot [K_f \delta K_f] = \sum_j (\sigma \otimes m)|\delta_j$$

induces a well-defined algebra homomorphism

$$\xi : \mathbf{T}(\Delta, K_f)_R \to \mathrm{End}_{\mathbf{Ch}(R)}(C^{ad}_\bullet(K_f, M)).$$

This action induces the usual Hecke action defined by correspondences on homology. Set $\tilde{T} = g_* \circ \xi(T) \circ f_* \in \mathrm{End}_{\mathbf{Ch}(R)}(C_\bullet(K_f, M))$. The map

$$\tilde{\xi} : \ \mathbf{T}(\Delta, K_f)_R \xrightarrow{T \mapsto \tilde{T}} \mathrm{End}_{\mathbf{Ch}(R)}(C_\bullet(K_f, M)) \twoheadrightarrow \mathrm{End}_{\mathbf{K}(R)}(C_\bullet(K_f, M))$$

is a well-defined ring homomorphism, since $g_* \circ \xi(T_1) \circ f_* \circ g_* \circ \xi(T_2) \circ f_*$ is homotopic to $g_* \circ \xi(T_1 T_2) \circ f_*$. Note that any individual lift $\tilde{T}$ is well-defined in $\mathrm{End}_{\mathbf{Ch}(R)}(C_\bullet(K_f, M))$, but if T_1 and T_2 commute in the abstract Hecke algebra, $\tilde{T}_1\tilde{T}_2$ and $\tilde{T}_2\tilde{T}_1$ will typically only commute up to homotopy.

[1] The ring structure on $\mathbf{T}(\Delta, K_f)_R$ is nicely explained in §3.1 of (Shi94).

Likewise, if N is a left $R[\Delta]$-module, the complex $\mathrm{Hom}_{\mathbf{Z}[\mathbf{G}(\mathbf{Q})]}(C_\bullet(D_{\mathbf{A}}), N)$ receives a natural Δ-action via the formula $\delta|\phi = \delta \cdot \phi(\sigma\delta)$, and $C^\bullet_{ad}(K_f, N)$ is naturally the K_f-invariants of this action. The formula

$$[K_f \delta K_f] \cdot \phi = \sum_j \delta_j | \phi$$

yields an algebra homomorphism $\xi : \mathbf{T}(\Delta, K_f)_R \to \mathrm{End}_R(C^\bullet_{ad}(K_f, N))$ which induces the usual Hecke action on cohomology, and $f^* \circ \xi \circ g^*$ defines an algebra homomorphism $\mathbf{T}(\Delta, K_f)_R \to \mathrm{End}_{\mathbf{K}(R)}(C^\bullet(K_f, M))$. It is extremely important for us that these Hecke actions are compatible with the duality isomorphism

$$\mathrm{Hom}_R(C_\bullet(K_f, M), P) \simeq C^\bullet(K_f, \mathrm{Hom}_R(M, P)),$$

where P is any R-module.

We shall be mostly concerned with the following Hecke algebras. For I, Λ and Δ as above, set $\mathcal{A}_p^+ = \mathbf{T}(\Delta, I)_{\mathbf{Q}_p}$. For any $t \in T^+$, the double coset operator $U_t = [ItI]$ defines an element of $\mathcal{A}_p^+$, and the map $\Lambda \ni t \mapsto U_t \in \mathcal{A}_p^+$ extends to a commutative ring isomorphism

$$\begin{array}{ccc} \mathbf{Q}_p[\Lambda] & \xrightarrow{\sim} & \mathcal{A}_p^+ \\ \sum c_t t & \mapsto & \sum c_t U_t. \end{array}$$

The operators $U_t = [ItI]$ are invertible in the full Iwahori-Hecke algebra $\mathbf{T}(\mathbf{G}(\mathbf{Q}_p), I)_{\mathbf{Q}_p}$ (IM65), and we define the *Atkin-Lehner algebra* $\mathcal{A}_p$ as the commutative subalgebra of $\mathbf{T}(\mathbf{G}(\mathbf{Q}_p), I)_{\mathbf{Q}_p}$ generated by elements of the form U_t and U_t^{-1} for $t \in \Lambda$. There is a natural ring isomorphism $\mathcal{A}_p \simeq \mathbf{Q}_p[T(\mathbf{Q}_p)/T(\mathbf{Z}_p)]$, though note that $t \cdot T(\mathbf{Z}_p)$ typically corresponds to the operator $U_{t_1} U_{t_2}^{-1}$ where $t_1, t_2 \in \Lambda$ are any elements with with $t_1 t_2^{-1} \in t \cdot T(\mathbf{Z}_p)$. A *controlling operator* is an element of $\mathcal{A}_p$ of the form U_t for $t \in \Lambda^+$.

Fix an open compact subgroup $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$. We say K^p is unramified at a prime $\ell \neq p$ if $\mathbf{G}/\mathbf{Q}_\ell$ is unramified and $K_\ell^p = K^p \cap \mathbf{G}(\mathbf{Q}_\ell)$ is a hyperspecial maximal compact subgroup of $\mathbf{G}(\mathbf{Q}_\ell)$, and we say K^p is ramified otherwise. Let $S = S(K^p)$ denote the finite set of places where K^p is ramified, and set $K_S^p = K^p \cap \prod_{\ell \in S} \mathbf{G}(\mathbf{Q}_\ell)$, so K^p admits a product decomposition $K^p = K^p_S \prod_{\ell \notin S} K^p_\ell$. We mainly work with the (commutative) Hecke algebras

$$\begin{aligned} \mathbf{T}^p(K^p) &= \bigotimes_{\ell \notin S(K^p)}{}' \mathbf{T}(\mathbf{G}(\mathbf{Q}_\ell), K^p_\ell)_{\mathbf{Q}_p}, \\ \mathbf{T}^+(K^p) &= \mathcal{A}_p^+ \otimes \mathbf{T}^p(K^p), \\ \mathbf{T}(K^p) &= \mathcal{A}_p \otimes \mathbf{T}^p(K^p). \end{aligned}$$

In words, $\mathbf{T}(K^p)$ takes into account the prime-to-p spherical Hecke operators together with the Atkin-Lehner operators at p; we write $\mathbf{T}_{\mathbf{G}}(K^p)$ if we need to emphasize $\mathbf{G}$. We also set $\mathbf{T}_{\mathrm{ram}}(K^p) = \mathbf{T}\left(\prod_{\ell \in S} \mathbf{G}(\mathbf{Q}_\ell), K^p_S\right)$.

2.2 Locally analytic modules

For each $s \geq 1$ fix an analytic isomorphism $\psi^s : \mathbf{Z}_p^d \simeq \overline{N}^s$, $d = \dim N$.[2]

Definition. *If R is any $\mathbf{Q}_p$-Banach algebra and s is a positive integer, the module $\mathbf{A}(\overline{N}^1, R)^s$ of* s-locally analytic R-valued functions on $\overline{N}^1$ *is the R-module of continuous functions $f : \overline{N}^1 \to R$ such that*

$$f\left(x\psi^s(z_1, \dots, z_d)\right) : \mathbf{Z}_p^d \to R$$

is given by an element of the d-variable Tate algebra $T_{d,R} = R\left\langle z_1, \dots, z_d \right\rangle$ for any fixed $x \in \overline{N}^1$.

Letting $\|\bullet\|_{T_{d,R}}$ denote the canonical norm on the Tate algebra, the norm $\|f(x\psi^s)\|_{T_{d,R}}$ depends only on the image of x in $\overline{N}^1/\overline{N}^s$, and the formula

$$\|f\|_s = \sup_{x \in \overline{N}^1} \|f(x\psi^s)\|_{T_{d,R}}$$

defines a Banach R-module structure on $\mathbf{A}(\overline{N}^1, R)^s$, with respect to which the canonical inclusion $\mathbf{A}(\overline{N}^1, R)^s \subset \mathbf{A}(\overline{N}^1, R)^{s+1}$ is compact.

Given a tame level group $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$, let $\overline{Z(K^pI)}$ be the p-adic closure of $Z(K^pI)$ in $T(\mathbf{Z}_p)$. The *weight space of level* K^p is the rigid analytic space $\mathscr{W} = \mathscr{W}_{K^p}$ over $\mathbf{Q}_p$

[2] Use the homomorphisms $x_\alpha : U_\alpha \to \mathbf{G}_a$ together with the product decomposition $\overline{N} \simeq \prod_{\alpha \in \Phi^-} U_\alpha$ for α in some fixed ordering.

such that for any $\mathbf{Q}_p$-affinoid algebra A, $\mathrm{Hom}(\mathrm{Sp}A, \mathscr{W}_{K^p})$ represents the functor which associates with A the set of p-adically continuous characters $\chi : T(\mathbf{Z}_p) \to A^\times$ trivial on $\overline{Z(K^pI)}$. It's not hard to check that $\mathscr{W}$ is the rigid space associated with the formal scheme $\mathrm{Spf}\mathbf{Z}_p[[T(\mathbf{Z}_p)/\overline{Z(K^pI)}]]$ via Raynaud's generic fiber functor (cf. §7 of (dJ95)). Given an admissible affinoid open $\Omega \subset \mathscr{W}$, we write $\chi_\Omega : T(\mathbf{Z}_p) \to \mathscr{O}(\Omega)^\times$ for the unique character it determines. We define $s[\Omega]$ as the minimal integer such that $\chi_\Omega|_{T^{s[\Omega]}}$ is analytic. For any integer $s \geq s[\Omega]$, we make the definition

$$\mathbf{A}^s_\Omega = \{f : I \to \mathscr{O}(\Omega),\ f\,\text{analytic on each}\,I^s-\text{coset},\ f(gtn) = \chi_\Omega(t)f(g)\,\forall n \in N(\mathbf{Z}_p),\ t \in T(\mathbf{Z}_p),\ g \in I\}\,.$$

By the Iwahori decomposition, restricting an element $f \in \mathbf{A}^s_\Omega$ to $\overline{N}^1$ induces an isomorphism

$$\begin{array}{rcl}\mathbf{A}^s_\Omega & \simeq & \mathbf{A}(\overline{N}^1, \mathscr{O}(\Omega))^s \\ f & \mapsto & f|_{\overline{N}^1},\end{array}$$

and we regard $\mathbf{A}^s_\Omega$ as a Banach $\mathscr{O}(\Omega)$-module via pulling back the Banach module structure on $\mathbf{A}(N(\mathbf{Z}_p), \mathscr{O}(\Omega))^s$ under this isomorphism. The rule $(f|\gamma)(g) = f(\gamma g)$ gives $\mathbf{A}^s_\Omega$ the structure of a continuous right $\mathscr{O}(\Omega)[I]$-module. More generally, the formula

$$\delta \star (nB(\mathbf{Z}_p)) = \delta n^\circ \delta^{-1}\sigma(\delta),\ n \in \overline{N}^1 \simeq I/B(\mathbf{Z}_p)\,\text{and}\,\delta \in T^+$$

yields a left action of Δ on $I/B(\mathbf{Z}_p)$ which extends the natural left translation action by I (cf. §2.5 of (AS08)) and induces a right Δ-action on $\mathbf{A}^s_\Omega$ which we denote by $f \star \delta$, $f \in \mathbf{A}^s_\Omega$. For any $\delta \in T^{++}$, the image of the operator $\delta \star - \in \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}^s_\Omega, \mathbf{A}^s_\Omega)$ factors through the inclusion $\mathbf{A}^{s-1}_\Omega \hookrightarrow \mathbf{A}^s_\Omega$, and so defines a compact operator on $\mathbf{A}^s_\Omega$. The Banach dual

$$\begin{array}{rcl}\mathbf{D}^s_\Omega & = & \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}^s_\Omega, \mathscr{O}(\Omega)) \\ & \simeq & \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}(\overline{N}^1, \mathbf{Q}_p)^s \widehat{\otimes}_{\mathbf{Q}_p} \mathscr{O}(\Omega), \mathscr{O}(\Omega)) \\ & \simeq & \mathcal{L}_{\mathbf{Q}_p}(\mathbf{A}(\overline{N}^1, \mathbf{Q}_p)^s, \mathscr{O}(\Omega))\end{array}$$

inherits a dual left action of Δ, and the operator $- \star \delta$ for $\delta \in T^{++}$ likewise factors through the inclusion $\mathbf{D}^{s+1}_\Omega \hookrightarrow \mathbf{D}^s_\Omega$.

We define an ind-Banach module

$$\mathcal{A}_\Omega = \lim_{s\to\infty} \mathbf{A}_\Omega^s$$

where the direct limit is taken with respect to the natural compact, injective transition maps $\mathbf{A}_\Omega^s \to \mathbf{A}_\Omega^{s+1}$. Note that $\mathcal{A}_\Omega$ is topologically isomorphic to the module of $\mathscr{O}(\Omega)$-valued locally analytic functions on $\overline{N}^1$, equipped with the finest locally convex topology for which the natural maps $\mathbf{A}_\Omega^s \hookrightarrow \mathcal{A}_\Omega$ are continuous. The Δ-actions on $\mathbf{A}_\Omega^s$ induce a continuous Δ-action on $\mathcal{A}_\Omega$. Set

$$\mathcal{D}_\Omega = \left\{\mu : \mathcal{A}_\Omega \to \mathscr{O}(\Omega),\, \mu\,\mathrm{is}\,\mathscr{O}(\Omega)-\mathrm{linear\,and\,continuous}\right\},$$

and topologize $\mathcal{D}_\Omega$ via the coarsest locally convex topology for which the natural maps $\mathcal{D}_\Omega \to \mathbf{D}_\Omega^s$ are continuous. In particular, the canonical map

$$\mathcal{D}_\Omega \to \lim_{\infty\leftarrow s} \mathbf{D}_\Omega^s$$

is a topological isomorphism of locally convex $\mathscr{O}(\Omega)$-modules, and $\mathcal{D}_\Omega$ is compact and Fréchet. Note that the transition maps $\mathbf{D}_\Omega^{s+1} \to \mathbf{D}_\Omega^s$ are *injective*, so $\mathcal{D}_\Omega = \cap_{s\gg 0}\mathbf{D}_\Omega^s$.

Suppose $\Sigma \subset \Omega$ is a Zariski closed subspace; by Corollary 9.5.2/8 of (BGR84), Σ arises from a surjection $\mathscr{O}(\Omega) \twoheadrightarrow \mathscr{O}(\Sigma)$ with $\mathscr{O}(\Sigma)$ an affinoid algebra. We make the definitions $\mathbf{D}_\Sigma^s = \mathbf{D}_\Omega^s \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Sigma)$ and $\mathcal{D}_\Sigma = \mathcal{D}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Sigma)$.

Proposition 2.2.1. *There are canonical topological isomorphisms* $\mathbf{D}_\Sigma^s \simeq \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Sigma))$ *and* $\mathcal{D}_\Sigma \simeq \mathcal{L}_{\mathscr{O}(\Omega)}(\mathcal{A}_\Omega, \mathscr{O}(\Sigma))$.

Proof. Set $\mathfrak{a}_\Sigma = \ker(\mathscr{O}(\Omega) \to \mathscr{O}(\Sigma))$, so $\mathscr{O}(\Sigma) \simeq \mathscr{O}(\Omega)/\mathfrak{a}_\Sigma$. The definitions immediately imply isomorphisms

$$\begin{aligned}\mathbf{D}_\Sigma^s &\simeq \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Omega))/\mathfrak{a}_\Sigma\mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Omega)) \\ &\simeq \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Omega))/\mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathfrak{a}_\Sigma),\end{aligned}$$

so the first isomorphism will follow if we can verify that the sequence

$$0 \to \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathfrak{a}_\Sigma) \to \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Omega)) \to \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Sigma))$$

is exact on the right. Given a $\mathbf{Q}_p$-Banach space E, write $b(E)$ for the Banach space of bounded sequences $\{(e_i)_{i\in\mathbb{N}}, \sup_{i\in\mathbb{N}} |e_i|_E < \infty\}$. Choosing an orthonormal basis of $\mathbf{A}(N(\mathbf{Z}_p), \mathscr{O}(\Omega))^s$ gives rise to an isometry $\mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}^s_\Omega, E) \simeq b(E)$ for E any Banach $\mathscr{O}(\Omega)$-module. Thus we need to show the surjectivity of the reduction map $b(\mathscr{O}(\Omega)) \to b(\mathscr{O}(\Sigma))$. Choose a presentation $\mathscr{O}(\Omega) = T_n/\mathfrak{b}_\Omega$, so $\mathscr{O}(\Sigma) = T_n/\mathfrak{b}_\Sigma$ with $\mathfrak{b}_\Omega \subseteq \mathfrak{b}_\Sigma$. Quite generally for any $\mathfrak{b} \subset T_n$, the function

$$f \in T_n/\mathfrak{b} \mapsto \|f\|_{\mathfrak{b}} = \inf_{\tilde{f}\in f+\mathfrak{b}} \left\|\tilde{f}\right\|_{T_n}$$

defines a norm on $T_n/\mathfrak{b}$. By Proposition 3.7.5/3 of (BGR84), there is a unique Banach algebra structure on any affinoid algebra. Hence for any sequence $(f_i)_{i\in\mathbb{N}} \in b(\mathscr{O}(\Sigma))$, we may choose a bounded sequence of lifts $(\widetilde{f_i})_{i\in\mathbb{N}} \in b(T_n)$; reducing the latter sequence modulo $\mathfrak{b}_\Omega$, we are done.

Taking inverse limits in the sequence we just proved to be exact, the second isomorphism follows. □

Suppose $\lambda \in X^*_+ \subset \mathcal{W}(\mathbf{Q}_p)$ is a dominant weight for B, with $\mathcal{L}_\lambda$ the corresponding irreducible left $G(\mathbf{Q}_p)$-representation of highest weight λ. We may realize $\mathcal{L}_\lambda$ explicitly as

$$\mathcal{L}_\lambda(L) = \left\{f : G \to L \,\text{algebraic},\, f(n'tg) = \lambda(t)f(g) \,\text{for}\, n' \in \overline{N}, t \in T, g \in G\right\}$$

with G acting by right translation. The function $f_\lambda(g)$ defined by $f_\lambda(n'tn) = \lambda(t)$ on the big cell extends uniquely to an algebraic function on G, and is the highest weight vector in $\mathcal{L}_\lambda$. For $g \in G(\mathbf{Q}_p)$ and $h \in I$, the function $f_\lambda(gh)$ defines an element of $\mathcal{L}_\lambda \otimes \mathcal{A}_\lambda$, and pairing it against $\mu \in \mathcal{D}_\lambda$ defines a map $i_\lambda : \mathcal{D}_\lambda \to \mathcal{L}_\lambda$ which we notate suggestively as

$$i_\lambda(\mu)(g) = \int f_\lambda(gh)\mu(h).$$

The map $\mu \mapsto i_\lambda(\mu)(g)$ satisfies the following intertwining relation for $\gamma \in I$:

$$\begin{aligned}\gamma \cdot i_\lambda(\mu)(g) &= i_\lambda(\mu)(g\gamma)\\ &= \int f_\lambda(g\gamma n^\circ)\mu(n^\circ)\\ &= \int f_\lambda(n^\circ g)(\gamma\cdot\mu)(n^\circ)\\ &= i_\lambda(\gamma\cdot\mu)(g).\end{aligned}$$

The case of $\mathrm{GL}_n/\mathbf{Q}_p$

We examine the case when $\mathbf{G}/\mathbf{Q}_p \simeq \mathrm{GL}_n/\mathbf{Q}_p$. We choose B and $\overline{B}$ as the upper and lower triangular Borel subgroups, respectively, and we identify T with diagonal matrices. The splitting σ is canonically induced from the homomorphism

$$\begin{array}{rcl} \mathbf{Q}_p^\times & \to & \mathbf{Z}_p^\times \\ x & \mapsto & p^{-v_p(x)}x. \end{array}$$

Since $T(\mathbf{Z}_p) \simeq (\mathbf{Z}_p^\times)^n$, we canonically identify a character $\lambda : T(\mathbf{Z}_p) \to R^\times$ with the n-tuple of characters $(\lambda_1, \dots, \lambda_n)$ where

$$\begin{array}{rcl} \lambda_i : \mathbf{Z}_p^\times & \to & R^\times \\ x & \mapsto & \lambda \circ \mathrm{diag}(\underbrace{1, \dots, 1}_{i-1}, x, 1, \dots, 1). \end{array}$$

Dominant weights are identified with characters $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i(x) = x^{k_i}$ for integers $k_1 \geq k_2 \geq \cdots \geq k_n$.

We want to explain how to "twist away" one dimension's worth of weights in a canonical fashion. For any $\lambda \in \mathscr{W}$, a simple calculation shows that the $\star$-action of Δ on $\mathbf{A}_\lambda^s$ is given explicitly by the formula

$$(f \star \delta)(x) = \lambda(\sigma(\mathrm{t}(\delta))\mathrm{t}(\delta)^{-1}\mathrm{t}(\delta x))f(\mathrm{n}(\delta x)),\ \delta \in \Delta,\ x \in \overline{N}^1,\ f \in \mathbf{A}(\overline{N}^1, k(\lambda))^s.$$

Given $1 \leq i \leq n$, let $m_i(g)$ denote the determinant of the upper-left i-by-i block of $g \in \mathrm{GL}_n$. For any $g \in \Delta$, a pleasant calculation left to the reader shows that

$$\mathrm{t}(g) = \mathrm{diag}(m_1(g), m_1(g)^{-1}m_2(g), \dots, m_i^{-1}(g)m_{i+1}(g), \dots, m_{n-1}(g)^{-1}\mathrm{det}g).$$

In particular, writing $\lambda^0 = (\lambda_1\lambda_n^{-1}, \lambda_2\lambda_n^{-1}, \dots, \lambda_{n-1}\lambda_n^{-1}, 1)$ yields a canonical isomorphism

$$\mathbf{A}_\lambda^s \simeq \mathbf{A}_{\lambda^0}^s \otimes \lambda_n(\det \cdot |\det|_p)$$

of Δ-modules, and likewise for $\mathbf{D}_\lambda^s$.

In the case of GL_2 we can be even more explicit. Here Δ is generated by the center of

$G(\mathbf{Q}_p)$ and by the monoid

$$\Sigma_0(p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p),\ c \in p\mathbf{Z}_p,\ a \in \mathbf{Z}_p^\times,\ ad - bc \neq 0 \right\}.$$

Another simple calculation shows that the center of $G(\mathbf{Q}_p)$ acts on $\mathbf{A}_\lambda^s$ through the character $z \mapsto \lambda(\sigma(z))$, while the monoid $\Sigma_0(p)$ acts via

$$(g \cdot f)(x) = (\lambda_1\lambda_2^{-1})(a+bx)\lambda_2(\det g|\det g|_p) f\left(\frac{c+dx}{a+bx}\right),\ f \in \mathbf{A}(\overline{N}^1, k)^s,\ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \in \overline{N}^1,$$

(almost) exactly as in (Ste94).

Remarks. There are some subtle differences between the different modules we have defined. The assignment $\Omega \mapsto \mathcal{A}_\Omega$ describes a presheaf over $\mathscr{W}$, and the modules $\mathbf{A}_\Omega^s$ are orthonormalizable. On the other hand, the modules $\mathbf{D}_\Omega^s$ are *not* obviously orthonormalizable, and $\mathcal{D}_\Omega$ doesn't obviously form a presheaf. There are alternate modules of distributions available, namely $\tilde{\mathbf{D}}_\Omega^s = \mathcal{L}_k(\mathbf{A}(\overline{N}^1, \mathbf{Q}_p)^s, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p}\mathscr{O}(\Omega)$ and $\tilde{\mathcal{D}}_\Omega = \lim_{\leftarrow s} \tilde{\mathcal{D}}_\Omega^s$, equipped with suitable actions such that there is a natural $\mathscr{O}(\Omega)[\Delta]$-equivariant embedding $\tilde{\mathcal{D}}_\Omega \hookrightarrow \mathcal{D}_\Omega$. The modules $\tilde{\mathbf{D}}_\Omega^s$ are orthonormalizable, and $\tilde{\mathcal{D}}_\Omega$ forms a presheaf over weight space, but of course is not the continuous dual of $\mathcal{A}_\Omega$. Despite these differences, the practical choice to work with one module or the other is really a matter of taste: for any $\lambda \in \Omega(\overline{\mathbf{Q}_p})$, there are isomorphisms

$$\mathcal{D}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda \simeq \tilde{\mathcal{D}}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda \simeq \mathcal{D}_\lambda,$$

and in point of fact the slope-$\leq h$ subspaces of $H^*(K^p, \mathcal{D}_\Omega)$ and $H^*(K^p, \tilde{\mathcal{D}}_\Omega)$, when they are defined, are canonically isomorphic as Hecke modules, as we show in Proposition 3.1.6 below. One of our goals is to demonstrate the feasibility of working successfully with the modules $\mathcal{D}_\Omega$ by treating the dual modules $\mathcal{A}_\Omega$ on an equal footing.

2.3 Slope decompositions of modules and complexes

Here we review the very general notion of slope decomposition introduced in (AS08). Let A be a $\mathbf{Q}_p$-Banach algebra, and let M be an A-module equipped with an A-linear endomorphism $u : M \to M$ (for short, "an $A[u]$-module"). Fix a rational number $h \in \mathbf{Q}_{\geq 0}$.

We say a polynomial $Q \in A[x]$ is *multiplicative* if the leading coefficient of Q is a unit in A, and that Q has *slope* $\leq h$ if every edge of the Newton polygon of Q has slope $\leq h$. Write $Q^*(x) = x^{\deg Q} Q(1/x)$. An element $m \in M$ has slope $\leq h$ if there is a multiplicative polynomial $Q \in A[T]$ of slope $\leq h$ such that $Q^*(u) \cdot m = 0$. Let $M_{\leq h}$ be the set of elements of M of slope $\leq h$; according to Proposition 4.6.2 of *loc. cit.*, $M_{\leq h}$ is an A-submodule of M.

Definition 2.3.1. A *slope-$\leq h$ decomposition* of M is an $A[u]$-module isomorphism

$$M \simeq M_{\leq h} \oplus M_{>h}$$

such that $M_{\leq h}$ is a finitely generated A-module and the map $Q^*(u) : M_{>h} \to M_{>h}$ is an A-module isomorphism for every multiplicative polynomial $Q \in A[T]$ of slope $\leq h$.

The following proposition summarizes the fundamental results on slope decompositions.

Proposition 2.3.2 (Ash-Stevens):

a) *Suppose M and N are both $A[u]$-modules with slope-$\leq h$ decompositions. If $\psi : M \to N$ is a morphism of $A[u]$-modules, then $\psi(M_{\leq h}) \subseteq N_{\leq h}$ and $\psi(M_{>h}) \subseteq N_{>h}$; in particular, a module can have at most one slope-$\leq h$ decomposition. Furthermore,* $\ker \psi$ *and* $\mathrm{im}\psi$ *inherit slope-$\leq h$ decompositions. Given a short exact sequence*

$$0 \to M \to N \to L \to 0$$

of $A[u]$-modules, if two of the modules admit slope-$\leq h$ decompositions then so does the third.

b) *If $C^\bullet$ is a complex of $A[u]$-modules, all with slope-$\leq h$ decompositions, then*

$$H^n(C^\bullet) \simeq H^n(C^\bullet_{\leq h}) \oplus H^n(C^\bullet_{>h})$$

is a slope-$\leq h$ decomposition of $H^n(C^\bullet)$.

Proof. This is a rephrasing of (a specific case of) Proposition 4.1.2 of (AS08). □

Suppose now that A is a reduced affinoid algebra, M is an orthonormalizable Banach A-module, and u is a compact operator. Let

$$F(T) = \det(1 - uT)|M \in A[[T]]$$

denote the Fredholm determinant for the u-action on M. We say F admits a *slope-$\leq h$ factorization* if we can write $F(T) = Q(T) \cdot R(T)$ where Q is a multiplicative polynomial of slope $\leq h$ and $R(T) \in A[[T]]$ is an entire power series of slope $> h$. Theorem 3.3 of (Buz07) guarantees that F admits a slope-$\leq h$ factorization if and only if M admits a slope-$\leq h$ decomposition. Furthermore, given a slope-$\leq h$ factorization $F(T) = Q(T) \cdot R(T)$, we obtain the slope-$\leq h$ decomposition of M upon setting $M_{\leq h} = \{m \in M | Q^*(u) \cdot m = 0\}$, and $M_{\leq h}$ in this case is a finite flat A-module upon which u acts invertibly.[3] Combining this with Theorem 4.5.1 of (AS08) and Proposition 2.3.1, we deduce:

Proposition 2.3.3. *If $C^\bullet$ is a bounded complex of orthonormalizable Banach $A[u]$-modules, and u acts compactly on the total complex $\oplus_i C^i$, then for any $x \in \mathrm{Max}(A)$ and any $h \in \mathbf{Q}_{\geq 0}$ there is an affinoid subdomain $\mathrm{Max}(A') \subset \mathrm{Max}(A)$ containing x such that the complex $C^\bullet \widehat{\otimes}_A A'$ of $A'[u]$-modules admits a slope-$\leq h$ decomposition, and $(C^\bullet \widehat{\otimes}_A A')_{\leq h}$ is a complex of finite flat A'-modules.*

Proposition 2.3.4. *If M is an orthonormalizable Banach A-module with a slope-$\leq h$ decomposition, and A' is a Banach A-algebra, then $M \widehat{\otimes}_A A'$ admits a slope-$\leq h$ decomposition and in fact*

$$(M \widehat{\otimes}_A A')_{\leq h} \simeq M_{\leq h} \otimes_A A'.$$

Proposition 2.3.5. *If $N \in \mathrm{Ban}_A$ is finite and $M \in \mathrm{Ban}_A$ is an $A[u]$-module with a slope-$\leq h$ decomposition, the $A[u]$-modules $M \widehat{\otimes}_A N$ and $\mathcal{L}_A(M, N)$ inherit slope-$\leq h$ decompositions.*

Proof. This is an immediate consequence of the A-linearity of the u-action and the fact that $-\widehat{\otimes}_A N$ and $\mathcal{L}_A(-, N)$ commute with finite direct sums. □

If A is a field and M is either an orthonormalizable Banach A-module or the cohomology of a complex of such, then M admits a slope-$\leq h$ decomposition for every h, and if $h < h'$ there is a natural decomposition

$$M_{\leq h'} \simeq M_{\leq h} \oplus (M_{> h})_{\leq h'}$$

and in particular a projection $M_{\leq h'} \to M_{\leq h}$. We set $M^{\mathrm{fs}} = \lim_{\infty \leftarrow h} M_{\leq h}$.

[3] Writing $Q^*(x) = a + x \cdot r(x)$ with $r \in A[x]$ and $a \in A^\times$, u^{-1} on $M_{\leq h}$ is given explicitly by $-a^{-1} r(u)$.

Chapter 3

Overconvergent cohomology

Fix a connected, reductive group $\mathbf{G}/\mathbf{Q}$ with $\mathbf{G}/\mathbf{Q}_p$ split. For any tame level group $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$, we abbreviate $H_*(K^pI, -)$ by $H_*(K^p, -)$, and likewise for cohomology.

3.1 Basic results

In this section we establish some foundational results on overconvergent cohomology. These results likely follow from the formalism introduced in Chapter 5 of (AS08), but we give different proofs. We use the notations introduced in §2.1-§2.3.

Fix an augmented Borel-Serre complex $C_\bullet(K^p, -) = C_\bullet(K^pI, -)$. Fix an element $t \in \Lambda^+$, and let $\tilde{U} = \tilde{U}_t$ denote the lifting of $U_t = [ItI]$ to an endomorphism of the complex $C_\bullet(K^p, -)$ defined in §2.1. Given a connected admissible open affinoid subset $\Omega \subset \mathscr{W}_{K^p}$ and any integer $s \geq s[\Omega]$, the endomorphism $\tilde{U}_t \in \mathrm{End}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}_\Omega^s))$ is compact; let

$$F_\Omega^s(X) = \det(1 - X\tilde{U}_t)|C_\bullet(K^p, \mathbf{A}_\Omega^s) \in \mathscr{O}(\Omega)[[X]]$$

denote its Fredholm determinant. We say (U_t, Ω, h) is a *slope datum* if $C_\bullet(K^p, \mathbf{A}_\Omega^s)$ admits a slope-$\leq h$ decomposition for the $\tilde{U}_t$ action for some $s \geq s[\Omega]$.

Proposition 3.1.1. *The function $F_\Omega^s(X)$ is independent of s.*

Proof. For any integer $s \geq s[\Omega]$ we write $C_\bullet^s = C_\bullet(K^p, \mathbf{A}_\Omega^s)$ for brevity. By construction, the operator $\tilde{U}_t$ factors into compositions $\rho_s \circ \check{U}_t$ and $\check{U}_t \circ \rho_{s+1}$ where $\check{U}_t : C_\bullet^s \to C_\bullet^{s-1}$ is continuous and $\rho_s : C_\bullet^{s-1} \to C_\bullet^s$ is compact. Now, considering the commutative diagram[1]

$$\begin{tikzcd}
C_\bullet^s \arrow[r, "\check{U}_t"] \arrow[d, "\tilde{U}_t"'] & C_\bullet^{s-1} \arrow[d, "\tilde{U}_t"] \arrow[dl, "\rho_s"'] \\
C_\bullet^s \arrow[r, "\check{U}_t"'] & C_\bullet^{s-1}
\end{tikzcd}$$

we calculate

$$\begin{aligned}
\det(1 - X\tilde{U}_t)|C_\bullet^s &= \det(1 - X\rho_s \circ \check{U}_t)|C_\bullet^s \\
&= \det(1 - X\check{U}_t \circ \rho_s)|C_\bullet^{s-1} \\
&= \det(1 - X\tilde{U}_t)|C_\bullet^{s-1},
\end{aligned}$$

where the second line follows from Lemma 2.7 of (Buz07), so $F_\Omega^s(X) = F_\Omega^{s-1}(X)$ for all $s > s[\Omega]$. $\square$

Proposition 3.1.2. *The slope-$\leq h$ subcomplex $C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h}$, if it exists, is independent of s. If Ω' is an affinoid subdomain of Ω, then the restriction map $\mathbf{A}_\Omega^s \to \mathbf{A}_{\Omega'}^s$ induces a canonical isomorphism*

$$C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \simeq C_\bullet(K^p, \mathbf{A}_{\Omega'}^s)_{\leq h}$$

for any $s \geq s[\Omega]$.

Proof. Since $F_\Omega^s(X)$ is independent of s, we simply write $F_\Omega(X)$. Suppose we are given a slope-$\leq h$ factorization $F_\Omega(X) = Q(X) \cdot R(X)$; by the remarks in §2.3, setting $C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h} = \ker Q^*(\tilde{U}_t)$ yields a slope-$\leq h$ decomposition of $C_\bullet(K^p, \mathbf{A}_\Omega^s)$ for any $s \geq s[\Omega]$. By Proposition 2.3.1, the injection $\rho_s : C_\bullet(K^p, \mathbf{A}_\Omega^{s-1}) \hookrightarrow C_\bullet(K^p, \mathbf{A}_\Omega^s)$ gives rise to a canonical injection

$$\rho_s : C_\bullet(K^p, \mathbf{A}_\Omega^{s-1})_{\leq h} \hookrightarrow C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h}$$

for any $s > s[\Omega]$. The operator $\tilde{U}_t$ acts invertibly on $C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h}$, and its image factors through ρ_s, so ρ_s is surjective and hence bijective. This proves the first claim.

[1]The importance of drawing diagrams like this seems to have first been realized by Hida; they have since been used extensively by Stevens and others.

For the second claim, by Proposition 2.3.3 we have

$$\begin{aligned} C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') &\simeq \left(C_\bullet(K^p, \mathbf{A}^s_\Omega)\widehat{\otimes}_{\mathscr{O}(\Omega)}\mathscr{O}(\Omega')\right)_{\leq h} \\ &\simeq C_\bullet(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}, \end{aligned}$$

so the result now follows from the first claim. □

Proposition 3.1.3. *Given a slope datum (U_t, Ω, h) and an affinoid subdomain $\Omega' \subset \Omega$, there is a canonical isomorphism*

$$H_*(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \simeq H_*(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}$$

for any $s \geq s[\Omega]$.

Proof. Since $\mathscr{O}(\Omega')$ is $\mathscr{O}(\Omega)$-flat, the functor $-\otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega')$ commutes with taking the homology of any complex of $\mathscr{O}(\Omega)$-modules. Thus we calculate

$$\begin{aligned} H_*(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') &\simeq H_*\left(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}\right) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \\ &\simeq H_*\left(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega')\right) \\ &\simeq H_*\left(C_\bullet(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}\right) \\ &\simeq H_*(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}, \end{aligned}$$

where the third line follows from Proposition 2.3.4. □

Proposition 3.1.4. *Given a slope datum (U_t, Ω, h), the complex $C_\bullet(K^p, \mathcal{A}_\Omega)$ and the homology module $H_*(K^p, \mathcal{A}_\Omega)$ admit slope-$\leq h$ decompositions, and there is an isomorphism*

$$H_*(K^p, \mathcal{A}_\Omega)_{\leq h} \simeq H_*(K^p, \mathbf{A}^s_\Omega)_{\leq h}$$

for any $s \geq s[\Omega]$. Furthermore, given an affinoid subdomain $\Omega' \subset \Omega$, there is a canonical isomorphism

$$H_*(K^p, \mathcal{A}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \simeq H_*(K^p, \mathcal{A}_{\Omega'})_{\leq h}.$$

Proof. For any fixed $s \geq s[\Omega]$, we calculate

$$\begin{aligned} C_\bullet(K^p, \mathcal{A}_\Omega) &\simeq \varinjlim_{s'} C_\bullet(K^p, \mathbf{A}_\Omega^{s'}) \\ &\simeq \varinjlim_{s'} C_\bullet(K^p, \mathbf{A}_\Omega^{s'})_{\leq h} \oplus C_\bullet(K^p, \mathbf{A}_\Omega^{s'})_{>h} \\ &\simeq C_\bullet(K^p, \mathbf{A}_\Omega^{s})_{\leq h} \oplus \varinjlim_{s'} C_\bullet(K^p, \mathbf{A}_\Omega^{s'})_{>h} \end{aligned}$$

with the third line following from Proposition 3.1.2. The two summands in the third line naturally form the components of a slope-$\leq h$ decomposition, so passing to homology yields the first sentence of the proposition, and the second sentence then follows immediately from Proposition 2.3.3. □

We're now in a position to prove the subtler cohomology analogue of Proposition 3.1.4.

Proposition 3.1.5. *Given a slope datum (U_t, Ω, h) and a rigid Zariski-closed subset $\Sigma \subset \Omega$, the complex $C^\bullet(K^p, \mathcal{D}_\Sigma)$ and the cohomology module $H^*(K^p, \mathcal{D}_\Sigma)$ admit slope-$\leq h$ decompositions, and there is an isomorphism*

$$H^*(K^p, \mathcal{D}_\Sigma)_{\leq h} \simeq H^*(K^p, \mathbf{D}_\Sigma^s)_{\leq h}$$

for any $s \geq s[\Omega]$. Furthermore, given an affinoid subdomain $\Omega' \subset \Omega$, there are canonical isomorphisms

$$C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \simeq C^\bullet(K^p, \mathcal{D}_{\Omega'})_{\leq h}$$

and

$$H^*(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \simeq H^*(K^p, \mathcal{D}_{\Omega'})_{\leq h}.$$

Proof. By a topological version of the duality stated in §2.1, we have a natural isomorphism

$$\begin{aligned} C^\bullet(K^p, \mathbf{D}_\Sigma^s) &= C^\bullet(K^p, \mathcal{L}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Sigma))) \\ &\simeq \mathcal{L}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}_\Omega^s), \mathscr{O}(\Sigma)) \end{aligned}$$

for any $s \geq s[\Omega]$. By assumption, $C_\bullet(K^p, \mathbf{A}_\Omega^s)$ admits a slope-$\leq h$ decomposition, so we calculate

$$\begin{aligned}C^\bullet(K^p, \mathbf{D}^s_\Sigma) &\simeq \mathcal{L}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma)) \\ &\simeq \mathcal{L}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Sigma)) \\ &\quad \oplus \mathcal{L}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{> h}, \mathscr{O}(\Sigma)).\end{aligned}$$

By Proposition 3.1.2, passing to the inverse limit over s in this isomorphism yields a slope-$\leq h$ decomposition of $C^\bullet(K^p, \mathcal{D}_\Sigma)$ together with a natural isomorphism

$$C^\bullet(K^p, \mathcal{D}_\Sigma)_{\leq h} \simeq C^\bullet(K^p, \mathbf{D}^s_\Sigma)_{\leq h} \simeq \mathcal{L}_{\mathscr{O}(\Omega)}(C^\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Sigma))$$

for any $s \geq s[\Omega]$. This proves the first sentence of the proposition.

For the second sentence, we first note that since $C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}$ is a complex of finite $\mathscr{O}(\Omega)$-modules, the natural map

$$\mathcal{L}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Omega)) \to \mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Omega))$$

is an isomorphism by Lemma 2.2 of (Buz07). Next, note that if R is a commutative ring, S is a flat R-algebra, and M, N are R-modules with M finitely presented, the natural map $\mathrm{Hom}_R(M, N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$ is an isomorphism. With these two facts in hand, we calculate as follows:

$$\begin{aligned}C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') &\simeq \mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Omega)) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \\ &\simeq \mathrm{Hom}_{\mathscr{O}(\Omega')}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega'), \mathscr{O}(\Omega')) \\ &\simeq \mathrm{Hom}_{\mathscr{O}(\Omega')}(C_\bullet(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}, \mathscr{O}(\Omega')) \\ &\simeq C^\bullet(K^p, \mathcal{D}_{\Omega'})_{\leq h},\end{aligned}$$

where the third line follows from Proposition 2.3.3. Passing to cohomology, the result follows as in the proof of Proposition 3.1.3. □

Recall from the end of §2.2 the alternate module of distributions $\tilde{\mathcal{D}}_\Omega$. For completeness's sake, we sketch the following result.

Proposition 3.1.6. *If the complexes $C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)$ and $C^\bullet(K^p, \mathcal{D}_\Omega)$ both admit slope-$\leq h$*

decompositions for the $\tilde{U}_t$-action, the natural map $\tilde{\mathcal{D}}_\Omega \to \mathcal{D}_\Omega$ induces an isomorphism

$$H^*(K^p, \tilde{\mathcal{D}}_\Omega)_{\leq h} \simeq H^*(K^p, \mathcal{D}_\Omega)_{\leq h}.$$

Proof. The morphism $f : \tilde{\mathcal{D}}_\Omega \to \mathcal{D}_\Omega$ induces a morphism $\phi : C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\leq h} \to C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$. Choose an arbitrary weight $\lambda \in \Omega(\overline{\mathbf{Q}_p})$; since

$$\mathcal{D}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda \simeq \tilde{\mathcal{D}}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda \simeq \mathcal{D}_\lambda,$$

we calculate

$$\begin{aligned} C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda &\simeq C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda)_{\leq h} \\ &\simeq C^\bullet(K^p, \mathcal{D}_\lambda)_{\leq h} \\ &\simeq C^\bullet(K^p, \mathcal{D}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda)_{\leq h} \\ &\simeq C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda, \end{aligned}$$

where the first and fourth lines follow by Proposition 2.3.3. Thus $\phi \bmod \mathfrak{m}_\lambda$ is an isomorphism in the category of $\mathscr{O}(\Omega)/\mathfrak{m}_\lambda$-module complexes. Writing $C^\bullet(\phi)$ for the cone of ϕ, we have an exact triangle

$$C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\leq h} \xrightarrow{\phi} C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \to C^\bullet(\phi) \to C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\leq h}[1]$$

in $\mathbf{D}^b(\mathscr{O}(\Omega))$. Since $C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\leq h}$ and $C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$ are complexes of projective $\mathscr{O}(\Omega)$-modules, the cone $C^\bullet(\phi)$ is a complex of projective modules as well; therefore, the functors $- \otimes^{\mathbf{L}}_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}$ and $- \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}$ agree on these three complexes (cf. (Wei94), 10.6). In particular, applying $- \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda$ yields an exact triangle

$$C^\bullet(K^p, \mathcal{D}_\lambda)_{\leq h} \xrightarrow{\phi \bmod \mathfrak{m}_\lambda} C^\bullet(K^p, \mathcal{D}_\lambda)_{\leq h} \to C^\bullet(\phi) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda \to C^\bullet(K^p, \mathcal{D}_\lambda)_{\leq h}[1]$$

in $\mathbf{D}^b(\mathscr{O}(\Omega)/\mathfrak{m}_\lambda)$. Since $\phi \bmod \mathfrak{m}_\lambda$ is an isomorphism, $C^\bullet(\phi) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)/\mathfrak{m}_\lambda$ is acyclic, which in turn implies that $\mathfrak{m}_\lambda$ is not contained in the support of $H^*(C^\bullet(\phi))$. But $\mathfrak{m}_\lambda$ is an arbitrary maximal ideal in $\mathscr{O}(\Omega)$, so Nakayama's lemma now shows that $H^*(C^\bullet(\phi))$ vanishes identically. Thus $C^\bullet(\phi)$ is acyclic and ϕ is a quasi-isomorphism as desired. □

3.2 Finite-slope eigenpackets and non-critical classes

In this section we explain two results which are fundamental in our analysis. First of all, we recall and summarize some of the work of Eichler, Shimura, Matsushima, Borel-Wallach, Franke, and Li-Schwermer on the cohomology of arithmetic groups. Next, we state a fundamental theorem of Ash-Stevens and Urban (Theorem 6.4.1 of (AS08), Proposition 4.3.10 of (Urb11)) relating overconvergent cohomology classes of small slope with classical automorphic forms. The possibility of such a result was largely the original *raison d'etre* of overconvergent cohomology; in the case $\mathbf{G} = \mathrm{GL}_2/\mathbf{Q}$, Stevens proved this theorem in a famous preprint (Ste94).

Let $\lambda \in X_+^*$ be a B-dominant algebraic weight, and let $K_f \subset \mathbf{G}(\mathbf{A}_f)$ be any open compact subgroup. By fundamental work of Franke, the cohomology $H^*(Y(K_f), \mathcal{L}_\lambda)_{\mathbf{C}} = H^*(Y(K_f), \mathcal{L}_\lambda) \otimes_{\mathbf{Q}_p, \iota} \mathbf{C}$ admits an analytically defined splitting

$$H^*(Y(K_f), \mathcal{L}_\lambda)_{\mathbf{C}} \simeq H^*_{\mathrm{cusp}}(Y(K_f), \mathcal{L}_\lambda)_{\mathbf{C}} \oplus H^*_{\mathrm{Eis}}(Y(K_f), \mathcal{L}_\lambda)_{\mathbf{C}}$$

into $\mathbf{T}(\mathbf{G}(\mathbf{A}_f), K_f)_{\mathbf{C}}$-stable submodules, which we refer to as the cuspidal and Eisenstein cohomology, respectively. The cuspidal cohomology admits an exact description in terms of cuspidal automorphic forms (BW00, Fra98, FS98):

Proposition 3.2.1. *There is a canonical isomorphism*

$$H^*_{\mathrm{cusp}}(Y(K_f), \mathcal{L}_\lambda)_{\mathbf{C}} \simeq \bigoplus_{\pi \in L^2_{\mathrm{cusp}}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A}))} m(\pi) \pi_f^{K_f} \otimes H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes \mathcal{L}_\lambda)$$

of graded $\mathbf{T}(\mathbf{G}(\mathbf{A}_f), K_f)$-modules, where $m(\pi)$ denotes the multiplicity of π in $L^2_{\mathrm{cusp}}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A}))$. If λ is a regular weight, the natural inclusion of $H^*_{\mathrm{cusp}}(Y(K_f), \mathcal{L}_\lambda)$ into $H^*_!(Y(K_f), \mathcal{L}_\lambda)$ is an isomorphism.

Note that if π contributes nontrivially to the direct sum decomposition of the previous proposition, the central and infininitesimal characters of π_∞ are necessarily equal to those of $\mathcal{L}_\lambda$.

For any weight $\lambda \in \mathscr{W}_{K^p}(L)$ and a given controlling operator U_t, we define $\mathbf{T}_{\lambda,h}(K^p)$ as the subalgebra of $\mathrm{End}_L\left(H^*(K^p, \mathcal{D}_\lambda)_{\leq h}\right)$ generated by the image of $\mathbf{T}(K^p) \otimes_{\mathbf{Q}_p} L$, and we set $\mathbf{T}_\lambda(K^p) = \lim_{\infty \leftarrow h} \mathbf{T}_{\lambda,h}(K^p)$. The algebra $\mathbf{T}_\lambda(K^p)$ is independent of the choice of controlling operator used in its definition.

Definition 3.2.2. *A* finite-slope eigenpacket of weight λ and level K^p *(or simply a* finite-slope eigenpacket*) is an algebra homomorphism* $\phi : \mathbf{T}_\lambda(K^p) \to \overline{\mathbf{Q}_p}$.

If ϕ is a finite-slope eigenpacket, we shall regard the contraction of $\ker \phi$ under the structure map $\mathbf{T}(K^p) \to \mathbf{T}_\lambda(K^p)$ as a maximal ideal in $\mathbf{T}(K^p)$, which we also denote by $\ker \phi$. Note that $\mathbf{T}_\lambda(K^p)$ is a countable direct product of zero-dimensional Artinian local rings, and the factors in this direct product are in natural bijection with the finite-slope eigenpackets.

A weight λ is *arithmetic* if it factors as the product of a finite-order character ε of $T(\mathbf{Z}_p)$ and an element λ^{alg} of X^*; if $\lambda^{\mathrm{alg}} \in X^*_+$ we say λ is *dominant arithmetic*. If $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ is a dominant arithmetic weight, we are going to formulate some comparisons between $H^*(K^p, \mathcal{D}_\lambda)$ and $H^*(Y(K^p I_1^s), \mathcal{L}_{\lambda^{\mathrm{alg}}})$. In order to do this, we need to twist the natural Hecke action on the latter module slightly. More precisely, if M is any $\mathcal{A}_p$-module and $\lambda \in X^*$, we define the $\star$*-action in weight* λ by $U_t \star_\lambda m = \lambda(t)^{-1} U_t m$ (AS08). The map i_λ defined in §2.2 induces a morphism

$$i_\lambda : H^*(K^p, \mathcal{D}_\lambda) \to H^*(K^p I_1^s, \mathcal{L}_{\lambda^{\mathrm{alg}}})$$

for any $s \geq s[\varepsilon]$ which intertwines the standard action of $\mathbf{T}(K^p)$ on the source with the $\star$-action on the target.

Definition 3.2.3. *If $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ is an arithmetic weight and $s = s[\varepsilon]$, a finite-slope eigenpacket is* classical *if the module $H^*(Y(K^p I_1^s), \mathcal{L}_{\lambda^{\mathrm{alg}}})$ is nonzero after localization at* $\ker \phi$*, and* noncritical *if the map*

$$i_\lambda : H^*(K^p, \mathcal{D}_\lambda) \to H^*(K^p I_1^s, \mathcal{L}_{\lambda^{\mathrm{alg}}})$$

becomes an isomorphism after localization at $\ker \phi$*. A classical eigenpacket is* interior *if $H^*_\partial(K^p I_1^s, \mathcal{L}_{\lambda^{\mathrm{alg}}})$ vanishes after localization at* $\ker \phi$*, and* strongly interior *if $H^*_\partial(K^p, \mathcal{D}_\lambda)$ vanishes after localization at* $\ker \phi$ *as well.*

Next we formulate a result which generalizes Stevens's control theorem (Ste94). Given $\lambda \in X^*$, we define an action of the Weyl group W by ${}^w\lambda = w \cdot (\lambda + \rho) - \rho$.

Definition 3.2.4. *Fix a controlling operator U_t, $t \in \Lambda^+$. Given an arithmetic weight*

$\lambda = \lambda^{\mathrm{alg}}\varepsilon$, *a rational number* h *is a* small slope for λ *if*

$$h < \inf_{w \in W \smallsetminus \{\mathrm{id}\}} v_p(w \cdot \lambda^{\mathrm{alg}}(t)) - v_p(\lambda^{\mathrm{alg}}(t)).$$

Theorem 3.2.5 (Ash-Stevens, Urban). *Fix an arithmetic weight* $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ *and a controlling operator* U_t. *If* h *is a small slope for* λ, *there is a natural isomorphism of Hecke modules*

$$H^*(K^p, \mathcal{D}_\lambda)_{\leq h} \simeq H^*(Y(K^p I_1^s), \mathcal{L}_{\lambda^{\mathrm{alg}}})_{\leq h}^{T(\mathbf{Z}/p^s\mathbf{Z})=\varepsilon}$$

for any $s \geq s[\varepsilon]$.

Proof (sketch). Suppose $\lambda = \lambda^{\mathrm{alg}}$ for simplicity. Urban constructs a second quadrant spectral sequence

$$E_1^{i,j} = \bigoplus_{w \in W,\, \ell(w) = -i} H^j(K^p, \mathcal{D}_{w \cdot \lambda})^{\mathrm{fs}} \Rightarrow H^{i+j}(K^p I, \mathcal{L}_\lambda)^{\mathrm{fs}}$$

which is equivariant for U_t if we twist the action as follows: U_t acts through the $\star$-action in weight λ on the target and $(\lambda(t)^{-1} w \cdot \lambda)(t))U_t$ acts on the w-summand of the E_1-page. In particular, taking the slope-$\leq h$ part yields a spectral sequence

$$E_1^{i,j} = \bigoplus_{w \in W,\, \ell(w) = -i} H^j(K^p, \mathcal{D}_{w \cdot \lambda^{\mathrm{alg}}})_{\leq h - v_p(w \cdot \lambda(t)) + v_p(\lambda(t))} \Rightarrow H^{i+j}(K^p I, \mathcal{L}_\lambda)_{\leq h}.$$

But any element of $\mathcal{A}_p^+$ is contractive on $H^j(K^p, \mathcal{D}_\lambda)$, so the w-summand of the E_1-page is now empty for $w \neq \mathrm{id}$. $\square$

3.3 The spectral sequences

In this section we introduce our main technical tool for analyzing overconvergent cohomology. Fix a choice of tame level K^p and an augmented Borel-Serre complex $C_\bullet(K^p, -)$.

Theorem 3.3.1. *Fix a slope datum* (U_t, Ω, h), *and let* $\Sigma \subseteq \Omega$ *be an arbitrary rigid Zariski closed subspace. Then* $H^*(K^p, \mathcal{D}_\Sigma)$ *admits a slope-*$\leq h$ *decomposition, and there is* *a convergent first quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathscr{O}(\Omega)}(H_j(K^p, \mathcal{A}_\Omega)_{\leq h}, \mathscr{O}(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\Sigma)_{\leq h}.$$

Furthermore, there is a convergent second quadrant spectral sequence

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\mathscr{O}(\Omega)}(H^j(K^p, \mathcal{D}_\Omega)_{\leq h}, \mathscr{O}(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\Sigma)_{\leq h}.$$

In addition, there are analogous spectral sequences relating Borel-Moore homology with compactly supported cohomology, and boundary homology with boundary cohomology, and there are morphisms between the spectral sequences compatible with the morphisms between these different cohomology theories. Finally, the spectral sequences and the morphisms between them are equivariant for the natural Hecke actions on their E_2 pages and abutments; more succinctly, they are spectral sequences of $\mathbf{T}(K^p)$-modules.

When no ambiguity is likely, we will refer to the two spectral sequences of Theorem 3.3.1 as "the Ext spectral sequence" and "the Tor spectral sequence." The Hecke-equivariance of these spectral sequences is crucial for applications, since it allows one to localize the entire spectral sequence at any ideal in the Hecke algebra.

Proof of Theorem 3.3.1. By the isomorphisms proved in §3.1, it suffices to construct a Hecke-equivariant spectral sequence $\mathrm{Ext}^i_{\mathscr{O}(\Omega)}(H_j(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathbf{D}^s_\Sigma)_{\leq h}$ for some $s \geq s[\Omega]$.

Consider the hyperext group $\mathbf{E}\mathrm{xt}^n_{\mathscr{O}(\Omega)}(C^{ad}_\bullet(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma))$. Since $C^{ad}_\bullet(K^p, \mathbf{A}^s_\Omega)$ is a complex of $\mathbf{T}(K^p)$-modules, this hyperext group is naturally a $\mathbf{T}(K^p)$-module, and the hypercohomology spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathscr{O}(\Omega)}(H_j(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma)) \Rightarrow \mathbf{E}\mathrm{xt}^{i+j}_{\mathscr{O}(\Omega)}(C^{ad}_\bullet(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma))$$

is a spectral sequence of $\mathbf{T}(K^p)$-modules. On the other hand, the quasi-isomorphism $C^{ad}_\bullet(K^p, \mathbf{A}^s_\Omega) \simeq C_\bullet(K^p, \mathbf{A}^s_\Omega)$ in $\mathbf{D}^b(A(\Omega))$ together with the slope-$\leq h$ decomposition $C_\bullet(K^p, \mathbf{A}^s_\Omega) \simeq C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \oplus C_\bullet(K^p, \mathbf{A}^s_\Omega)_{>h}$ induces Hecke-stable slope-$\leq h$-decompositions of the abutment and of the entries on the E_2 page. By Proposition 2.3.2, the slope decomposition of the E_2 page induces slope decompositions of all entries on all higher pages of the spectral sequence. In other words, we may pass to the "slope-$\leq h$ part" of the hypercohomology spectral sequence in a Hecke-equivariant way, getting a spectral sequence

$$'E_2^{i,j} = \mathrm{Ext}^i_{\mathscr{O}(\Omega)}(H_j(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma))_{\leq h} \Rightarrow \mathbf{E}\mathrm{xt}^{i+j}_{\mathscr{O}(\Omega)}(C^{ad}_\bullet(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma))_{\leq h}$$

of $\mathbf{T}(K^p)$-modules. But $\mathrm{Ext}^i_{\mathscr{O}(\Omega)}(H_j(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma))_{\leq h} \simeq \mathrm{Ext}^i_{\mathscr{O}(\Omega)}(H_j(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Sigma))$, and

$$\begin{aligned}
\mathbf{E}\mathrm{xt}^n_{\mathscr{O}(\Omega)}(C^{ad}_\bullet(K^p, \mathbf{A}^s_\Omega), \mathscr{O}(\Sigma))_{\leq h} &\simeq \mathbf{E}\mathrm{xt}^n_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Sigma)) \\
&\simeq H^n\left(\mathbf{R}\mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Sigma))\right) \\
&\simeq H^n\left(\mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Sigma))\right) \\
&\simeq H^n(K^p, \mathbf{D}^s_\Sigma)_{\leq h},
\end{aligned}$$

where the third line follows from the projectivity of each $C_i(K^p, \mathbf{A}^s_\Omega)_{\leq h}$ and the fourth line follows from the proof of Proposition 3.1.5.

For the Tor spectral sequence, the isomorphism

$$C^\bullet(K^p, \mathbf{D}^s_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Sigma) \simeq C^\bullet(K^p, \mathbf{D}^s_\Sigma)_{\leq h}$$

yields an isomorphism

$$C^\bullet(K^p, \mathbf{D}^s_\Omega)_{\leq h} \otimes^{\mathbf{L}}_{\mathscr{O}(\Omega)} \mathscr{O}(\Sigma) \simeq C^\bullet(K^p, \mathbf{D}^s_\Sigma)_{\leq h}$$

of $\mathbf{T}(K^p)$-module complexes in $\mathbf{D}^b(\mathscr{O}(\Omega))$, and the result follows analogously from the hypertor spectral sequence

$$\mathrm{Tor}^R_{-i}(H^j(C^\bullet), N) \Rightarrow \mathbf{T}\mathrm{or}^R_{-i-j}(C^\bullet, N).$$

Remark 3.1.1. If (Ω, h) is a slope datum, Σ_1 is Zariski-closed in Ω, and Σ_2 is Zariski-closed in Σ_1, the transitivity of the derived tensor product yields an isomorphism

$$\begin{aligned}
C^\bullet(K^p, \mathcal{D}_{\Sigma_2})_{\leq h} &\simeq C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes^{\mathbf{L}}_{\mathscr{O}(\Omega)} \mathscr{O}(\Sigma_2) \\
&\simeq C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes^{\mathbf{L}}_{\mathscr{O}(\Omega)} \mathscr{O}(\Sigma_1) \otimes^{\mathbf{L}}_{\mathscr{O}(\Sigma_1)} \mathscr{O}(\Sigma_2) \\
&\simeq C^\bullet(K^p, \mathcal{D}_{\Sigma_1})_{\leq h} \otimes^{\mathbf{L}}_{\mathscr{O}(\Sigma_1)} \mathscr{O}(\Sigma_2)
\end{aligned}$$

which induces a relative version of the Tor spectral sequence, namely

$$E_2^{i,j} = \mathrm{Tor}^i_{A(\Sigma_1)}(H^j(K^p, \mathcal{D}_{\Sigma_1})_{\leq h}, A(\Sigma_2)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_{\Sigma_2})_{\leq h}.$$

This spectral sequence plays an important role in Newton's proof of Theorem 4.5.3 (cf. the appendix of (Han12b)).

The boundary and Borel-Moore/compactly supported spectral sequences Notation as in §2.1, let $\overline{D_\infty}$ denote the Borel-Serre bordification of D_∞, and let $\partial\overline{D_\infty} = \overline{D_\infty} \smallsetminus D_\infty$. Setting $\overline{D_{\mathbf{A}}} = \overline{D_\infty} \times \mathbf{G}(\mathbf{A}_f)$ and $\partial\overline{D_{\mathbf{A}}} = \partial\overline{D_\infty} \times \mathbf{G}(\mathbf{A}_f)$, the natural map $C_\bullet(D_{\mathbf{A}}) \to C_\bullet(\overline{D_{\mathbf{A}}})$ induces a functorial isomorphism $H_*(K_f, M) \simeq H_*\left(C_\bullet(\overline{D_{\mathbf{A}}}) \otimes_{\mathbf{Z}[\mathbf{G}(\mathbf{Q})\times K_f]} M\right)$, so we may redefine $C_\bullet^{ad}(K_f, M)$ as $C_\bullet(\overline{D_{\mathbf{A}}}) \otimes_{\mathbf{Z}[\mathbf{G}(\mathbf{Q})\times K_f]} M$. Setting $C_\bullet^{\partial,ad}(K_f, M) = C_\bullet(\partial\overline{D_{\mathbf{A}}}) \otimes_{\mathbf{Z}[\mathbf{G}(\mathbf{Q})\times K_f]} M$, the natural inclusion induces a map $C_\bullet^{\partial,ad}(K_f, M) \to C_\bullet^{ad}(K_f, M)$, and we define $C_\bullet^{\mathrm{BM},ad}(K_f, M)$ as the cone of this map. Not surprisingly, the homology of $C_\bullet^{\partial}$ (resp. $C_\bullet^{\mathrm{BM}}$) computes boundary (resp. Borel-Moore) homology. Choosing a triangulation of $\overline{Y(K_f)}$ induces a triangulation on the boundary, and yields a complex $C_\bullet^{\partial}(K_f, , M)$ together with a map $C_\bullet^{\partial}(K_f, , M) \to C_\bullet^{\partial}(K_f, , M)$; defining $C_\bullet^{\mathrm{BM}}$ as the cone of this map, these complexes all fit into a big diagram

$$\begin{tikzcd}
C_\bullet^{\partial,ad}(K_f, -) \arrow[r] & C_\bullet^{ad}(K_f, -) \arrow[r] & C_\bullet^{\mathrm{BM},ad}(K_f, -) \\
C_\bullet^{\partial}(K_f, -) \arrow[r] \arrow[u] & C_\bullet(K_f, -) \arrow[r] \arrow[u] & C_\bullet^{\mathrm{BM}}(K_f, -) \arrow[u]
\end{tikzcd}$$

in which the rows are exact triangles functorial in M, and the vertical arrows are quasi-isomorphisms. We make analogous definitions of $C^\bullet_{c,ad}(K_f, M)$, etc.

The boundary and Borel-Moore/compactly supported sequences, and the morphisms between them, follow from taking the slope-$\leq h$ part of the diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet^{\mathrm{BM},ad}(K^p, \mathbf{A}_\Omega^s), \mathscr{O}(\Sigma)) \arrow[r] \arrow[d] & C^\bullet_{c,ad}(K^p, \mathrm{Hom}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Sigma))) \arrow[d] \\
\mathbf{R}\mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet^{ad}(K^p, \mathbf{A}_\Omega^s), \mathscr{O}(\Sigma)) \arrow[r] \arrow[d] & C^\bullet_{ad}(K^p, \mathrm{Hom}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Sigma))) \arrow[d] \\
\mathbf{R}\mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet^{\partial,ad}(K^p, \mathbf{A}_\Omega^s), \mathscr{O}(\Sigma)) \arrow[r] & C^\bullet_{\partial,ad}(K^p, \mathrm{Hom}_{\mathscr{O}(\Omega)}(\mathbf{A}_\Omega^s, \mathscr{O}(\Sigma)))
\end{tikzcd}$$

in which the horizontal arrows are quasi-isomorphisms, the columns are exact triangles in $\mathbf{D}^b(\mathscr{O}(\Omega))$, and the diagram commutes for the natural action of $\mathbf{T}(K^p)$.

Chapter 4

The geometry of eigenvarieties

In this chapter we begin to use global rigid analytic geometry in a more serious way; the bible of the subject is (BGR84), and (Con08) is a nice survey of the main ideas. We shall repeatedly and tacitly use the fact that if Ω' is an affinoid subdomain of an affinoid space Ω, $\mathscr{O}(\Omega')$ is a flat $\mathscr{O}(\Omega)$-module; this is an easy consequence of the universal property of an affinoid subdomain together with the local criterion for flatness.

4.1 Fredholm hypersurfaces

Let A be an affinoid integral domain. We say that such an A is *relatively factorial* if for any $f = \sum_{n=0}^{\infty} a_n X^n \in A\langle X\rangle$ with $a_0 = 1$, (f) factors uniquely as a product of principal prime ideals (f_i) where each f_i may be chosen with constant term 1. A rigid analytic space $\mathscr{W}$ is relatively factorial if it has an admissible covering by relatively factorial affinoids. Throughout the remainder of this thesis, we reserve the letter $\mathscr{W}$ for a relatively factorial rigid analytic space.

Definition 4.1.1. *A* Fredholm series *is a global section $f \in \mathscr{O}(\mathscr{W} \times \mathbf{A}^1)$ such that under the map $\mathscr{O}(\mathscr{W} \times \mathbf{A}^1) \xrightarrow{j^*} \mathscr{O}(\mathscr{W})$ induced by $j : \mathscr{W} \times \{0\} \to \mathscr{W} \times \mathbf{A}^1$ we have $j^*f = 1$. A* Fredholm hypersurface *is a closed immersion $\mathscr{Z} \subset \mathscr{W} \times \mathbf{A}^1$ such that the ideal sheaf of $\mathscr{Z}$ is generated by a Fredholm series f, in which case we write $\mathscr{Z} = \mathscr{Z}(f)$.*

Note that the natural projection $\mathscr{W} \times \mathbf{A}^1 \to \mathscr{W}$ induces a map $w : \mathscr{Z} \to \mathscr{W}$. Let $\mathscr{O}(\mathscr{W})\{\{X\}\}$ denote the subring of $\mathscr{O}(\mathscr{W})[[X]]$ consisting of series $\sum_{n=0}^{\infty} a_n X^n$ such that $|a_n|_\Omega r^n \to 0$ as $n \to \infty$ for any affinoid $\Omega \subset \mathscr{W}$ and any $r \in \mathbf{R}_{>0}$. The natural injection $\mathscr{O}(\mathscr{W} \times \mathbf{A}^1) \simeq \mathscr{O}(\mathscr{W})\{\{X\}\} \hookrightarrow \mathscr{O}(\mathscr{W})[[X]]$ identifies the monoid of Fredholm series with elements of $\mathscr{O}(\mathscr{W})\{\{X\}\}$ such that $a_0 = 1$. When $\mathscr{W}$ is relatively factorial, the ring $\mathscr{O}(\mathscr{W})\{\{X\}\}$ admits a good factorization theory, and we may speak of irreducible Fredholm series without ambiguity. We say a collection of distinct irreducible Fredholm series $\{f_i\}_{i \in I}$ is *locally finite* if $\mathscr{Z}(f_i) \cap U = \emptyset$ for all but finitely many $i \in I$ and any quasi-compact admissible open subset $U \subset \mathscr{W} \times \mathbf{A}^1$.

Proposition 4.1.2 (Coleman-Mazur, Conrad). *If $\mathscr{W}$ is relatively factorial, any Fredholm series f admits a factorization $f = \prod_{i \in I} f_i^{n_i}$ as a product of irreducible Fredholm series with $n_i \geq 1$; any such factorization is unique up to reordering the terms, the collection $\{f_i\}_{i \in I}$ is locally finite, and the irreducible components of $\mathscr{Z}(f)$ are exactly the Fredholm hypersurfaces $\mathscr{Z}(f_i^{n_i})$. The nilreduction of $\mathscr{Z}(f)$ is $\mathscr{Z}(\prod_{i \in I} f_i)$.*

Proof. See §1 of (CM98) and §4 of (Con99) (especially Theorems 4.2.2 and 4.3.2). □

Proposition 4.1.3. *If $\mathscr{Z}$ is a Fredholm hypersurface, the image $w(\mathscr{Z})$ is Zariski-open in $\mathscr{W}$.*

Proof. We may assume $\mathscr{Z} = \mathscr{Z}(f)$ with $f = 1 + \sum_{n=1}^{\infty} a_n X^n$ irreducible. By Lemma 1.3.2 of (CM98), the fiber of $\mathscr{Z}$ over $\lambda \in \mathscr{W}(\overline{\mathbf{Q}_p})$ is empty if and only if $a_n \in \mathfrak{m}_\lambda$ for all n, if and only if $\mathscr{I} = (a_1, a_2, a_3, \dots) \subset \mathfrak{m}_\lambda$. The ideal $\mathscr{I}$ is naturally identified with the global sections of a coherent ideal sheaf, which cuts out a closed immersion $V(\mathscr{I}) \hookrightarrow \mathscr{W}$ in the usual way, and $w(\mathscr{Z})$ is exactly the complement of $V(\mathscr{I})$. □

Given a Fredholm hypersurface $\mathscr{Z} = \mathscr{Z}(f)$, a rational number $h \in \mathbf{Q}$, and an affinoid $\Omega \subset \mathscr{W}$, we define $\mathscr{Z}_{\Omega,h} = \mathscr{O}(\Omega)\left\langle p^h X \right\rangle / (f(X))$ regarded as an admissible affinoid open subset of $\mathscr{Z}$. The natural map $\mathscr{Z}_{\Omega,h} \to \Omega$ is flat but not necessarily finite, and we define an affinoid of the form $\mathscr{Z}_{\Omega,h}$ to be *slope-adapted* if $\mathscr{Z}_{\Omega,h} \to \Omega$ is a finite flat map. The affinoid $\mathscr{Z}_{\Omega,h}$ is slope-adapted if and only if $f_\Omega = f|_{\mathscr{O}(\Omega)\{\{X\}\}}$ admits a slope-$\leq h$ factorization $Q(X) \cdot R(X)$, in which case $\mathscr{O}(\mathscr{Z}_{\Omega,h}) \simeq \mathscr{O}(\Omega)[X]/(Q(X))$.

Proposition 4.1.4. *For any Fredholm hypersurface $\mathscr{Z}$, the collection of slope-adapted affinoids forms an admissible cover of $\mathscr{Z}$.*

Proof. See §4 of (Buz07).

4.2 Eigenvariety data

Definition 4.2.1. *An* eigenvariety datum *is a tuple $\mathfrak{D} = (\mathscr{W}, \mathscr{Z}, \mathscr{M}, \mathbf{T}, \psi)$ where $\mathscr{W}$ is a separated, reduced, equidimensional, relatively factorial rigid analytic space, $\mathscr{Z} \subset \mathscr{W} \times \mathbf{A}^1$*

is a Fredholm hypersurface, $\mathscr{M}$ is a coherent analytic sheaf on $\mathscr{Z}$, $\mathbf{T}$ is a commutative $\mathbf{Q}_p$-algebra, and ψ is a $\mathbf{Q}_p$-algebra homomorphism $\psi : \mathbf{T} \to \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M})$.

In practice $\mathbf{T}$ will be a Hecke algebra, $\mathscr{Z}$ will be a "spectral variety" parametrizing the eigenvalues of some distinguished operator $U \in \mathbf{T}$ on some graded module M^* of p-adic automorphic forms or on a complex whose cohomology yields M^*, and $\mathscr{M}$ will be the natural "spreading out" of M^* to a coherent sheaf over $\mathscr{Z}$. We do *not* require that $\mathscr{M}$ be locally free on $\mathscr{Z}$.

Theorem 4.2.2. *Given an eigenvariety datum $\mathfrak{D}$, there is a rigid analytic space $\mathscr{X} = \mathscr{X}(\mathfrak{D})$ together with a finite morphism $\pi : \mathscr{X} \to \mathscr{Z}$, a morphism $w : \mathscr{X} \to \mathscr{W}$, an algebra homomorphism $\phi_{\mathscr{X}} : \mathbf{T} \to \mathscr{O}(\mathscr{X})$, and a coherent sheaf $\mathscr{M}^\dagger$ on $\mathscr{X}$ such that $\pi_*\mathscr{M}^\dagger \simeq \mathscr{M}$ and the diagram*

$$\begin{tikzcd} \mathbf{T} \arrow[r, "\psi"] \arrow[d, "\phi_{\mathscr{X}}"'] & \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M}) \\ \mathscr{O}(\mathscr{X}) \arrow[r] & \mathrm{End}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{M}^\dagger) \arrow[u, "\pi_*"'] \end{tikzcd}$$

commutes. The points of $\mathscr{X}$ lying over $z \in \mathscr{Z}$ are in bijection with the generalized eigenspaces for the action of $\mathbf{T}$ on $\mathscr{M}(z)$.

Proof. Let $\mathscr{C}\mathrm{ov} = \{\Omega_i\}_{i \in I}$ be an admissible affinoid cover of $\mathscr{Z}$; we abbreviate $\Omega_i \cap \Omega_j$ by Ω_{ij}. For any Ω_i we let $\mathbf{T}_{\Omega_i}$ be the finite $\mathscr{O}(\Omega_i)$-subalgebra of $\mathrm{End}_{\mathscr{O}(\Omega_i)}(\mathscr{M}(\Omega_i))$ generated by $\mathrm{im}\psi$, with structure map $\phi_{\Omega_i} : \mathbf{T} \to \mathbf{T}_{\Omega_i}$. Let $\mathscr{X}_{\Omega_i}$ be the affinoid rigid space $\mathrm{Sp}\mathbf{T}_{\Omega_i}$, with $\pi : \mathscr{X}_{\Omega_i} \to \Omega_i$ the natural morphism. The canonical morphisms $\mathbf{T}_{\Omega_i} \otimes_{\mathscr{O}(\Omega_i)} \mathscr{O}(\Omega_{ij}) \to \mathbf{T}_{\Omega_{ij}}$ are isomorphisms, and so we may glue the affinoid rigid spaces $\mathscr{X}_{\Omega_i}$ together via their overlaps $\mathscr{X}_{\Omega_{ij}}$ into a rigid space $\mathscr{X}$ together with a finite map $\pi : \mathscr{X} \to \mathscr{Z}$. The $\mathbf{T}_{\Omega_i}$-module structure on $\mathscr{M}(\Omega_i)$ is compatible with the transition maps, and so these modules glue to a coherent sheaf $\mathscr{M}^\dagger$. The structure maps ϕ_{Ω_i} glue to a map $\phi : \mathbf{T} \to \mathscr{O}(\mathscr{X})$ which is easily seen to be an algebra homomorphism. The remainder of the theorem is tautological from the construction. □

The space $\mathscr{X}$ is the **eigenvariety** associated with the given eigenvariety datum. For any point $x \in \mathscr{X}$, we write $\phi_{\mathscr{X}}(x) : \mathbf{T} \to k(x)$ for the composite map

$$\big(\mathscr{O}(\mathscr{X}) \to \mathscr{O}_{\mathscr{X},x} \twoheadrightarrow k(x)\big) \circ \phi_{\mathscr{X}},$$

and we say $\phi_{\mathscr{X}}(x)$ is the **eigenpacket** parametrized by the point x. If the map $x \mapsto \phi_{\mathscr{X}}(x)$

determines a bijection of $\mathscr{X}(\overline{\mathbf{Q}_p})$ with a set of eigenpackets of a certain type, we write $\phi \mapsto x_\phi$ for the inverse map.

4.3 Eigenvariety data from overconvergent cohomology

Fix $\mathbf{G}$, K^p, a controlling operator U_t, and an augmented Borel-Serre complex $C_\bullet(K^p, -)$. For $\Omega \subset \mathscr{W}_{K^p}$ an affinoid open, the Fredholm series $f_\Omega(X) = \det(1 - \tilde{U}_t X)|C_\bullet(K^p, \mathbf{A}^s_\Omega)$ is well-defined independently of $s \geq s[\Omega]$ by Proposition 3.1.1, and if $\Omega' \subset \Omega$ is open then $f_\Omega(X)|_{\Omega'} = f_{\Omega'}(X)$. By Tate's acyclicity theorem, there is a unique $f(X) \in \mathscr{O}(\mathscr{W})\{\{X\}\}$ with $f(X)|_\Omega = f_\Omega(X)$ for all Ω. Set $\mathscr{Z} = \mathscr{Z}_f$. If $\mathscr{Z}_{\Omega,h} \subset \mathscr{Z}$ is a slope-adapted affinoid, then $C^\bullet(K^p, \mathcal{D}_\Omega)$ admits a slope-$\leq h$ decomposition, with $C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \simeq \mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}, \mathscr{O}(\Omega))$ for any $s \geq s[\Omega]$, and $C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$ is naturally a (graded) module over $\mathscr{O}(\mathscr{Z}_{\Omega,h}) \cong \mathscr{O}(\Omega)[X]/(Q_{\Omega,h}(X))$ via the map $X \mapsto \tilde{U}_t^{-1}$; here $Q_{\Omega,h}(X)$ denotes the slope-$\leq h$ factor of f_Ω.

Proposition 4.3.1. *There is a unique complex $\mathscr{K}^\bullet$ of coherent analytic sheaves on $\mathscr{Z}$ such that $\mathscr{K}^\bullet(\mathscr{Z}_{\Omega,h}) \cong C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$ for any slope-adapted affinoid $\mathscr{Z}_{\Omega,h}$.*

Proof. For $\mathscr{Z}_{\Omega,h}$ a slope-adapted affinoid, we simply set $\mathscr{K}^\bullet(\mathscr{Z}_{\Omega,h}) \cong C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$, with $\mathscr{K}^\bullet(\mathscr{Z}_{\Omega,h})$ regarded as an $\mathscr{O}(\mathscr{Z}_{\Omega,h})$-module as in the previous paragraph. We are going to show that the formation of $\mathscr{K}^\bullet(\mathscr{Z}_{\Omega,h})$ is compatible with overlaps of slope-adapted affinoids; since slope-adapted affinoids form a base for the ambient G-topology on $\mathscr{Z}$, this immediately implies that the $\mathscr{K}^\bullet(\mathscr{Z}_{\Omega,h})$'s glue together into a sheaf over $\mathscr{Z}$.

If $\mathscr{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$ and $\Omega' \subset \Omega$ with Ω' connected, a calculation gives $\mathscr{O}(\mathscr{Z}_{\Omega',h}) \simeq \mathscr{O}(\mathscr{Z}_{\Omega,h}) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega')$, so then $\mathscr{Z}_{\Omega',h} \in \mathscr{C}\mathrm{ov}$. Fix $\mathscr{Z}_{\Omega',h'} \subseteq \mathscr{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$ with $\mathscr{Z}_{\Omega',h'} \in \mathscr{C}\mathrm{ov}$; we necessarily have $\Omega' \subseteq \Omega$, and we may assume $h' \leq h$. Set $C_{\Omega,h} = C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$. We now trace through the following sequence of canonical isomorphisms:

$$\begin{aligned} C_{\Omega,h} \otimes_{\mathscr{O}(\mathscr{Z}_{\Omega,h})} \mathscr{O}(\mathscr{Z}_{\Omega',h'}) &\simeq C_{\Omega,h} \otimes_{\mathscr{O}(\mathscr{Z}_{\Omega,h})} \mathscr{O}(\mathscr{Z}_{\Omega',h}) \otimes_{\mathscr{O}(\mathscr{Z}_{\Omega',h})} \mathscr{O}(\mathscr{Z}_{\Omega',h'}) \\ &\simeq \Big(C_{\Omega,h} \otimes_{\mathscr{O}(\mathscr{Z}_{\Omega,h})} \mathscr{O}(\mathscr{Z}_{\Omega,h}) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \Big) \otimes_{\mathscr{O}(\mathscr{Z}_{\Omega',h})} \mathscr{O}(\mathscr{Z}_{\Omega',h'}) \\ &\simeq \big(C_{\Omega,h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \big) \otimes_{\mathscr{O}(\mathscr{Z}_{\Omega',h})} \mathscr{O}(\mathscr{Z}_{\Omega',h'}) \\ &\simeq C_{\Omega',h} \otimes_{\mathscr{O}(\mathscr{Z}_{\Omega',h})} \mathscr{O}(\mathscr{Z}_{\Omega',h'}) \\ &\simeq C_{\Omega',h'}. \end{aligned}$$

The fourth line here follows from Proposition 3.1.5. □

Taking the cohomology of $\mathscr{K}^\bullet$ yields a graded sheaf $\mathscr{M}^*$ on $\mathscr{Z}$ together with canonical isomorphisms $\mathscr{M}^*(\mathscr{Z}_{\Omega,h}) \cong H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ for any slope-adapted affinoid $\mathscr{Z}_{\Omega,h}$. By Proposition 3.1.5 the natural maps $\mathbf{T}(K^p) \to \mathrm{End}_{\mathscr{O}(\mathscr{Z}_{\Omega,h})}(H^*(K^p, \mathcal{D}_\Omega)_{\leq h})$ glue together into a degree-preserving algebra homomorphism $\psi : \mathbf{T}(K^p) \to \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M}^*)$.

Definition 4.3.2. *The eigenvariety $\mathscr{X}_{\mathbf{G}}(K^p)$ is the eigenvariety associated with the eigenvariety datum $(\mathscr{W}_{K^p}, \mathscr{Z}, \mathscr{M}^*, \mathbf{T}(K^p), \psi)$. For n a given integer, $\mathscr{X}^n_{\mathbf{G}}(K^p)$ is the eigenvariety associated with the eigenvariety datum $(\mathscr{W}_{K^p}, \mathscr{Z}, \mathscr{M}^n, \mathbf{T}(K^p), \psi)$.*

Note that $\mathscr{Z}$ is highly noncanonical, depending as it does on a choice of augmented Borel-Serre complex; the idea of lifting the U_t action to the level of chain complexes is due to Ash (AS08). However, $\mathscr{X}_{\mathbf{G}}(K^p)$ is completely canonical and independent of this choice: setting $\mathscr{J} = \mathrm{ann}_{\mathscr{O}_{\mathscr{Z}}} \mathscr{M}^* \subset \mathscr{O}_{\mathscr{Z}}$, the closed immersion $Z \hookrightarrow \mathscr{W} \times \mathbf{A}^1$ cut out by

$$\mathscr{O}_{\mathscr{W} \times \mathbf{A}^1} \twoheadrightarrow \mathscr{O}_{\mathscr{Z}} \twoheadrightarrow \mathscr{O}_{\mathscr{Z}} / \mathscr{J} = \mathscr{O}_Z$$

is independent of all choices. Note also that in practice, the eigenvarieties $\mathscr{X}_{\mathbf{G}}(K^p)$ carry some extra structure which we don't really exploit in this thesis: in particular, the sheaves $\mathscr{M}^\dagger$ are sheaves of $\mathbf{T}_{\mathrm{ram}}(K^p)$-modules. Our first main result on the eigenvarieties $\mathscr{X}_{\mathbf{G}}(K^p)$ is the following.

Theorem 4.3.3. *The points $x \in \mathscr{X}_{\mathbf{G}}(K^p)(\overline{\mathbf{Q}_p})$ lying over a given weight $\lambda \in \mathscr{W}_{K^p}(\overline{\mathbf{Q}_p})$ are in bijection with the finite-slope eigenpackets for $\mathbf{G}$ of weight λ and level K^p, and this bijection is realized by sending $x \in \mathscr{X}$ to the eigenpacket $\phi_{\mathscr{X}}(x)$.*

This theorem is due to Ash and Stevens (Theorem 6.2.1 of (AS08)), but the following proof is new.

Proof. Given a finite-slope eigenpacket ϕ of weight λ, fix a slope-adapted affinoid $\mathscr{Z}_{\Omega,h}$ with $\lambda \in \Omega$ and $h > v_p(\phi(U_t))$, and let $\mathbf{T}_{\Omega,h}$ be the $\mathscr{O}(\Omega)$-subalgebra of $\mathrm{End}_{\mathscr{O}(\Omega)}(H^*(K^p, \mathcal{D}_\Omega)_{\leq h})$ generated by $\mathbf{T}(K^p) \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega)$. Let $\mathfrak{M}$ be the maximal ideal of $\mathbf{T}(K^p) \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega)$ defined by

$$\mathfrak{M} = (T \otimes 1 + 1 \otimes x),\, T \in \ker \phi \,\mathrm{and}\, x \in \mathfrak{m}_\lambda.$$

After localizing the spectral sequence

$$E_2^{i,j} = \mathrm{Tor}_i^{\mathscr{O}(\Omega)}(H^j(K^p, \mathcal{D}_\Omega)_{\leq h}, k(\lambda)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\lambda)_{\leq h}$$

at $\mathfrak{M}$, the abutment is nonzero by assumption, so the source must be nonzero as well. Therefore $\mathfrak{M}$ determines a maximal ideal of $\mathbf{T}_{\Omega,h}$ lying over $\mathfrak{m}_\lambda$, or equivalently a point $x \in \mathscr{X}_{\Omega,h}$ with $w(x) = \lambda$.

On the other hand, given a point $x \in \mathscr{X}_{\Omega,h}$ with $w(x) = \lambda$, let $\mathfrak{M} = \mathfrak{M}_x \subset \mathbf{T}_{\Omega,h}$ be the maximal ideal associated with x, and let d be the largest degree for which $H^j(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}} \neq 0$. Localizing the spectral sequence at $\mathfrak{M}$, the entry $E_2^{0,d}$ is nonzero and stable, so the spectral sequence induces an isomorphism

$$0 \neq H^d(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}} \otimes_{\mathscr{O}(\Omega)} k(\lambda) \simeq H^d(K^p, \mathcal{D}_\lambda)_{\leq h,\mathfrak{M}},$$

and thus $\mathfrak{M}$ induces a finite-slope eigenpacket in weight λ as desired. $\square$

4.4 The support of overconvergent cohomology modules

As in the previous section, fix $\mathbf{G}$, K^p, and an augmented Borel-Serre complex $C_\bullet(K^p, -)$. We are going to prove the following theorem.

Theorem 4.4.1. *Fix a slope datum (U_t, Ω, h).*

i. *For any i, $H_i(K^p, \mathcal{A}_\Omega)_{\leq h}$ is a faithful $\mathscr{O}(\Omega)$-module if and only if $H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ is faithful.*

ii. *If the derived group of $\mathbf{G}$ is $\mathbf{Q}$-anisotropic, $H_i(K^p, \mathcal{A}_\Omega)_{\leq h}$ and $H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ are torsion $\mathscr{O}(\Omega)$-modules for all i, unless $\mathbf{G}^{\mathrm{der}}(\mathbf{R})$ has a discrete series, in which case they are torsion for all $i \neq \frac{1}{2}\dim\mathbf{G}(\mathbf{R})/K_\infty Z_\infty$.*

Let R be a Noetherian ring, and let M be a finite R-module. We say M has *full support* if $\mathrm{Supp}(M) = \mathrm{Spec}(R)$, and that M is *torsion* if $\mathrm{ann}(M) \neq 0$. We shall repeatedly use the following basic result.

Proposition 4.4.2. *If $\mathrm{Spec}(R)$ is reduced and irreducible, the following are equivalent:*

i) M is faithful (i.e. $\mathrm{ann}(M) = 0$),

ii) M has full support,

iii) M has nonempty open support,

iv) $\mathrm{Hom}_R(M, R) \neq 0$,

v) $M \otimes_R K \neq 0$, $K = \mathrm{Frac}(R)$.

Proof. Since M is finite, $\mathrm{Supp}(M)$ is the underlying topological space of $\mathrm{Spec}(R/\mathrm{ann}(M))$, so i) obviously implies ii). If $\mathrm{Spec}(R/\mathrm{ann}(M)) = \mathrm{Spec}(R)$ as topological spaces, then

$\mathrm{ann}(M) \subset \sqrt{(0)} = (0)$ since R is reduced, so ii) implies i). The set $\mathrm{Supp}(M) = \mathrm{Spec}(R/\mathrm{ann}(M))$ is *a priori* closed; since $\mathrm{Spec}(R)$ is irreducible by assumption, the only nonempty simultaneously open and closed subset of $\mathrm{Spec}(R)$ is all of $\mathrm{Spec}(R)$, so ii) and iii) are equivalent. By finiteness, M has full support if and only if (0) is an associated prime of M, if and only if there is an injection $R \hookrightarrow M$; tensoring with K implies the equivalence of ii) and v). Finally, $\mathrm{Hom}_R(M,R) \otimes_R K \simeq \mathrm{Hom}_K(M \otimes_R K, K)$, so $M \otimes_R K \neq 0$ if and only if $\mathrm{Hom}_R(M,R) \neq 0$, whence iv) and v) are equivalent. □

Proof of Theorem 4.1.1.i. (I'm grateful to Jack Thorne for suggesting this proof.) Tensoring the Ext spectral sequence with $K(\Omega) = \mathrm{Frac}(\mathscr{O}(\Omega))$, it degenerates to isomorphisms

$$\mathrm{Hom}_{K(\Omega)}(H_i(K^p, \mathcal{A}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} K(\Omega), K(\Omega)) \simeq H^i(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} K(\Omega),$$

so the claim is immediate from the preceding proposition. □

Proof of Theorem 4.1.1.ii. We give the proof in two steps, with the first step naturally breaking into two cases. In the first step, we prove the result assuming Ω contains an arithmetic weight. In the second step, we eliminate this assumption via analytic continuation.

Step One, Case One: G doesn't have a discrete series. Let $\mathscr{W}^{\mathrm{sd}}$ be the rigid Zariski closure in $\mathscr{W}$ of the arithmetic weights whose algebraic parts are the highest weights of irreducible $\mathbf{G}$-representations with nonvanishing $(\mathfrak{g}, K_\infty)$-cohomology. A simple calculation using §II.6 of (BW00) shows that $\mathscr{W}^{\mathrm{sd}}$ is the union of its countable set of irreducible components, each of dimension $< \dim \mathscr{W}$. An arithmetic weight is *non-self-dual* if $\lambda \notin \mathscr{W}^{\mathrm{sd}}$.

Now, by assumption Ω contains an arithmetic weight, so Ω automatically contains a Zariski dense set $\mathcal{N}_h \subset \Omega \smallsetminus \Omega \cap \mathscr{W}^{\mathrm{sd}}$ of non-self-dual arithmetic weights for which h is a small slope. By Theorem 3.2.5, $H^*(K^p, \mathcal{D}_\lambda)_{\leq h}$ vanishes identically for any $\lambda \in \mathcal{N}_h$. For any fixed $\lambda \in \mathcal{N}_h$, suppose $\mathfrak{m}_\lambda \in \mathrm{Supp}_\Omega H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$; let d be the largest integer with $\mathfrak{m}_\lambda \in \mathrm{Supp}_\Omega H^d(K^p, \mathcal{D}_\Omega)_{\leq h}$. Taking $\Sigma = \lambda$ in the Tor spectral sequence gives

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\mathscr{O}(\Omega)}(H^j(K^p, \mathcal{D}_\Omega)_{\leq h}, k(\lambda)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\lambda)_{\leq h}.$$

The entry $E_2^{0,d} = H^d(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathscr{O}(\Omega)} k(\lambda)$ is nonzero by Nakayama's lemma, and is stable since every row of the E_2-page above the dth row vanishes by assumption. In particular, $E_2^{0,d}$ contributes a nonzero summand to the grading on $H^d(K^p, \mathcal{D}_\lambda)_{\leq h}$ - but this module is zero, contradicting our assumption that $\mathfrak{m}_\lambda \in \mathrm{Supp}_\Omega H^*(K^p, \mathcal{D}_\Omega)$. Therefore, $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$

does *not* have full support, so is not a faithful $\mathscr{O}(\Omega)$-module.

Step One, Case Two: G has a discrete series. The idea is the same as Case One, but with $\mathcal{N}_h$ replaced by $\mathcal{R}_h$, the set of arithmetic weights with *regular* algebraic part for which h is a small slope. For these weights, Proposition 3.2.5 together with known results on $(\mathfrak{g}, K_\infty)$-cohomology (see e.g. Sections 4-5 of (LS04)) implies that $H^i(K^p, \mathcal{D}_\lambda)_{\leq h}$, $\lambda \in \mathcal{R}_h$ vanishes for $i \neq d_{\mathbf{G}} = \frac{1}{2}\dim \mathbf{G}(\mathbf{R})/Z_\infty K_\infty$. The Tor spectral sequence with $\Sigma = \lambda \in \mathcal{R}_h$ then shows that $\mathcal{R}_h$ doesn't meet $\mathrm{Supp}_\Omega H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ for any $i > d_{\mathbf{G}}$. On the other hand, the Ext spectral sequence with $\Sigma = \lambda \in \mathcal{R}_h$ then shows that $\mathcal{R}_h$ doesn't meet $\mathrm{Supp}_\Omega H_i(K^p, \mathcal{A}_\Omega)_{\leq h}$ for any $i < d_{\mathbf{G}}$, whence the Ext spectral sequence with $\Sigma = \Omega$ shows that $\mathcal{R}_h$ doesn't meet $\mathrm{Supp}_\Omega H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ for any $i < d_{\mathbf{G}}$. The result follows.

Step Two. We maintain the notation of §4.3. As in that subsection, $H^n(K^p, \mathcal{D}_\Omega)_{\leq h}$ glues together over the affinoids $\mathscr{X}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$ into a coherent $\mathscr{O}_{\mathscr{X}}$-module sheaf $\mathscr{M}^n$, and in particular, the support of $\mathscr{M}^n$ is a closed analytic subset of $\mathscr{X}$. Let $w : \mathscr{X} \to \mathscr{W}$ denote the natural projection. For any $\mathscr{X}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$, we have

$$w_* \mathrm{Supp}_{\mathscr{X}_{\Omega,h}} \mathscr{M}^n(\mathscr{X}_{\Omega,h}) = \mathrm{Supp}_\Omega H^n(K^p, \mathcal{D}_\Omega)_{\leq h}.$$

Suppose $\mathrm{Supp}_\Omega H^n(K^p, \mathcal{D}_\Omega)_{\leq h} = \Omega$ for some $\mathscr{X}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$. This implies that $\mathrm{Supp}_{\mathscr{X}_{\Omega,h}} \mathscr{M}^n(\mathscr{X}_{\Omega,h})$ contains a closed subset of dimension equal to $\dim \mathscr{X}$, so contains an irreducible component of $\mathscr{X}_{\Omega,h}$. Since $\mathrm{Supp}_{\mathscr{X}} \mathscr{M}^n$ is a priori closed, Corollary 2.2.6 of (Con99) implies that $\mathrm{Supp}_{\mathscr{X}} \mathscr{M}^n$ contains an entire irreducible component of $\mathscr{X}$, say $\mathscr{X}_0$. By Proposition 4.1.3, the image of $\mathscr{X}_0$ is Zariski-open in $\mathscr{W}$, so we may choose an arithmetic weight $\lambda_0 \in w_* \mathscr{X}_0$. For some sufficiently large h_0 and some affinoid Ω_0 containing λ_0, $\mathscr{X}_{\Omega_0,h_0}$ will contain $\mathscr{X}_{\Omega_0,h_0} \cap \mathscr{X}_0$ as a nonempty union of irreducible components, and the latter intersection will be finite flat over Ω_0. Since $\mathscr{M}^n(\mathscr{X}_{\Omega_0,h_0}) \simeq H^n(K^p, \mathcal{D}_{\Omega_0})_{\leq h_0}$, we deduce that $\mathrm{Supp}_{\Omega_0} H^n(K^p, \mathcal{D}_{\Omega_0})_{\leq h_0} = \Omega_0$, whence $H^n(K^p, \mathcal{D}_{\Omega_0})_{\leq h_0}$ is faithful, so by Step One $\mathbf{G}^{\mathrm{der}}(\mathbf{R})$ has a discrete series and $n = \frac{1}{2}\dim \mathbf{G}(\mathbf{R})/Z_\infty K_\infty$.

4.5 Eigenvarieties at noncritical interior points.

Define the *defect* and the *amplitude* of $\mathbf{G}$, respectively, as the integers $l(\mathbf{G}) = \mathrm{rank}\mathbf{G} - \mathrm{rank}K_\infty Z_\infty$[1] and $q(\mathbf{G}) = \frac{1}{2}(\dim(\mathbf{G}(\mathbf{R})/K_\infty Z_\infty) - l(\mathbf{G}))$. Note that $l(\mathbf{G})$ is zero if and only if $\mathbf{G}^{\mathrm{der}}(\mathbf{R})$ has a discrete series, and that algebraic representations with regular highest weight contribute to $(\mathfrak{g}, K_\infty)$-cohomology exactly in the unbroken range of degrees $[q(\mathbf{G}), q(\mathbf{G}) + l(\mathbf{G})]$ (LS04).

In this section we prove the following result; part i. of this theorem is a generalization of "Coleman families".

Theorem 4.5.1. *Let $x = x_\phi \in \mathscr{X}_{\mathbf{G}}(K^p)$ be a point associated with a classical, noncritical, interior eigenpacket ϕ such that $\lambda = w(x)$ has regular algebraic part.*

i. *If $l(\mathbf{G}) = 0$, every irreducible component of $\mathscr{X}$ containing x has dimension equal to $\dim \mathscr{W}$.*

ii. *If $l(\mathbf{G}) \geq 1$ and ϕ is strongly interior, then every irreducible component of $\mathscr{X}_{\mathbf{G}}(K^p)$ containing x has dimension $\leq \dim\mathscr{W} - 1$, with equality if $l(\mathbf{G}) = 1$.*

Proof. By the basic properties of irreducible components together with the construction given in §4.2-4.3, it suffices to work locally over a fixed $\mathscr{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$. Suppose $x \in \mathscr{X}_{\Omega,h}$ is as in the theorem, with $\phi : \mathbf{T}_{\Omega,h} \to \overline{\mathbf{Q}_p}$ the corresponding eigenpacket. Set $\mathfrak{M} = \ker\phi$, and let $\mathfrak{m} = \mathfrak{m}_\lambda$ be the contraction of $\mathfrak{M}$ to $\mathscr{O}(\Omega)$. Let $\mathfrak{P} \subset \mathbf{T}_{\Omega,h}$ be any minimal prime contained in $\mathfrak{M}$, and let $\mathfrak{p}$ be its contraction to a prime in $\mathscr{O}(\Omega)$. The ring $\mathbf{T}_{\Omega,h}/\mathfrak{P}$ is a finite integral extension of $\mathscr{O}(\Omega)/\mathfrak{p}$, so both rings have the same dimension. Note also that $\mathfrak{p}$ is an associated prime of $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$.

Proposition 4.5.2. *The largest degrees for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ and $H^*(K^p, \mathcal{D}_\lambda)_{\leq h}$ coincide, and the smallest degrees for which ϕ occurs in $H^*(K^p, \mathcal{D}_\lambda)_{\leq h}$ and $H_*(K^p, \mathcal{A}_\Omega)_{\leq h}$ coincide. Finally, the smallest degree for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ is greater than or equal to the smallest degree for which ϕ occurs in $H_*(K^p, \mathcal{A}_\Omega)_{\leq h}$.*

Proof. For the first claim, localize the Tor spectral sequence at $\mathfrak{M}$, with $\Sigma = \lambda$. If ϕ occurs in $H^i(K^p, \mathcal{D}_\lambda)_{\leq h}$ then it occurs in a subquotient of $\mathrm{Tor}_j^{\mathscr{O}(\Omega)}(H^{i+j}(K^p, \mathcal{D}_\Omega)_{\leq h}, k(\lambda))$ for some $j \geq 0$. On the other hand, if d is the largest degree for which ϕ occurs in $H^d(K^p, \mathcal{D}_\Omega)_{\leq h}$, the entry $E_2^{0,d}$ of the Tor spectral sequence is stable and nonzero after localizing at $\mathfrak{M}$, and

[1] Here "rank" denotes the absolute rank, i.e. the dimension of any maximal torus, split or otherwise.

it contributes to the grading on $H^d(K^p, \mathcal{D}_\lambda)_{\leq h,\mathfrak{M}}$. The second and third claims follow from an analogous treatment of the Ext spectral sequence. □

First we treat the case where $l(\mathbf{G}) = 0$, so $\mathbf{G}^{\mathrm{der}}(\mathbf{R})$ has a discrete series. By the noncriticality of ϕ together with the results recalled in §3.2, the only degree for which ϕ occurs in $H^i(K^p, \mathcal{D}_\lambda)_{\leq h}$ is the middle degree $d = \frac{1}{2}\mathrm{dim}\mathbf{G}(\mathbf{R})/Z_\infty K_\infty$, so Proposition 4.5.2 implies that the only degree for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ is the middle degree as well. The Tor spectral sequence localized at $\mathfrak{M}$ now degenerates, and yields

$$\mathrm{Tor}_i^{\mathscr{O}(\Omega)_\mathfrak{m}}(H^d(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}}, k(\lambda)) = 0 \,\mathrm{for\,all}\, i \geq 1,$$

so $H^d(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}}$ is a *free* module over $\mathscr{O}(\Omega)_\mathfrak{m}$ by Proposition A.3. Since $\mathscr{O}(\Omega)_\mathfrak{m}$ is a domain and $\mathfrak{p}$ is (locally at $\mathfrak{m}$) an associated prime of a free module, $\mathfrak{p} = 0$ and thus $\mathrm{dim}\mathbf{T}_{\Omega,h}/\mathfrak{P} = \mathrm{dim}\mathscr{O}(\Omega)_\mathfrak{m} = \mathrm{dim}\mathscr{W}$.

Now we turn to the case $l(\mathbf{G}) \geq 1$. First we demonstrate the existence of an affinoid open $\mathscr{Y} \subset \mathscr{X}_{\Omega,h}$ containing x, and meeting every component of $\mathscr{X}_{\Omega,h}$ containing x, such that every regular classical non-critical point in $\mathscr{Y}$ is cuspidal. By our assumptions, ϕ does not occur in $H^*_\partial(K^p, \mathcal{D}_\lambda)_{\leq h}$, so by the boundary spectral sequence ϕ does not occur in $H^*_\partial(K^p, \mathcal{D}_\Omega)_{\leq h}$. Since $\mathrm{Supp}_{\mathbf{T}_{\Omega,h}} H^*_\partial(K^p, \mathcal{D}_\Omega)_{\leq h}$ is closed in $\mathscr{X}_{\Omega,h}$ and does not meet x, the existence of a suitable $\mathscr{Y}$ now follows easily. Shrinking Ω and $\mathscr{Y}$ as necessary, we may assume that $\mathscr{O}(\mathscr{Y})$ is finite over $\mathscr{O}(\Omega)$, and thus $\mathscr{M}^*(\mathscr{Y}) = H^*(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathbf{T}_{\Omega,h}} \mathscr{O}(\mathscr{Y})$ is finite over $\mathscr{O}(\Omega)$ as well. Exactly as in the proof of Theorem 4.4.1, the Tor spectral sequence shows that $\mathrm{Supp}_\Omega \mathscr{M}^*(\mathscr{Y})$ doesn't contain any regular non-self-dual weights for which h is a small slope, so $\mathscr{M}^*(\mathscr{Y})$ and $\mathscr{O}(\mathscr{Y})$ are torsion $\mathscr{O}(\Omega)$-modules.

Finally, suppose $l(\mathbf{G}) = 1$. Set $d = q(\mathbf{G})$, so ϕ occurs in $H^*(K^p, \mathcal{D}_\lambda)_\mathfrak{M} \simeq H^*(K^p, \mathcal{L}_\lambda)_\mathfrak{M}$ only in degrees d and $d+1$. By the argument of the previous paragraph, $H^*(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}}$ is a torsion $\mathscr{O}(\Omega)_\mathfrak{m}$-module. Taking $\Sigma = \lambda$ in the Ext spectral sequence and localizing at $\mathfrak{M}$, Proposition 4.5.2 yields

$$H^d(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}} \simeq \mathrm{Hom}_{\mathscr{O}(\Omega)_\mathfrak{m}}(H_d(K^p, \mathcal{A}_\Omega)_{\leq h,\mathfrak{M}}, \mathscr{O}(\Omega)_\mathfrak{m}).$$

Since the left-hand term is a torsion $\mathscr{O}(\Omega)_\mathfrak{m}$-module, Proposition 4.4.2 implies that both modules vanish identically. Proposition 4.5.2 now shows that $d+1$ is the only degree for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$. Taking $\Sigma = \lambda$ in the Tor spectral sequence and localizing at $\mathfrak{M}$, the only nonvanishing entries are $E_2^{0,d+1}$ and $E_2^{-1,d+1}$. In particular, $\mathrm{Tor}_i^{\mathscr{O}(\Omega)_\mathfrak{m}}(H^{d+1}(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}}, \mathscr{O}(\Omega)/\mathfrak{m}) = 0$ for all $i \geq 2$, so $H^{d+1}(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}}$ has projective dimension at most one by Proposition A.3. Summarizing, we've shown that $H^i(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}}$ vanishes in degrees $\neq q(G)+1$, and that $H^{q(G)+1}(K^p, \mathcal{D}_\Omega)_{\leq h,\mathfrak{M}}$ is a torsion $\mathscr{O}(\Omega)_\mathfrak{m}$-module of projective dimension one, so $\mathrm{ht}\mathfrak{p} = 1$ by Proposition A.6. □

More generally, for any point $x \in \mathscr{X}_\mathbf{G}(K^p)$ we define

$$l(x) = \sup\left\{i | H^i(K^p, \mathcal{D}_{w(x)}) \otimes_{\mathbf{T}(K^p),\phi(x)} \overline{\mathbf{Q}_p} \neq 0\right\} - \inf\left\{i | H^i(K^p, \mathcal{D}_{w(x)}) \otimes_{\mathbf{T}(K^p),\phi(x)} \overline{\mathbf{Q}_p} \neq 0\right\}.$$

If x is associated with a noncritical, strongly interior, regular eigenpacket, then $l(x) = l(\mathbf{G})$ by the aforementioned results of Wallach and Li-Schwermer. Using techniques similar to those above, James Newton has established the following result (cf. the appendix to (Han12b))

Theorem 4.5.3 (Newton). *Every irreducible component of $\mathscr{X}_\mathbf{G}(K^p)$ containing x has dimension $\geq \dim \mathscr{W} - l(x)$.*

Results of this type were established by Stevens and Urban in unpublished work: however, their proofs required an a priori assumption that the ideal $\wp$ (in the notation of the proof of Theorem 4.5.1) is generated by a regular sequence locally in $\mathscr{O}(\Omega)_\mathfrak{m}$.

4.6 General linear groups

In this section we examine the special case when $\mathbf{G} \simeq \mathrm{Res}_{F/\mathbf{Q}}\mathbf{H}$ for some number field F and some F-inner form $\mathbf{H}$ of GL_n/F, introducing notation which will remain in effect throughout Chapters 5 and 6. Our running assumptions require p split completely in F and $\mathbf{H}/F_v \simeq \mathrm{GL}_n/F_v$ for all $v|p$; as such, we identify $\mathbf{G}/\mathbf{Q}_p$ with $(\mathrm{GL}_n/\mathbf{Q}_p)^{[F:\mathbf{Q}]}$ and we take B to be the product of the upper-triangular Borel subgroups.

Next, we work with a canonical family of tame level subgroups suggested by the theory of new vectors. More precisely, given an integral ideal $\mathfrak{n} = \prod \mathfrak{p}_v^{e_v(\mathfrak{n})} \subset \mathcal{O}_F$ with $e_v = 0$ if $v|p$ or if $\mathbf{H}(F_v) \not\cong \mathrm{GL}_n(F_v)$, set

$$K(\mathfrak{n}) = \prod_{v \text{ with } e_v(\mathfrak{n})>0} K_v(\varpi_v^{e_v(\mathfrak{n})}) \prod_{v \text{ with } e_v(\mathfrak{n})=0} K_v$$

where $K_v(\varpi_v^e)$ denotes the open compact subgroup of $\mathrm{GL}_n(\mathcal{O}_v)$ consisting of matrices with lowest row congruent to $(0, \ldots, 0, 1) \bmod \varpi_v^e$, and K_v denotes a fixed maximal compact subgroup of $\mathbf{H}(F_v)$. The Hecke algebra $\mathbf{T}(K(\mathfrak{n}))$ then contains the usual operators $T_{v,i}$ corresponding to the double cosets of the matrices $\mathrm{diag}\left(\underbrace{\varpi_v, \ldots, \varpi_v}_{i}, 1, \ldots, 1\right)$ for $1 \leq i \leq n$ and v a place of F such that $e_v(\mathfrak{n}) = 0$ and $\mathbf{H}(F_v) \simeq \mathrm{GL}_n(F_v)$. For a place $v|p$ we write $U_{v,i}$ for the element of $\mathcal{A}_p^+$ corresponding to the double coset of $\mathrm{diag}\left(1, \ldots, 1, \underbrace{\varpi_v, \ldots, \varpi_v}_{i}\right)$, and we set $u_{v,i} = U_{v,i-1}^{-1} U_{v,i} \in \mathcal{A}_p$. The product $\prod_{v|p} \prod_{i=1}^{n-1} U_{v,i}$ is a canonical controlling operator, which we denote by U_p. If $n = 2$ we adopt the more classical notation, writing $T_v = T_{v,1}$ and $S_v = T_{v,2}$. We write $\mathbf{T}_\lambda(\mathfrak{n})$ for the finite-slope Hecke algebra of weight λ and tame level $K(\mathfrak{n})$ as in §3.2.

If $F = \mathbf{Q}$, we define $\mathscr{W}_{K^p}^0$ as the subspace of $\mathscr{W}_{K^p}$ parametrizing characters trivial on the one-parameter subgroup $\mathrm{diag}(1, \ldots, 1, t_n)$, and we set $\mathscr{X}_{\mathbf{G}}(K^p)^0 = \mathscr{X}_{\mathbf{G}}(K^p) \cap w^{-1}\left(\mathscr{W}_{K^p}^0\right)$. By the remarks in §2.2, $\mathscr{X}_{\mathbf{G}}(K^p)$ is a disc bundle over $\mathscr{X}_{\mathbf{G}}(K^p)^0$: for any point $x \in \mathscr{X}_{\mathbf{G}}(K^p)$ with $w(x) = \lambda = (\lambda_1, \ldots, \lambda_n)$, there is a unique point $x^0 \in \mathscr{X}_{\mathbf{G}}(K^p)^0$ with $\lambda^0 = (\lambda_1 \lambda_n^{-1}, \ldots, \lambda_{n-1} \lambda_n^{-1}, 1)$ such that $\phi(x)(T_{\ell,i}) = \lambda_n(\ell)^i \phi(x_0)(T_{\ell,i})$. Restricting attention to $\mathscr{X}_{\mathbf{G}}(K^p)^0$ amounts to factoring out "wild twists": in particular, $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q}}^1(K(N))^0$ "is" the Coleman-Mazur-Buzzard eigencurve of tame level N (this is essentially a theorem of Bellaïche (Bel12)).

Chapter 5

p-adic Langlands functoriality

Buzzard (Buz04, Buz07) constructed an eigencurve $\mathscr{C}_D$ using overconvergent algebraic modular forms associated with an $\mathbf{R}$-nonsplit quaternion algebra $D/\mathbf{Q}$, and raised the question of whether there exists a closed immersion $\iota_{\mathrm{JL}} : \mathscr{C}_D \hookrightarrow \mathscr{C}(\mathrm{disc}D)$ into the Coleman-Mazur-Buzzard eigencurve interpolating the Jacquet-Langlands correspondence on classical points. Chenevier affirmatively answered this question in a beautiful paper (Che05) as a consequence of a more general interpolation theorem. Chenevier's interpolation theorem applies to eigenvarieties which arise from sheaves of orthonormalizable Banach modules over $\mathscr{W}$, and thus is limited in applications to eigenvarieties arising either from overconvergent modular forms on Shimura varieties or from groups compact-mod-center at infinity.

In Section 5.1, by shifting the emphasis from sheaves of orthonormalizable Banach modules over $\mathscr{W}$ to *coherent* sheaves over Fredholm hypersurfaces, we establish a significantly more flexible interpolation theorem (Theorems 5.1.2 and 5.1.5). The remainder of Chapter 5 consists of applications of this theorem: we establish an analogue of Chenevier's Jacquet-Langlands morphism for $\mathbf{R}$-*split* quaternion algebras, as well as p-adic analogues of the symmetric square lifting of Gelbart-Jacquet and the Rankin-Selberg lifting of Ramakrishnan. The latter two results will be used in Chapter 6. There are many other significant application of these results; an application to cyclic base change is given in (Han12a).

5.1 An interpolation theorem

Definition 5.1.1. *Given an eigenvariety datum $\mathfrak{D} = (\mathscr{W}, \mathscr{Z}, \mathscr{M}, \mathbf{T}, \psi)$ with associated eigenvariety $\mathscr{X}$, the* core *of $\mathscr{X}$, denoted $\mathscr{X}^\circ$, is the union of the* $\dim \mathscr{W}$*-dimensional irreducible components of the nilreduction $\mathscr{X}^{\mathrm{red}}$, regarded as a closed subspace of $\mathscr{X}$. An eigenvariety $\mathscr{X}$ is* unmixed *if $\mathscr{X}^\circ \simeq \mathscr{X}$.*

Let $\mathscr{Z}^\circ$ denote the subspace of points in $\mathscr{Z}$ whose preimage in $\mathscr{X}$ meets the core of $\mathscr{X}$, with its reduced rigid subspace structure; $\mathscr{Z}^\circ$ is naturally a union of irreducible components of $\mathscr{Z}^{\mathrm{red}}$. We will see below that $\mathscr{X}^\circ$ really is an eigenvariety, in the sense of being associated with an eigenvariety datum.

Suppose we are given two eigenvariety data $\mathfrak{D}_i = (\mathscr{W}_i, \mathscr{Z}_i, \mathscr{M}_i, \mathbf{T}, \psi_i)$ for $i = 1, 2$, together with a closed immersion $\jmath : \mathscr{W}_1 \hookrightarrow \mathscr{W}_2$; we write j for the natural extension of $\jmath$ to a closed immersion $\jmath \times \mathrm{id} : \mathscr{W}_1 \times \mathbf{A}^1 \hookrightarrow \mathscr{W}_2 \times \mathbf{A}^1$. Given a point $z \in \mathscr{Z}_i$ and any $T \in \mathbf{T}$, we write $D_i(T, X)(z) \in k(z)[X]$ for the characteristic polynomial $\det(1 - \psi_i(T)X)|\mathscr{M}_i(z)$.

Theorem 5.1.2. *Notation and assumptions as in the previous paragraph, suppose there is some very Zariski-dense set $\mathscr{Z}^{\mathrm{cl}} \subset \mathscr{Z}_1^\circ$ with $j(\mathscr{Z}^{\mathrm{cl}}) \subset \mathscr{Z}_2$ such that the polynomial $D_1(T, X)(z)$ divides $D_2(T, X)(j(z))$ in $k(z)[X]$ for all $T \in \mathbf{T}$ and all $z \in \mathscr{Z}^{\mathrm{cl}}$. Then j induces a closed immersion $\zeta : \mathscr{Z}_1^\circ \hookrightarrow \mathscr{Z}_2$, and there is a canonical closed immersion $i : \mathscr{X}_1^\circ \hookrightarrow \mathscr{X}_2$ such that the diagrams*

$$\begin{tikzcd}
\mathscr{X}_1^\circ \arrow[r, hook, "i"] \arrow[d, "w_1"'] & \mathscr{X}_2 \arrow[d, "w_2"] \\
\mathscr{W}_1 \arrow[r, hook, "\jmath"'] & \mathscr{W}_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{T} \arrow[rr, "\phi_1^\circ"] \arrow[dr, "\phi_2"'] & & \mathscr{O}(\mathscr{X}_1^\circ) \\
& \mathscr{O}(\mathscr{X}_2) \arrow[ur, "i^*"'] &
\end{tikzcd}$$

commute.

First we prove two lemmas.

Lemma 5.1.3. *Suppose A is an affinoid algebra, B is a module-finite A-algebra, and S*

is a Zariski-dense subset of $\mathrm{Max}A$. *Then*

$$I = \bigcap_{\mathfrak{m}\in\mathrm{Max}B \text{ with } \mathfrak{m} \text{ lying over some } \mathfrak{n}\in S} \mathfrak{m} \subset B$$

is contained in every minimal prime $\mathfrak{p}$ *of* B *with* $\mathrm{dim}B/\mathfrak{p} = \mathrm{dim}A$.

Proof. Translated into geometric language, this is the self-evident statement that the preimage of S under $\pi : \mathrm{Sp}B \to \mathrm{Sp}A$ is Zariski-dense in any irreducible component of $\mathrm{Sp}B$ with Zariski-open image in $\mathrm{Sp}A$. □

Lemma 5.1.4. *Suppose* A *and* B *are affinoid algebra, with* $\mathrm{Max}B$ *an affinoid subdomain of* $\mathrm{Max}A$. *Let* A° *be the maximal reduced quotient of* A *which is equidimensional of dimension* $\mathrm{dim}A$. *Then* $A^\circ \otimes_A B$ *is the maximal reduced quotient of* B *which is equidimensional of dimension* $\mathrm{dim}A$ *if* $\dim B = \dim A$, *and is zero if* $\mathrm{dim}B < \dim A$.

Proof. Set $d = \mathrm{dim}A$. The kernel of $A \to A^\circ$ is the ideal $I^\circ = \cap_{\mathfrak{p}\in\mathrm{Spec}A,\mathrm{coht}\mathfrak{p}=d}\mathfrak{p}$, so tensoring the sequence

$$0 \to I^\circ \to A \to A^\circ \to 0$$

with B yields

$$0 \to I^\circ \otimes_A B \to B \to A^\circ \otimes_A B \to 0$$

by the A-flatness of B. It thus suffices to prove an isomorphism

$$I^\circ \otimes_A B \simeq \bigcap_{\mathfrak{p}\in\mathrm{Spec}B,\, \mathrm{coht}\mathfrak{p}=d} \mathfrak{p}.$$

By the Jacobson property of affinoid algebras, we can rewrite I° as follows:

$$\begin{aligned} I^\circ &= \bigcap_{\mathrm{coht}\mathfrak{p}=d} \mathfrak{p} \\ &= \bigcap_{\substack{\mathfrak{m}\in\mathrm{Max}(A) \text{ with} \\ \mathfrak{m}\supset\mathfrak{p} \text{ and } \mathrm{coht}\mathfrak{p}=d}} \mathfrak{m} \\ &= \bigcap_{\substack{\mathfrak{m}\in\mathrm{Max}(A) \text{ with} \\ \mathrm{ht}\mathfrak{m}=d}} \mathfrak{m}. \end{aligned}$$

Since B is A-flat, we have $(I_1 \cap I_2) \otimes_A B = I_1 B \cap I_2 B$ for any ideals $I_i \subset A$, so

$$\begin{aligned} I^\circ \otimes_A B &= \bigcap_{\substack{\mathfrak{m}\in \mathrm{Max}(A) \text{ with} \\ \mathrm{ht}\mathfrak{m}=d}} \mathfrak{m}B \\ &= \bigcap_{\substack{\mathfrak{m}\in \mathrm{Max}(B) \text{ with} \\ \mathrm{ht}\mathfrak{m}=d}} \mathfrak{m} \\ &= \bigcap_{\mathfrak{p}\in \mathrm{Spec} B,\, \mathrm{coht}\mathfrak{p}=d} \mathfrak{p}, \end{aligned}$$

and we're done if $\dim B = \dim A$. But if $\dim B < \dim A$, then $\mathfrak{m}B = B$ for all $\mathfrak{m} \in \mathrm{Max}(A)$ of height d, so $I^\circ \otimes_A B = B$ as desired.

Proof of Theorem 5.1.2. Set $d = \dim \mathscr{W}_1$. First, we establish the theorem in the special case when $\mathscr{W}_1 = \mathscr{W}_2$, $\jmath = \mathrm{id}$, $\mathscr{Z}_1^\circ \simeq \mathscr{Z}_1 \simeq \mathscr{Z}_2$, and $\mathscr{X}_1^\circ \simeq \mathscr{X}_1$; we refer to this as the *narrow case.* For brevity we write $\mathscr{W} = \mathscr{W}_1$ and $\mathscr{Z} = \mathscr{Z}_1$. As in §4.2, let $\mathscr{C}\mathrm{ov} = \{\Omega_i\}_{i\in I}$ be an admissible affinoid covering of $\mathscr{Z}$. For any $\Omega \in \mathscr{C}\mathrm{ov}$ and $i \in \{1, 2\}$, let $\mathbf{T}_{\Omega,i}$ denote the $\mathscr{O}(\Omega)$-subalgebra of $\mathrm{End}_{\mathscr{O}(\Omega)}(\mathscr{M}_i(\Omega))$ generated by the image of ψ_i, and let $I_{\Omega,i}$ be the kernel of the natural surjection

$$\phi_{\Omega,i} : \mathbf{T} \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega) \twoheadrightarrow \mathbf{T}_{\Omega,i}.$$

We are going to establish an inclusion $I_{\Omega,2} \subseteq I_{\Omega,1}$ for all $\Omega \in \mathscr{C}\mathrm{ov}$. Granting this inclusion, let $\mathscr{I}_\Omega \subset \mathbf{T}_{\Omega,2}$ be the kernel of the induced surjection $\mathbf{T}_{\Omega,2} \twoheadrightarrow \mathbf{T}_{\Omega,1}$. If $\Omega' \subset \Omega$ is an affinoid subdomain, then applying $- \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega')$ to the sequence

$$0 \to I_{\Omega,i} \to \mathbf{T} \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega) \overset{\phi_{\Omega,i}}{\to} \mathbf{T}_{\Omega,i} \to 0$$

yields a canonical isomorphism $I_{\Omega,i} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \cong I_{\Omega',i}$, so applying $- \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega')$ to the canonical isomorphism $\mathscr{I}_\Omega \cong I_{\Omega,2}/I_{\Omega,1}$ yields an isomorphism $\mathscr{I}_\Omega \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \cong \mathscr{I}_{\Omega'}$. Therefore the assignments $\Omega \mapsto \mathscr{I}_\Omega$ glue together into a coherent ideal subsheaf of the structure sheaf of $\mathscr{X}_2$ cutting out $\mathscr{X}_1$; equivalently, the surjections $\mathbf{T}_{\Omega,2} \twoheadrightarrow \mathbf{T}_{\Omega,1}$ glue together over $\Omega \in \mathscr{C}\mathrm{ov}$ into the desired closed immersion.

It remains to establish the inclusion $I_{\Omega,2} \subseteq I_{\Omega,1}$. Let $\mathscr{Z}^{\mathrm{reg}}$ be the maximal subset of $\mathscr{Z}$ such that $\mathscr{O}_{\mathscr{Z},z}$ is regular for all $z \in \mathscr{Z}^{\mathrm{reg}}$ and the sheaves $\mathscr{M}_1$ and $\mathscr{M}_2$ are locally free after restriction to $\mathscr{Z}^{\mathrm{reg}}$; since $\mathscr{Z}^{\mathrm{reg}}$ is naturally the intersection

$$\mathrm{Reg}(\mathscr{Z}) \bigcap \left(\mathscr{Z} \smallsetminus \mathrm{Supp} \oplus_{i=1}^{d} \mathscr{E}\mathrm{xt}^{i}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M}_1 \oplus \mathscr{M}_2, \mathscr{O}_{\mathscr{Z}}) \right),$$

and $\mathrm{Reg}(\mathscr{Z})$ is Zariski-open by the excellence of affinoid algebras, $\mathscr{Z}^{\mathrm{reg}}$ is naturally a Zariski-open and Zariski-dense rigid subspace of $\mathscr{Z}$. For any $T \in \mathbf{T}$, let $D_i(T,X) \in \mathscr{O}(\mathscr{Z}^{\mathrm{reg}})[X]$ be the determinant of $1 - \psi_i(T)X$ acting on $\mathscr{M}_i|_{\mathscr{Z}^{\mathrm{reg}}}$, defined in the usual way (this is why we need local freeness). For any $z \in \mathscr{Z}^{\mathrm{reg}}$, the image of $D_i(T,X)$ in the residue ring $k(z)[X]$ is simply $D_i(T,X)(z)$. By our hypotheses, the formal quotient

$$Q(T,X) = D_2(T,X)/D_1(T,X) = \sum_{n \geq 0} a_n X^n \in \mathscr{O}(\mathscr{Z}^{\mathrm{reg}})[[X]]$$

reduces for any $z \in \mathscr{Z}^{\mathrm{reg}} \cap \mathscr{Z}^{\mathrm{cl}}$ to an element of $k(z)[X]$ with degree bounded uniformly above as a function of z on any given irreducible component of $\mathscr{Z}^{\mathrm{reg}}$. In particular, the restriction of a_n to any given irreducible component of $\mathscr{Z}^{\mathrm{reg}}$ is contained in a Zariski-dense set of maximal ideals for sufficiently large n, and so is zero. Thus $D_1(T,X)(z)$ divides $D_2(T,X)(z)$ in $k(z)[X]$ for *any* $z \in \mathscr{Z}^{\mathrm{reg}}$ and any $T \in \mathbf{T}$. This extends by an explicit calculation with generalized eigenspaces to the same divisibility but for any $T \in \mathbf{T} \otimes_{\mathbf{Q}_p} k(z)$.

Suppose now that $T \in \mathbf{T} \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega)$ is contained in $I_{\Omega,2}$. Since $D_2(T,X)(z) = 1$ for any $z \in \Omega \cap \mathscr{Z}^{\mathrm{reg}}$, the deduction in the previous paragraph shows that $D_1(T,X)(z) = 1$ for any $z \in \Omega \cap \mathscr{Z}^{\mathrm{reg}}$. But then

$$\phi_1(T) \in \bigcap_{x \in \mathrm{Sp}\mathbf{T}_{\Omega,1} \text{ with } \pi(x) \in \Omega \cap \mathscr{Z}^{\mathrm{reg}}} \mathfrak{m}_x \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Spec}\mathbf{T}_{\Omega,1},\, \mathrm{coht}\mathfrak{p} = d} \mathfrak{p} = 0,$$

where the middle inclusion follows from Lemma 5.1.3 and the rightmost equality follows since $\mathbf{T}_{\Omega,1}$ is reduced and equidimensional of dimension d by assumption. This establishes the narrow case.

It remains to establish the general case. By the hypotheses of the theorem, $j(\mathscr{Z}^{\mathrm{cl}}) \in j(\mathscr{Z}_1^{\circ}) \cap \mathscr{Z}_2$, so j induces the closed immersion $\zeta : \mathscr{Z}_1^{\circ} \hookrightarrow \mathscr{Z}_2$ by the Zariski-density of $\mathscr{Z}^{\mathrm{cl}}$ in $\mathscr{Z}_1^{\circ}$. Let $\mathscr{X}_2'$ denote the fiber product $\mathscr{X}_2 \times_{\mathscr{Z}_2, \zeta} \mathscr{Z}_1^{\circ}$; note that $\mathscr{X}_2'$ is the eigenvariety associated with the eigenvariety datum $\mathfrak{D}_2' = (\mathscr{W}_1, \mathscr{Z}_1^{\circ}, \zeta^* \mathscr{M}_2, \mathbf{T}, \zeta^{\sharp}\psi_2)$, and there is a canonical closed immersion $i' : \mathscr{X}_2' \hookrightarrow \mathscr{X}_2$ by construction. For $\Omega \subseteq \mathscr{Z}_1$ an affinoid open we define an ideal $\mathscr{J}(\Omega) \subset \mathbf{T}_{\Omega,1}$ by the rule

$$\mathscr{J}(\Omega) = \begin{cases} \mathbf{T}_{\Omega,1} & \text{if } \dim \mathbf{T}_{\Omega,1} < d \\ \bigcap_{x \in \mathrm{Sp}\mathbf{T}_{\Omega,1},\, \mathrm{ht}\mathfrak{m}_x = d} \mathfrak{m}_x & \text{if } \dim \mathbf{T}_{\Omega,1} = d. \end{cases}$$

By Lemma 5.1.4 the ideals $\mathscr{J}(\Omega)$ glue into a coherent ideal sheaf $\mathscr{J} \subset \mathscr{O}_{\mathscr{X}_1}$. The support of $\mathscr{O}_{\mathscr{X}_1}/\mathscr{J}$ in $\mathscr{Z}_1$ is exactly $\mathscr{Z}_1^\circ$, and in fact the closed immersion cut out by $\mathscr{J}$ is exactly the core of $\mathscr{X}_1$. In particular, the core of $\mathscr{X}_1$ is the eigenvariety associated with the (somewhat tautological) eigenvariety datum $\mathfrak{D}_1^\circ = \left(\mathscr{W}_1, \mathscr{Z}_1^\circ, \pi_*\left(\mathscr{M}_1^\dagger \otimes_{\mathscr{O}_{\mathscr{X}_1}} \mathscr{O}_{\mathscr{X}_1}/\mathscr{J}\right), \mathbf{T}, \psi \bmod \mathscr{J}\right)$. The narrow case of Theorem 5.1.2 applies to the pair of eigenvariety data $\mathfrak{D}_1^\circ$ and $\mathfrak{D}_2'$, producing a closed immersion $i'' : \mathscr{X}_1^\circ \hookrightarrow \mathscr{X}_2'$, and the general case follows upon setting $i = i' \circ i''$. □

From Theorem 5.1.2, it's easy to deduce the following more flexible interpolation theorem.

Theorem 5.1.5. *Suppose we are given two eigenvariety data* $\mathfrak{D}_i = (\mathscr{W}_i, \mathscr{Z}_i, \mathscr{M}_i, \mathbf{T}_i, \psi_i)$ *for* $i = 1, 2$, *together with the following additional data:*

i) A closed immersion $\jmath : \mathscr{W}_1 \hookrightarrow \mathscr{W}_2$.

ii) An algebra homomorphism $\sigma : \mathbf{T}_2 \to \mathbf{T}_1$.

iii) A very Zariski-dense set $\mathscr{Z}^{\mathrm{cl}} \subset \mathscr{Z}_1^\circ$ *with* $j(\mathscr{Z}^{\mathrm{cl}}) \subset \mathscr{Z}_2$ *such that* $D_1(\sigma(T), X)(z)$ *divides* $D_2(T, X)(j(z))$ *in* $k(z)[X]$ *for all* $z \in \mathscr{Z}^{\mathrm{cl}}$ *and all* $T \in \mathbf{T}_2$.

Then there exists a morphism $i : \mathscr{X}_1^\circ \to \mathscr{X}_2$ *such that the diagrams*

$$\begin{tikzcd} \mathscr{X}_1^\circ \arrow[r, "i"] \arrow[d, "w_1"'] & \mathscr{X}_2 \arrow[d, "w_2"] \\ \mathscr{W}_1 \arrow[r, hook, "\jmath"'] & \mathscr{W}_2 \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{O}(\mathscr{X}_2) \arrow[r, "i^*"] & \mathscr{O}(\mathscr{X}_1^\circ) \\ \mathbf{T}_2 \arrow[r, "\sigma"'] \arrow[u, "\phi_2"] & \mathbf{T}_1 \arrow[u, "\phi_1^\circ"'] \end{tikzcd}$$

commute, and i *may be written as a composite* $i_c \circ i_f$ *where* i_f *is a finite morphism and* i_c *is a closed immersion.*

Proof. Let $\mathfrak{D}_1^\sigma$ be the eigenvariety datum $(\mathscr{W}_1, \mathscr{Z}_1, \mathscr{M}_1, \mathbf{T}_2, \psi_1 \circ \sigma)$. Theorem 5.1.2 produces a closed immersion $i'_c : \mathscr{X}(\mathfrak{D}_1^\sigma)^\circ \hookrightarrow \mathscr{X}(\mathfrak{D}_2)$. The inclusion $\mathrm{im}(\psi_1 \circ \sigma)(\mathbf{T}_2) \subset \mathrm{im}\psi_1(\mathbf{T}_1) \subset \mathrm{End}_{\mathscr{O}(\Omega)}(\mathscr{M}_1(\Omega))$ induces a finite morphism $i'_f = \mathscr{X}(\mathfrak{D}_1) \to \mathscr{X}(\mathfrak{D}_1^\sigma)$. The ideal subsheaf of $\mathscr{O}_{\mathscr{X}(\mathfrak{D}_1^\sigma)}$ cut out by the kernel of the composite $\mathscr{O}_{\mathscr{X}(\mathfrak{D}_1^\sigma)} \to \mathscr{O}_{\mathscr{X}(\mathfrak{D}_1)} \twoheadrightarrow \mathscr{O}_{\mathscr{X}(\mathfrak{D}_1)^\circ}$ determines a closed immersion $\mathscr{Y} \hookrightarrow \mathscr{X}(\mathfrak{D}_1^\sigma)$ fitting into a diagram

$$\begin{tikzcd}
\mathscr{X}(\mathfrak{D}_1)^\circ \arrow[r, hook] \arrow[d, "i_f"'] & \mathscr{X}(\mathfrak{D}_1) \arrow[dr, "i'_f"] & \\
\mathscr{Y} \arrow[r, hook, "i''_c"'] & \mathscr{X}(\mathfrak{D}_1^\sigma)^\circ \arrow[r, hook] & \mathscr{X}(\mathfrak{D}_1^\sigma),
\end{tikzcd}$$

and taking $i_c = i'_c \circ i''_c$ concludes. $\square$

The template for applying these results is as follows. Consider a pair of connected, reductive groups $\mathbf{G}$ and $\mathbf{H}$ over $\mathbf{Q}$, together with an L-homomorphism ${}^L\sigma : {}^L\mathbf{G} \to^L \mathbf{H}$ which is *known* to induce a Langlands functoriality map. The L-homomorphism induces a morphism $\mathbf{T_H} \to \mathbf{T_G}$ of unramified Hecke algebras in the usual way. When $\mathbf{G}$ and $\mathbf{H}$ are inner forms of each other, ${}^L\sigma$ is an isomorphism, and Theorem 5.1.2 (with $\mathscr{W}_1 = \mathscr{W}_2$ and $\jmath = \mathrm{id}$) gives rise to closed immersions of eigenvarieties interpolating correspondences of Jacquet-Langlands type and/or comparing different theories of overconvergent automorphic forms. In the general case, the homomorphism $X^*(\widehat{T}_\mathbf{H}) \to X^*(\widehat{T}_\mathbf{G})$ together with the natural identification $T(A) \simeq X^*(\widehat{T}) \otimes_\mathbf{Z} A$ induces a homomorphism $\tau : T_\mathbf{H}(\mathbf{Z}_p) \to T_\mathbf{G}(\mathbf{Z}_p)$, and $\jmath$ is given by sending a character λ of $T_\mathbf{G}$ to the character $(\tau \circ \lambda) \cdot \xi_\sigma$ for some *fixed* character ξ_σ of $T_\mathbf{H}$ which may or may not be trivial. In this case, Theorem 5.1.5 then induces morphisms of eigenvarieties interpolating the functoriality associated with ${}^L\sigma$. In practice, one must carefully choose the character ξ_σ, compatible tame levels for $\mathbf{G}$ and $\mathbf{H}$, and an extension of the map on unramified Hecke algebras to include the Atkin-Lehner operators.

5.2 Refinements of unramified representations

In order to apply the interpolation theorems of the previous section, we need a systematic way of producing sets $\mathscr{Z}^{\mathrm{cl}}$ such that $\mathscr{M}_1(z)$ consists entirely of classical automorphic forms for $z \in \mathscr{Z}^{\mathrm{cl}}$. The key is Theorem 3.2.5 together with Proposition 5.2.1 below.

Let $G \simeq \mathrm{GL}_n/\mathbf{Q}_p$, with B the upper-triangular Borel and $\overline{B}$ the lower-triangular Borel.

In this case we may canonically parametrize L-valued characters of $\mathcal{A}_p$ and unramified characters of $T(\mathbf{Q}_p)$ by ordered n-tuples $\mathbf{a} = (a_1, \ldots, a_n) \in (L^\times)^n$, the former via the map

$$\mathbf{a} \mapsto \chi_{\mathbf{a}}(u_{p,i}) = a_{n+1-i}$$

and the latter via the map

$$\mathbf{a} \mapsto \chi_{\mathbf{a}}(t_1, \ldots, t_n) = \prod_{i=1}^{n} a_i^{v_p(t_i)}.$$

Let π be an unramified generic irreducible representation of $G = \mathrm{GL}_n$ defined over L, and let $r : W_{\mathbf{Q}_p} \to \mathrm{GL}_n(L)$ be the unramified Weil-Deligne representation satisfying $r \simeq \mathrm{rec}(\pi_p \otimes |\det|^{\frac{1-n}{2}})$. Let $\varphi_1, \ldots, \varphi_n$ be any fixed ordering on the eigenvalues of $r(\mathrm{Frob}_p)$, and let χ_σ, $\sigma \in S_n$ be the character of $\mathcal{A}_p$ defined by $\mathcal{A}_p(u_i) = p^{1-i}\varphi_{\sigma(i)}$.

Proposition 5.2.1. *For every $\sigma \in S_n$, the module π_p^I contains a nonzero vector v_σ such that $\mathcal{A}_p$ acts on v_σ through the character χ_σ.*

Proof. Assembling some results of Casselman (cf. §3.2.2 of (Taï12)), there is a natural isomorphism of Λ-modules

$$\pi^I \xrightarrow{\sim} (\pi_{\overline{N}})^{T(\mathbf{Z}_p)} \otimes \delta_{\overline{B}}^{-1}$$

where $t \in \Lambda$ acts on the left-hand side by U_t. By Theorem 4.2 of (BZ77) and Satake's classification of unramified representations, we may write π as the full normalized induction

$$\pi = \mathrm{Ind}_{\overline{B}}^{G}\chi$$

where χ is the character of T associated with the tuple $(p^{\frac{1-n}{2}}\varphi_{\sigma(n)}, \ldots, p^{\frac{1-n}{2}}\varphi_{\sigma(1)})$. By Frobenius reciprocity, there is an embedding of T-modules

$$L(\chi\delta_{\overline{B}}^{\frac{1}{2}}) \hookrightarrow (\mathrm{Ind}_{\overline{B}}^{G}\chi_\varphi)_{\overline{N}},$$

so $L(\chi\delta_{\overline{B}}^{-\frac{1}{2}}) \hookrightarrow \pi^I$, and $\chi_\sigma = \chi\delta_{\overline{B}}^{-\frac{1}{2}}$ upon noting that $\delta_{\overline{B}}^{-\frac{1}{2}}$ corresponds to the tuple $(p^{\frac{1-n}{2}}, \ldots p^{\frac{n-1}{2}})$. □

5.3 Some quaternionic eigencurves

Fix a squarefree positive integer $\delta \geq 2$, a positive integer N prime to δ, and a prime p with $p \nmid N\delta$. Let D be the quaternion division algebra over $\mathbf{Q}$ ramified at exactly the finite places dividing δ, and ramified or split over $\mathbf{R}$ according to whether δ has an odd or even number of distinct prime divisors. Let $\mathbf{G}$ be the inner form of $\mathrm{GL}_2/\mathbf{Q}$ associated with D, and let $\mathscr{X}_D$ be the eigenvariety $\mathscr{X}^i_{\mathbf{G}}(N)^0$ as defined in §4.2, where $i = 0$ or 1 according to whether D is ramified or split at infinity. Let $\mathscr{X}$ be the eigenvariety $\mathscr{X}^1_{\mathrm{GL}_2/\mathbf{Q}}(N\delta)^0$. The eigenvarieties $\mathscr{X}$ and $\mathscr{X}_D$ are both unmixed of dimension one.

Theorem 5.3.1. *There is a canonical closed immersion $\iota_{\mathrm{JL}} : \mathscr{X}_D \hookrightarrow \mathscr{X}$ interpolating the Jacquet-Langlands correspondence on non-critical classical points.*

Proof. Let $\mathscr{M}^\dagger$ and $\mathscr{M}^\dagger_D$ be the sheaves of automorphic forms carried by the eigenvarieties $\mathscr{X}$ and $\mathscr{X}_D$, respectively. Let $\mathscr{Z}^{\mathrm{ncc}} \subset (\mathscr{W} \times \mathbf{A}^1)(\overline{\mathbf{Q}_p})$ be the set of points $z = (\lambda, \alpha^{-1})$ with $\lambda \in \mathscr{W}(\mathbf{Q}_p)$ of the form $\lambda(x) = x^k$, $k \in \mathbf{Z}_{\geq 1}$, and with $v_p(\alpha) < k+1$. For any $z \in \mathscr{Z}^{\mathrm{ncc}}$ Theorem 3.2.5 together with the classical Eichler-Shimura isomorphism induces an isomorphisms of Hecke modules

$$\mathscr{M}^\dagger(z) \simeq \left(S_{k+2}(\Gamma_1(N\delta) \cap \Gamma_0(p)) \oplus M_{k+2}(\Gamma_1(N\delta) \cap \Gamma_0(p))\right)^{U_p = \iota(\alpha)}$$

and

$$\mathscr{M}^\dagger_D(z) \simeq \left(S^D_{k+2}(\Gamma_1(N) \cap \Gamma_0(p))^2\right)^{U_p = i(\alpha)}.$$

Let $\mathscr{Z}^{\mathrm{cl}}$ be the set of points $z \in \mathscr{Z}^{\mathrm{ncc}}$ for which $\mathscr{M}^\dagger_D(z)$ is nonzero. The set $\mathscr{Z}^{\mathrm{cl}}$ forms a Zariski-dense accumulation subset of $\mathscr{Z}^\circ_1$, and $D_1(T, X)(z)$ divides $D_2(T, X)(z)$ in $k(z)[X]$ for any $z \in \mathscr{Z}^{\mathrm{cl}}$ by the classical Jacquet-Langlands correspondence. Theorem 5.1.2 now applies.

5.4 A symmetric square lifting

Let $\mathscr{C}_0(N)$ be the cuspidal locus of the Coleman-Mazur-Buzzard eigencurve of tame level N. Given a non-CM cuspidal modular eigenform $f \in S_k(\Gamma_1(N))$, Gelbart and Jacquet constructed an cuspidal automorphic representation $\Pi(\mathrm{sym}^2 f)$ of $\mathrm{GL}_3/\mathbf{Q}$ characterized by the isomorphism

$$\iota\mathrm{WD}(\mathrm{sym}^2 V_{f,\iota}|G_{\mathbf{Q}_\ell}) \simeq \mathrm{rec}\left(\Pi(\mathrm{sym}^2 f)_\ell \otimes |\det|_\ell^{-1}\right)$$

for all primes ℓ. We are going to interpolate this map into a morphism $i_{\mathrm{GJ}} : \mathscr{C}_0^{\mathrm{ncm}}(N) \to \mathscr{X}$, where $\mathscr{C}_0^{\mathrm{ncm}}(N)$ is the Zariski-closure inside $\mathscr{C}_0(N)$ of the classical points associated with non-CM eigenforms and $\mathscr{X}$ is an eigenvariety arising from overconvergent cohomology on GL_3.

To construct the portion of the eigencurve we need, set $\mathbf{G} = \mathrm{GL}_2/\mathbf{Q}$ and $\mathbf{T}_1 = \mathbf{T}_{\mathbf{G}}(K(N))$. The eigencurve $\mathscr{C}_0(N)$ arises from an eigenvariety datum $\mathfrak{D}_1' = (\mathscr{W}_1, \mathscr{Z}_1, \mathscr{M}_1, \mathbf{T}_1, \psi_1)$ with $\mathscr{W}_1 = \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{G}_m)$, $\mathscr{Z}_1$ the Fredholm series of U_p^4 acting on cuspidal overconvergent modular forms of tame level N, and $\mathscr{M}_1$ the natural spreading out of overconvergent cuspidal modular forms of tame level N. Let $\mathscr{I} \subset \mathscr{O}_{\mathscr{C}_0(N)}$ be the coherent ideal sheaf cutting out the closed immersion $\mathscr{C}_0^{\mathrm{ncm}}(N) \hookrightarrow \mathscr{C}_0(N)$, and set $\mathfrak{D}_1 = (\mathscr{W}_1, \mathscr{Z}_1, \mathscr{M}_1 \otimes_{\mathscr{O}_{\mathscr{Z}}} \pi_*(\mathscr{O}_{\mathscr{C}_0(N)}/\mathscr{I}), \mathbf{T}_1, \psi)$, so $\mathscr{C}_0^{\mathrm{ncm}}(N)$ arises from the eigenvariety datum $\mathfrak{D}_1$. The eigencurve $\mathscr{C}_0^{\mathrm{ncm}}(N)$ is unmixed of dimension one. For any eigenform $f \in S_{k+2}^{\mathrm{ncm}}(\Gamma_1(N))$ and α either root of the Hecke polynomial $X^2 - a_f(p) + p^{k+1}\varepsilon(p)$, we define $\phi_{f,\alpha} : \mathbf{T}_1 \to \overline{\mathbf{Q}_p}$ as the eigenpacket associated with the point $x_{f,\alpha} \in \mathscr{C}_0(N)$. We normalize the weight map $\mathscr{C}_0(N) \to \mathscr{W}_1$ so that for f a classical cuspidal eigenform of weight $k+2$ whose nebentype has p-part ε, $w(x_f)$ corresponds to the character $t \mapsto t^k \varepsilon(t)$.

Now take $\mathbf{H} = \mathrm{GL}_3/\mathbf{Q}$ and $\mathbf{T}_2 = \mathbf{T}_{\mathbf{H}}(K(N^2))$. Let $\mathfrak{D}_2$ be the eigenvariety datum from Definition 4.3.2, with $\mathscr{X} = \mathscr{X}_{\mathbf{H}}(K(N^2))$ the associated eigenvariety.

Theorem 5.4.1. *Under the hypotheses above, there is a morphism $i_{\mathrm{GJ}} : \mathscr{C}_0^{\mathrm{ncm}}(N) \to \mathscr{X}$ interpolating the symmetric square lift on classical points.*

Let $\jmath : \mathscr{W}_1 \hookrightarrow \mathscr{W}_2$ be the closed immersion sending a character λ to the character

$\jmath(\lambda)(t_1, t_2, t_3) = \lambda(t_1^2 t_2)$. Let $\sigma : \mathbf{T}_2 \to \mathbf{T}_1$ be the map defined on generators by

$$\begin{aligned}
\sigma(T_{\ell,1}) &= T_\ell^2 - \ell S_\ell, \\
\sigma(T_{\ell,2}) &= T_\ell^2 S_\ell - \ell S_\ell^2, \\
\sigma(T_{\ell,3}) &= S_\ell^3, \\
\sigma(U_{p,1}) &= U_p^2, \\
\sigma(U_{p,2}) &= U_p^2 S_p, \\
\sigma(U_{p,3}) &= S_p^3.
\end{aligned}$$

Lemma 5.4.2. *If $f \in S_{k+2}^{\mathrm{ncm}}(\Gamma_1(N))$ has nebentype ε_f and $X^2 - a_f(p)X + p^{k+1}\varepsilon_f(p)$ has a root α with $v_p(\alpha) < \frac{k+1}{4}$, then $\Pi(\mathrm{sym}^2 f)$ contributes to $H^*(K(N^2), \mathcal{L}_{(2k,k,0)})$, and $H^*(K(N^2), \mathcal{D}_{(2k,k,0)})$ contains a nonzero vector v such that every $T \in \mathbf{T}_2$ acts on v through the scalar $(\phi_{f,\alpha} \circ \sigma)(T)$.*

Proof. Fix f and α as in the lemma, and let λ be the highest weight $(2k, k, 0)$. A calculation with the local Langlands correspondence then shows that $\Pi(\mathrm{sym}^2 f)_\ell$ has conductor at most $\ell^{2v_\ell(N)}$. Since f is non-CM, $\Pi(\mathrm{sym}^2 f)$ is cuspidal. The Hecke module $\Pi(\mathrm{sym}^2 f)^{K(N^2)I}$ occurs in $H^*(K(N^2), \mathcal{L}_\lambda)$ by the Gelbart-Jacquet lifting and the calculations in (Clo90). At primes $\ell \nmid Np$, $\Pi(\mathrm{sym}^2 f)_\ell$ is unramified, and $T_{\ell,i}$ acts on the unramified line via the scalar $(\phi_f \circ \sigma)(T_{\ell,i})$. A simple calculation using Proposition 5.2.1 shows that $\Pi(\mathrm{sym}^2 f)^{K(N^2)I}$ contains a vector on which $\mathcal{A}_p$ acts through the character associated with the tuple $(p^{-2}\beta^2, p^{-1}\alpha\beta, \alpha^2)$, so there is a vector v' in the $\Pi(\mathrm{sym}^2 f)$-isotypic component of $H^*(K(N^2)I, \mathcal{L}_\lambda)$ such that the $\star$-action of $\mathcal{A}_p$ is given by the character associated with the tuple $(\alpha^{-2}\varepsilon_f(p)^2, \varepsilon_f(p), \alpha^2)$. In particular, U_p acts on v' through the scalar $\alpha^4\varepsilon(p)$. By Proposition 3.2.5, the integration map $i_\lambda : H^*(K^p, \mathcal{D}_\lambda) \to H^*(K^p, \mathcal{L}_\lambda)$ is an isomorphism on the subspace where U_p acts with slope $< k+1$, so $v = i_\lambda^{-1}(v')$ does the job. □

Now we take $\mathscr{Z}^{\mathrm{cl}}$ to be the set of points in $\mathscr{Z}_1$ of the form (λ, α^{-1}), where λ is a character of the form $\lambda(x) = x^k, k \in \mathbf{Z}_{\geq 3}$ and α satisfies $v_p(\alpha) < \frac{k+1}{4}$. This is a Zariski-dense accumulation subset of $\mathscr{Z}_1$. By Coleman's classicality criterion, there is a natural isomorphism $\mathscr{M}_1(z) \simeq S_{k+2}^{\mathrm{ncm}}(\Gamma_1(N) \cap \Gamma_0(p))^{U_p = \iota(\alpha)}$ of $\mathbf{T}_1$-modules, so Theorem 5.1.5 now applies, with the divisibility hypothesis following from Lemma 5.4.2. We thus conclude. □

It's not hard to show that the image of i_{GJ} is actually a union of irreducible components of $\mathscr{X}^0$.

5.5 A Rankin-Selberg lifting

Let f and g be a pair of level one holomorphic cuspidal eigenforms of weights k_f+2, k_g+2 with associated Galois representations $V_{f,\iota}$ and $V_{g,\iota}$. By a deep theorem of Ramakrishnan (Ram00), there is a unique isobaric automorphic representation $\Pi(f\otimes g)$ of $\mathrm{GL}_4(\mathbf{A}_{\mathbf{Q}})$ characterized by the equality

$$\mathrm{rec}_\ell\left(\Pi(f\otimes g)_\ell\otimes|\det|_\ell^{-\frac{3}{2}}\right)\simeq\iota\mathrm{WD}(V_{f,\iota}\otimes V_{g,\iota}|G_{\mathbf{Q}_\ell})$$

for all primes ℓ. We are going to interpolate the map $(f,g)\mapsto\Pi(f\otimes g)$ into a morphism of eigenvarieties $\mathscr{C}_0\times\mathscr{C}_0\to\mathscr{X}$, where $\mathscr{C}_0$ denotes the cuspidal locus of the Coleman-Mazur eigencurve and $\mathscr{X}$ denotes an eigenvariety associated with overconvergent cohomology on $\mathrm{GL}_4/\mathbf{Q}$.

Set $\mathbf{G}=\mathrm{GL}_2/\mathbf{Q}$, $\mathbf{T}_1=\mathbf{T}_{\mathbf{G}}(K(1))\otimes\mathbf{T}_{\mathbf{G}}(K(1))$, and $\mathscr{W}_1=\mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times\times\mathbf{Z}_p^\times,\mathbf{G}_m)$; we regard an A-point of $\mathscr{W}_1$ as a pair of characters $\lambda_1,\lambda_2:\mathbf{Z}_p^\times\to A^\times$ in the obvious way. The product $\mathscr{C}_0\times\mathscr{C}_0$ arises from an eigenvariety datum $\mathfrak{D}_1=(\mathscr{W}_1,\mathscr{Z}_1,\mathscr{M}_1,\mathbf{T}_1,\psi_1)$, where $(\lambda_1,\lambda_2,\alpha^{-1})\in\mathscr{Z}_1(\overline{\mathbf{Q}_p})$ if and only if there exist cuspidal overconvergent eigenforms f_1 and f_2 of weights λ_1 and λ_2 such that $U_p^4\otimes U_p^2-\alpha$ annihilates $f_1\otimes f_2$.

Set $\mathbf{H}=\mathrm{GL}_4/\mathbf{Q}$, $\mathbf{T}_2=\mathbf{T}_{\mathbf{H}}(K(1))$, and let $\mathfrak{D}_2$ be the eigenvariety datum from Definition 4.3.2, with $\mathscr{X}$ the associated eigenvariety.

Theorem 5.4.1. *Under the hypotheses above, there is a morphism $i_{\mathrm{RS}}:\mathscr{C}_0\times\mathscr{C}_0\to\mathscr{X}$ interpolating the Rankin-Selberg lift on classical points.*

Let $\jmath:\mathscr{W}_1\hookrightarrow\mathscr{W}_2$ be the closed immersion defined by sending a character $\lambda\in\mathscr{W}_1$ to the character

$$\jmath(\lambda)(t)=(t_1t_2)^{-1}\lambda_1(t_1t_2)\lambda_2(t_1t_3),\ t=\mathrm{diag}(t_1,t_2,t_3,t_4)\in T_{\mathbf{H}}.$$

Define a map $\sigma : \mathbf{T}_2 \to \mathbf{T}_1$ on generators by

$$\begin{aligned}
\sigma(T_{\ell,1}) &= T_\ell \otimes T_\ell, \\
\sigma(T_{\ell,2}) &= S_\ell \otimes T_\ell^2 + T_\ell^2 \otimes S_\ell - 2\ell S_\ell \otimes S_\ell, \\
\sigma(T_{\ell,3}) &= \ell^{-1} S_\ell T_\ell \otimes S_\ell T_\ell, \\
\sigma(T_{\ell,4}) &= \ell^{-2} S_\ell^2 \otimes S_\ell^2 \\
\sigma(U_{p,1}) &= U_p \otimes U_p, \\
\sigma(U_{p,2}) &= U_p^2 \otimes S_p, \\
\sigma(U_{p,3}) &= U_p S_p \otimes S_p, \\
\sigma(U_{p,4}) &= S_p^2 \otimes S_p^2
\end{aligned}$$

Let (f,g) be an ordered pair as above with $k_f - 1 > k_g > 0$. Set $\lambda = \lambda(f \otimes g) : (x_1, x_2) \mapsto x_1^{k_f} x_2^{k_g} \in \mathscr{W}_1$. Let α_f, β_f be the roots of the Hecke polynomial $X^2 - a_f(p)X + p^{k_f+1}$, and likewise for g.

Lemma 5.5.2. *The module $\Pi(f \otimes g)_p^I$ contains a vector v on which $\mathcal{A}_p$ acts through the character associated with the tuple $(p^{-3}\beta_f\beta_g, p^{-2}\beta_f\alpha_g, p^{-1}\alpha_f\beta_g, \alpha_f\alpha_g)$; in particular, U_p acts via the scalar $p^{2k_g+k_f-1}\alpha_f^4\alpha_g^2$.*

Proof. This is a direct consequence of Proposition 5.2.1, together with the characterization of $\Pi(f \otimes g)$ given above. □

Lemma 5.5.3. *If α_f and α_g satisfy $v_p(\alpha_f^4\alpha_g^2) < \min(k_f - k_g, k_g + 1)$, then $H^*(K(1), \mathcal{D}_{j(\lambda)})$ contains a nonzero vector v such that every $T \in \mathbf{T}_2$ acts on v through the scalar $(\phi_{f,\alpha_f} \otimes \phi_{g,\alpha_g})(\sigma(T))$.*

Proof. Dominance of $j(\lambda)$ is obvious, so $\Pi(f \otimes g)$ is cohomological in the weight $j(\lambda)$. For primes $\ell \nmid p$, $\Pi(f \otimes g)_\ell$ is unramified and $T_{\ell,i}$ acts on the unramified line via the scalar $(\phi_{f,\alpha_f} \otimes \phi_{g,\alpha_g})(\sigma(T_{\ell,i}))$.

Next, recall that the $\star$-action of $\mathcal{A}_p$ on $\Pi(f \otimes g)_p^I$ is simply the usual action rescaled by $\lambda(1, p, p^2, p^3)^{-1}$, and λ corresponds to the highest weight $(k_f + k_g - 1, k_f - 1, k_g, 0)$, so $\lambda(1, p, p^2, p^3)^{-1} = p^{1-2k_g-k_f}$. In particular, $\Pi(f \otimes g)_p^I$ contains a vector on which U_p acts through the scalar $\alpha_f^4\alpha_g^2$ by Lemma 5.5.2. Writing $\kappa = \min(k_f - k_g, k_g + 1)$, the integration map $i_{j(\lambda)}$ induces an isomorphism

$$H^*(K(1), \mathcal{D}_{j(\lambda)})_{<\kappa} \xrightarrow{\sim} H^*(K(1), \mathcal{L}_{j(\lambda)})_{<\kappa},$$

and the target contains a vector satisfying the claim of the theorem. □

Finally, we take $\mathscr{X}^{\mathrm{cl}}$ to be the set of points in $\mathscr{X}_1$ of the form $(\lambda_1, \lambda_2, \alpha^{-1})$ where $\lambda_1(x) = x^{k_1}$ and $\lambda_2(x) = x^{k_2}$ with $k_i \in \mathbf{Z}$ and $0 < k_2 < k_1 - 1$, and α satisfies $\alpha < \min(k_1 - k_2, k_2 + 1)$. This is a Zariski-dense accumulation subset, and Theorem 5.1.5 applies with the divisibility hypothesis following from Lemma 5.5.3. This proves Theorem 5.5.1. □

Chapter 6

A modularity conjecture for triangulline Galois representations

6.1 The conjecture

Fix a prime p, and let $L/\mathbf{Q}_p$ be a finite extension. Let V be a continuous representation of $G_{\mathbf{Q}}$ on a finite-dimensional L-vector space such that $V|G_{\mathbf{Q}_\ell}$ is unramified for all but finitely many primes. When $V|G_{\mathbf{Q}_p}$ is potentially semistable, a fundamental conjecture of Langlands, Clozel, and Fontaine-Mazur predicts the existence of an automorphic representation $\pi \simeq \otimes'_\ell \pi_\ell$ of $\mathrm{GL}_n(\mathbf{A}_{\mathbf{Q}})$ such that $\pi_\ell \otimes |\det|_\ell^{\frac{1-n}{2}}$ matches $\mathrm{WD}(V|G_{\mathbf{Q}_\ell})$ under the local Langlands correspondence for all primes ℓ. By the theory of Eisenstein series, there is no loss in formulating this conjecture for V which are absolutely irreducible, in which case the predicted π should be cuspidal. If the representation V is regular, π will be visible in the cohomology of a local system. More precisely, suppose the Hodge-Tate weights of $V|G_{\mathbf{Q}_p}$ are $k_1 < k_2 < \cdots < k_n$, and let $\mathcal{L}_\lambda$ be the GL_n-representation of highest weight $(k_n + 1 - n, k_{n-1} + 2 - n, \ldots, k_1)$. Let N be the conductor of V, and let $\mathbf{T}_\lambda^{\mathrm{cl}}(N)$ be the commutative subalgebra of

$$\mathrm{End}_{\mathbf{C}}\left(H^*(Y(K_1(N)), \mathcal{L}_\lambda(\mathbf{C}))\right)$$

generated by the Hecke operators $T_{\ell,i}$ for $1 \le i \le n$ and primes $\ell \nmid N$. For any isomorphism $\iota : \overline{L} \xrightarrow{\sim} \mathbf{C}$, the Fontaine-Mazur-Langlands conjecture asserts the existence of a $\mathbf{C}$-algebra homomorphism $\phi_V : \mathbf{T}^{\mathrm{cl}}_\lambda(N) \to \mathbf{C}$ such that the equality

$$\iota \det(X \cdot \mathrm{Id} - \mathrm{Frob}_\ell)|V = \sum_{i=0}^{n} (-1)^i \ell^{\frac{i(i-1)}{2}} \phi_V(T_{\ell,i}) X^{n-i} \text{ in } \mathbf{C}[X]$$

holds for all $\ell \nmid pN$, where Frob_ℓ denotes a geometric Frobenius at ℓ.

In this chapter we formulate a generalization of the above conjecture encompassing all *trianguline* Galois representations. On the one hand, if $V|G_{\mathbf{Q}_p(\zeta_{p^j})}$ is semistable for some j, V is trianguline; on the other hand, most trianguline representations aren't even Hodge-Tate. In light of this, any such conjecture must go beyond finite-dimensional local systems. Perhaps unsurprisingly, we formulate a conjecture relating trianguline Galois representations with eigenclasses in overcovergent cohomology. In the setting of conjugate-self-dual Galois representations and unitary group eigenvarieties, an analogous conjecture is formulated in (Hel12).

To state the conjecture precisely, we need to recall some concepts from p-adic Hodge theory (for a nice introduction to this circle of ideas, see (Ber11)). Let $\mathcal{R} = \mathcal{R}_L$ be the subring of $L[[X, X^{-1}]]$ consisting of series which converge on some open annulus $r < |X|_p < 1$.[1] Set $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$, and let $\chi : \Gamma \xrightarrow{\sim} \mathbf{Z}_p^\times$ be the cyclotomic character.[2] The formulas $(\varphi \cdot f)(X) = f\left((1+X)^p - 1\right)$ and $(\gamma \cdot f)(X) = f\left((1+X)^{\chi(\gamma)} - 1\right)$ for $\gamma \in \Gamma$ define commuting actions on $\mathcal{R}$. A (φ, Γ)*-module of rank* n is a free $\mathcal{R}$-module $\mathbf{D}$ of rank n equipped with commuting $\mathcal{R}$-semilinear actions of φ and Γ, such that the matrix of φ lies in $\mathrm{GL}_n(\mathcal{R})$ for some basis of $\mathbf{D}$. A (φ, Γ)-module is *étale* if it admits a basis for which the matrix of φ lies in $\mathrm{GL}_n(\mathcal{R} \cap \mathcal{O}_L[[X, X^{-1}]])$. By work of Berger, Fontaine-Wintenberger, Cherbonnier-Colmez, and Kedlaya, there is an equivalence of categories between continuous L-linear $G_{\mathbf{Q}_p}$-representations and étale (φ, Γ)-modules. If V is a representation of $G_{\mathbf{Q}_p}$, we denote its associated (φ, Γ)-module by $\mathbf{D}(V)$.[3] The functor $V \mapsto \mathbf{D}(V)$ can be realized explicitly á la Fontaine: there is a ring $\mathbf{B}^\dagger_{\mathrm{rig}}$ equipped with commuting actions of $G_{\mathbf{Q}_p}$ and an operator φ such that

$$\mathbf{D}(V) = \left(V \otimes_{\mathbf{Q}_p} \mathbf{B}^\dagger_{\mathrm{rig}}\right)^{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p(\zeta_{p^\infty}))}.$$

[1] Throughout this chapter we tacitly enlarge L whenever convenient; in Berger's terminology, we make no distinction between "trianguline" and "split trianguline" representations.

[2] We shall occasionally regard χ as a character of $G_{\mathbf{Q}}$ in the obvious way.

[3] This functor is often denoted $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ in the literature.

As a simple but crucial example of this correspondence, there is a bijection between rank one (φ, Γ)-modules and continuous characters $\delta : \mathbf{Q}_p^\times \to L^\times$: the character δ corresponds to the rank one (φ, Γ)-module $\mathcal{R}(\delta)$ with φ and Γ acting on a basis element e_δ by $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$. The (φ, Γ)-module $\mathcal{R}(\delta)$ is étale if and only if $\delta(p) \in \mathcal{O}_L^\times$, in which case $\mathcal{R}(\delta)$ corresponds to the character of $G_{\mathbf{Q}_p}$ whose restriction to $W_{\mathbf{Q}_p}$ is $\mathrm{Art}_{\mathbf{Q}_p}^{-1} \circ \delta$. Any δ may be decomposed uniquely as $\mu_{\delta(p)}\delta(x_0)$ where $\mu_\alpha(x) = \alpha^{v_p(x)}$ and $x_0 = x|x|_p$.

Given a continuous n-dimensional $G_{\mathbf{Q}_p}$-representation V, an ordered n-tuple $\delta = (\delta_1, \ldots, \delta_n)$ of continuous characters $\delta_i : \mathbf{Q}_p^\times \to L^\times$ is a *parameter of* V if $\mathbf{D}(V)$ admits a filtration

$$0 = \mathrm{Fil}^0 \subset \mathrm{Fil}^1 \subset \cdots \subset \mathrm{Fil}^n = \mathbf{D}(V)$$

by (φ, Γ)-stable free $\mathcal{R}$-direct summands such that $\mathrm{Fil}^i/\mathrm{Fil}^{i-1} \simeq \mathcal{R}(\delta_i)$ for $1 \le i \le n$. Let $\mathscr{P}\mathrm{ar}(V)$ denote the set of parameters of V. Note that a given representation V may not admit any parameters at all.

Definition 6.1.1 (Colmez). *A $G_{\mathbf{Q}_p}$-representation V is* trianguline *if $\mathscr{P}\mathrm{ar}(V)$ is nonempty.*

The most well-studied trianguline representations are the *nearly ordinary representations,* in which case V itself admits a $G_{\mathbf{Q}_p}$-stable full flag $0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(n)} = V$ such that $\mathrm{Fil}^i = \mathbf{D}(V^{(i)})$; a parameter δ gives a nearly ordinary structure on V if and only if $\delta_i(p) \in \mathcal{O}_L^\times$ for $1 \le i \le n$. However, most trianguline representations are irreducible qua Galois representations.

Let $\mathbf{G} = \mathrm{GL}_n/\mathbf{Q}$, with T the standard diagonal maximal torus. Given an n-dimensional trianguline representation V with $\delta \in \mathscr{P}\mathrm{ar}(V)$, define a character $\lambda(\delta) : T(\mathbf{Z}_p) \to L^\times$ by the formula

$$\lambda(\delta)(t_1, \ldots, t_n) = \prod_{i=1}^{n} \delta_{n+1-i}(t_i^{-1})t_i^{i-n}.$$

As in §4.6, let $\mathbf{T}_\lambda(N)$ be the finite-slope Hecke algebra of weight λ and level $K(N)$.

Conjecture 6.1.2. *Let V be a continuous n-dimensional L-linear representation of $G_{\mathbf{Q}}$ which is odd, absolutely irreducible, unramified almost everywhere, and trianguline at p, with prime-to-p Artin conductor N. Then for any parameter $\delta \in \mathscr{P}\mathrm{ar}(V)$, there is a finite-slope* *eigenpacket* $\phi_{V,\delta} : \mathbf{T}_{\lambda(\delta)}(N) \to L$ *such that*

$$\det\left(X \cdot \mathrm{Id} - \mathrm{Frob}_\ell\right)|V = \sum_{i=0}^{n} (-1)^i \ell^{\frac{i(i-1)}{2}} \phi_{V,\delta}(T_{\ell,i}) X^{n-i} \text{ in } L[X]$$

for all primes $\ell \nmid Np$, *with* $\phi_{V,\delta}(u_{p,i}) = \delta_i(p)$ *for* $1 \leq i \leq n$.

Hard evidence aside, there are certain suggestive similarities between the Galois and automorphic sides of the picture. For example, any finite-slope eigenpacket satisfies the inequality $v_p(\phi(U_{p,i})) \geq 0$, while Kedlaya's theory of Frobenius slopes implies the inequality $v_p(\delta_1(p)\delta_2(p)\cdots\delta_i(p)) \geq 0$ for any parameter of any trianguline representation.

In light of Theorem 4.3.3, the truth of Conjecture 6.1.2 for any given pair (V, δ) is equivalent to the existence of a point $x_{V,\delta} \in \mathscr{X} = \mathscr{X}_{\mathrm{GL}_{n}/\mathbf{Q}}(K(N))$ with $w(x_{V,\delta}) = \lambda(\delta)$ such that $\phi_{\mathscr{X}}(x_{V,\delta})$ satisfies the putative properties of $\phi_{V,\delta}$. This simple observation turns out to be a powerful tool for establishing special cases of the conjecture.

Theorem 6.1.3. *If* p *is odd and* V *is a two-dimensional representation such that* $\overline{V}|G_{\mathbf{Q}(\zeta_p)}$ *is absolutely irreducible and* $\overline{V}|G_{\mathbf{Q}_p}$ *is not of the form* $\begin{pmatrix} \eta & * \\ & \eta \end{pmatrix}$ *or* $\begin{pmatrix} \eta & * \\ & \eta\overline{\chi} \end{pmatrix}$, *then Conjecture 6.1.2 is true for every parameter of* V.

This theorem is almost entirely due to others, and the proof is simply a matter of assembling their results. More precisely, under the hypotheses of Theorem 6.1.3, Emerton proved that V is a twist of the Galois representation V_f associated with an overconvergent cuspidal eigenform f of finite slope and some tame level N_f; the result then follows from work of Stevens and Bellaïche (Bel12; Ste00) showing that the Hecke data associated with overconvergent eigenforms appears in overconvergent cohomology. The situation is most interesting when the weight of f is an integer $k \geq 2$ and $U_p f = \alpha f$ with $v_p(\alpha) > k - 1$, in which case Conjecture 6.1.2 predicts the existence of the *companion form of* f: a form g of weight $2 - k$ such that $V_f \simeq V_g \otimes \chi^{1-k}$ (Col96). This case also demonstrates the falsity of a naive converse to Conjecture 6.1.2.

Theorem 6.1.4. *Conjecture 6.1.2 is true for pairs* $(\mathrm{sym}^2 V_f, \delta)$ *where* V_f *is the Galois representation associated with an overconvergent cuspidal eigenform* f *of finite slope and* $\delta = (\delta_1^2, \delta_1\delta_2, \delta_2^2)$ *with* $(\delta_1, \delta_2) \in \mathscr{P}\mathrm{ar}(V_f)$.

Next we turn to an example involving crystalline representations. Quite generally, if V is a crystalline representation, then V is automatically trianguline, and any $\delta \in \mathscr{P}\mathrm{ar}(V)$

has the form

$$\delta = \left(\mu_{\alpha_1} x_0^{-w_1}, \dots, \mu_{\alpha_n} x_0^{-w_n}\right)$$

for some integers w_i and scalars $\alpha_i \in L^\times$. Any parameter reveals a great deal about V: the integers w_i are in fact the Hodge-Tate weights of V in some fixed ordering, and the eigenvalues of the crystalline Frobenius on $\mathbf{D}_{\mathrm{crys}}(V)$ are $p^{w_i}\alpha_i$. Following Bellaiche and Chenevier, we say a parameter of a crystalline representation is *noncritical* if $w_1 < w_2 < \cdots < w_n$, and *critical* otherwise. Note that $\lambda(\delta)$ is a B-dominant weight if and only if δ is noncritical.

Theorem 6.1.5. *Let f, g be two distinct classical cusp forms of level one and weights k_f, k_g. Then Conjecture 6.1.2 holds for the representation $V_{f,\iota} \otimes V_{g,\iota}$ and for the parameters*

$$\left(\mu_{\alpha_f\alpha_g}, \mu_{\alpha_f\alpha_g^{-1}} x_0^{1-k_g}, \mu_{\alpha_g\alpha_f^{-1}} x_0^{1-k_f}, \mu_{\alpha_f^{-1}\alpha_g^{-1}} x_0^{2-k_f-k_g}\right)$$

and

$$\left(\mu_{\alpha_f\alpha_g}, \mu_{\alpha_g\alpha_f^{-1}} x_0^{1-k_f}, \mu_{\alpha_f\alpha_g^{-1}} x_0^{1-k_g}, \mu_{\alpha_f^{-1}\alpha_g^{-1}} x_0^{2-k_f-k_g}\right),$$

where α_f (resp. α_g) denotes either root of the Hecke polynomial $X^2 - a_f(p)X + p^{k_f-1}$ (resp. $X^2 - a_g(p)X + p^{k_g-1}$).

Note that if $k_f > k_g$ (resp. $k_g > k_f$), only the first (resp. second) parameters here is noncritical, and *all* of these parameters are critical if $k_f = k_g$.

We end this introductory section with a refined conjecture on the structure of the smooth $\mathrm{GL}_n(\mathbf{A}_f^p)$-representation

$$H^*(\mathcal{D}_\lambda) = \lim_{K^p \to \{1\}} H^*(K^p, \mathcal{D}_\lambda).$$

Given V as above and a prime $\ell \nmid p$, let $\pi_\ell(V)$ denote (a slight renormalization of) the generic representation of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ defined over L associated with the Weil-Deligne representation $\mathrm{WD}(V|G_{\mathbf{Q}_\ell})^{\mathrm{Fr-ss}}$ by the construction of Breuil-Schneider (BS07); this representation satisfies $\pi_\ell(V) \otimes |\det|_\ell^{\frac{1-n}{2}} \simeq \mathrm{rec}^{-1}\left(\mathrm{WD}(V|G_{\mathbf{Q}_\ell})^{\mathrm{Fr-ss}}\right)$ *if* the right-hand side is generic, but in general $\pi_\ell(V)$ is reducible.

Conjecture 6.1.6. *Notation and assumptions as in Conjecture 6.1.2, the module*

$$\mathrm{Hom}_{L[\mathrm{GL}_n(\mathbf{A}_f^p)]}\left(\otimes'_{\ell \neq p} \pi_\ell(V), H^*(\mathcal{D}_{\lambda(\delta)})\right)$$

contains a nonzero vector v such that $u_{p,i}v = \delta_i(p)v$ for $1 \leq i \leq n$.

6.2 Compatibility with Fontaine-Mazur

Let $\Pi = \Pi_\infty \otimes \Pi_f$ be a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_\mathbf{Q})$ with conductor N, and choose a prime $p \nmid N$. Let $V_{\Pi,\iota}$ be the Galois representation associated with Π in the usual sense, and suppose $V_{\Pi,\iota}|G_{\mathbf{Q}_p}$ is crystalline with

$$\iota \det{(X-\varphi)}\,|\mathbf{D}_{\mathrm{crys}}(V_{\Pi,\iota}) = \det\left(X - \mathrm{rec}(\Pi_p \otimes |\det|^{\frac{1-n}{2}})(\mathrm{Frob}_p)\right)$$

(this is known if Π is essentially self-dual). Suppose $V_{\Pi,\iota}|G_{\mathbf{Q}_p}$ admits a parameter $\delta = (\mu_{\alpha_1}x_0^{-k_1}, \dots, \mu_{\alpha_n}x_0^{-k_n})$ such that $k_1 < k_2 < \cdots < k_n$. The eigenvalues of the crystalline Frobenius on $\mathbf{D}_{\mathrm{crys}}(V_{\Pi,\iota}|G_{\mathbf{Q}_p})$ are simply $\varphi_i = p^{k_i}\alpha_i$. Since Π_p is generic, $\Pi_p^{I_p}$ contains a vector v' on which $\mathcal{A}_p$ acts through the character $(p^{1-n}\varphi_n, p^{2-n}\varphi_{n-1}, \dots, \varphi_1)$. The weight $\lambda(\delta)$ in this case is the dominant highest weight $(k_n+1-n, k_{n-1}+2-n, \dots, k_1)$, so $u_{p,i} \star_{\lambda(\delta)} v' = \alpha_i v'$. But the map

$$i_{\lambda(\delta)} : H^*(K(N), \mathcal{D}_{\lambda(\delta)}) \to H^*(K(N)I, \mathcal{L}_{\lambda(\delta)})$$

intertwines the standard $\mathcal{A}_p$-action on the source with the $\star$-action on the target, so the vector v predicted in the source by Conjecture 6.1.2 is compatible under this map with the contribution of Π in the target.

6.3 Evidence for two-dimensional Galois representations

In this section we prove Theorem 6.1.3. By a simple twisting argument, we may suppose $V \simeq V_f$ where f is an overconvergent cuspidal eigenform of finite slope and tame level N. Set $\mathbf{D}_f = \mathbf{D}(V_f|G_{\mathbf{Q}_p})$. We define the *weight* of f as the unique continuous character $w : \mathbf{Z}_p^\times \to L^\times$ such that $\det V_f|I_{\mathbf{Q}_p} \simeq w(\chi^{-1})\chi^{-1}$. Let $S_w^\dagger(\Gamma_1(N))$ denote the linear span of weight w overconvergent cuspidal finite-slope eigenforms. For any $\lambda = (\lambda_1, \lambda_2)$ in the $\mathrm{GL}_2/\mathbf{Q}$ weight space, work of Stevens (Ste00) and Bellaiche (Bel12) yields a noncanonical injection of Hecke modules

$$\beta_\lambda : S^\dagger_{\lambda_1\lambda_2^{-1}}(\Gamma_1(N)) \otimes \lambda_2 \circ (\det|\det|) \hookrightarrow H^1(K(N), \mathcal{D}_\lambda)$$

where of course the right-hand side denotes overconvergent cohomology for GL_2.

Let α be the U_p-eigenvalue of f, and set $k = 2 + \frac{\log w(1+p)}{\log(1+p)}$, so the Sen weights of V_f are exactly 0 and $k-1$. We now partition the set of overconvergent modular eigenforms f into four types:

1a. $k \in \mathbf{Z}_{\geq 2}$ and $0 \leq v_p(\alpha) < k-1$,

2. $k \in \mathbf{Z}_{\geq 1}$ and $v_p(\alpha) = k-1$,

3a. $k \in \mathbf{Z}_{\geq 1}$ and $v_p(\alpha) > k-1$,

3b. $k \notin \mathbf{Z}_{\geq 1}$.

Forms of type 1 are always classical, while forms of type 3 are never classical. If f is of type 1 or 3b, then $\mathbf{D}_f$ has parameter

$$\delta = (\mu_\alpha, \mu_{\alpha^{-1}\eta} w(x_0)^{-1} x_0^{-1})$$

where $\eta = \det \mathrm{Frob}_p$. (See Proposition 5.2 of (Che08).) In this case the truth of Conjecture 6.1.2 follows immediately from the appearance of f in $S_w^\dagger$.

If f is of type 3a, then $\mathbf{D}_f$ has parameter

$$\delta = (\mu_{p^{1-k}\alpha} x_0^{1-k}, \mu_{\alpha^{-1}\eta} \varepsilon(x_0)^{-1})$$

where $w(x) = x^{k-2}\varepsilon(x)$ with ε of finite order. In particular, Conjecture 6.1.2 predicts the weight $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1(x) = x^{-1}\varepsilon(x)$ and $\lambda_2(x) = x^{k-1}$. By deep work of Coleman, every form f of type 3a has a *companion form* g (Col96), namely a form of type 3b such that $V_f \simeq V_g \otimes \chi_p^{1-k}$ and $\alpha_f = p^{k-1}\alpha_g$. Since $(\lambda_1\lambda_2^{-1})(x) = x^{-k}\varepsilon(x)$ is the weight of g, the image of g under the map β_λ matches the prediction of Conjecture 6.1.2 exactly.

Forms of type 2 have a companion form if and only if $V_f|G_{\mathbf{Q}_p}$ is a direct sum of two characters; otherwise their parameter is the same as for type 1 forms. We omit the details.

6.4 Evidence for three-dimensional Galois representations

In this section we prove Theorem 6.1.4. Notation as in the theorem, we may realize $\mathbf{D}_{\mathrm{sym}^2 f} = \mathbf{D}(\mathrm{sym}^2 V_f|G_{\mathbf{Q}_p})$ as the $\mathcal{R}$-span of symmetric tensors in $\mathbf{D}_f \otimes_{\mathcal{R}} \mathbf{D}_f$. By definition, $\mathbf{D}_f$ has a triangulation

$$0 \to \mathcal{R}(\delta_1) \to \mathbf{D}_f \to \mathcal{R}(\delta_2) \to 0.$$

Let v_1, v_2 be a basis for $\mathbf{D}_f$ with v_1 spanning $\mathcal{R}(\delta_1)$. The filtration

$$0 \subset \mathcal{R}(\delta_1^2) \simeq \mathrm{Span}_{\mathcal{R}}(v_1 \otimes v_1) \subset \mathrm{Span}_{\mathcal{R}}(v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1) \\ \subset \mathrm{Span}_{\mathcal{R}}(v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1, v_2 \otimes v_2) \simeq \mathbf{D}_{\mathrm{sym}^2 f}$$

then exhibits $(\delta_1^2, \delta_1\delta_2, \delta_2^2)$ as an element of $\mathscr{P}\mathrm{ar}(\mathrm{sym}^2 V_f)$.

Since $\mathrm{sym}^2 V_f$ is assumed irreducible, V_f is neither reducible or dihedral up to twist, and f defines a unique point $x_f \in \mathscr{C}_0^{\mathrm{ncm}}(N)$ (with notation as in §5.4). If f is as in the theorem with f of type 1 or 3b, the eigenpacket associated with the point $x_{V,\delta} = i_{\mathrm{GJ}}(x_f) \in \mathscr{X}_{\mathrm{GL}_3/\mathbf{Q}}(N^2)$ satisfies Conjecture 6.1.2. If f is of type 3a with companion form g, we take $x_{V,\delta}$ to be a suitable twist of $i_{\mathrm{GJ}}(x_g)$.

6.5 Evidence for four-dimensional Galois representations

In this section we prove Theorem 6.1.5. Let the notation be as in Theorem 6.1.5; $\mathbf{D}_f = \mathbf{D}(V_f|G_{\mathbf{Q}_p})$ admits a triangulation

$$0 \to \mathcal{R}(\mu_{\alpha_f}) \to \mathbf{D}_f \to \mathcal{R}(\mu_{\alpha_f^{-1}} x_0^{1-k_f}) \to 0$$

and likewise for $\mathbf{D}_g$. Let v_1, v_2 be a basis for $\mathbf{D}_f$ with v_1 generating $\mathcal{R}(\mu_{\alpha_f})$, and let w_1, w_2 be an analogous basis for $\mathbf{D}_g$. We are ready to exhibit the claimed parameters on $\mathbf{D}(V_f \otimes V_g) = \mathbf{D}_f \otimes_{\mathcal{R}} \mathbf{D}_g$. The first set corresponds to the triangulations

$$0 \subset \mathcal{R}(\mu_{\alpha_f}) \otimes \mathcal{R}(\mu_{\alpha_g}) \subset \mathcal{R}(\mu_{\alpha_f}) \otimes \mathbf{D}_g \subset \mathrm{Span}_{\mathcal{R}}\,(v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1) \subset \mathbf{D}_f \otimes \mathbf{D}_g,$$

and the second corresponds to the triangulations

$$0 \subset \mathcal{R}(\mu_{\alpha_f}) \otimes \mathcal{R}(\mu_{\alpha_g}) \subset \mathbf{D}_f \otimes \mathcal{R}(\mu_{\alpha_g}) \subset \mathrm{Span}_{\mathcal{R}}\,(v_1 \otimes w_1, v_2 \otimes w_1, v_1 \otimes w_2) \subset \mathbf{D}_f \otimes \mathbf{D}_g.$$

Having proved Theorem 5.5.1, Theorem 6.1.5 follows quickly: notation as in §5.5, we take $x = i_{\mathrm{RS}}(x_{f,\alpha_f}, x_{g,\alpha_g})$ for δ in the first set of parameters, and $x = i_{\mathrm{RS}}(x_{g,\alpha_g}, x_{f,\alpha_f})$ for δ in the second set of parameters.

Appendix A

Some commutative algebra

In this appendix we collect some results relating the projective dimension of a module M and its localizations, the nonvanishing of certain Tor and Ext groups, and the heights of the associated primes of M. We also briefly recall the definition of a perfect module, and explain their basic properties. These results are presumably well-known to experts, but they are not given in our basic reference (Mat89).

Throughout this subsection, R is a commutative Noetherian ring and M is a finite R-module. Our notations follow (Mat89), with one addition: we write $\mathrm{mSupp}(M)$ for the set of maximal ideals in $\mathrm{Supp}(M)$.

Proposition A.1. *There is an equivalence*

$$\mathrm{projdim}_R(M) \geq n \Leftrightarrow \mathrm{Ext}^n_R(M,N) \neq 0 \,\text{for some}\, N \in \mathrm{Mod}_R.$$

See e.g. p. 280 of (Mat89) for a proof.

Proposition A.2. *The equality*

$$\mathrm{projdim}_R(M) = \sup_{\mathfrak{m}\in \mathrm{mSupp}(M)} \mathrm{projdim}_{R_\mathfrak{m}}(M_\mathfrak{m})$$

holds.

Proof. Any projective resolution of M localizes to a projective resolution of $M_\mathfrak{m}$, so $\mathrm{projdim}_{R_\mathfrak{m}}(M_\mathfrak{m}) \leq \mathrm{projdim}_R(M)$ for all $\mathfrak{m}$. On the other hand, if $\mathrm{projdim}_R(M) \geq n$, then $\mathrm{Ext}^n_R(M,N) \neq 0$ for some N, so $\mathrm{Ext}^n_R(M,N)_\mathfrak{m} \neq 0$ for some $\mathfrak{m}$; but $\mathrm{Ext}^n_R(M,N)_\mathfrak{m} \simeq \mathrm{Ext}^n_{R_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m})$, so $\mathrm{projdim}_{R_\mathfrak{m}}(M_\mathfrak{m}) \geq n$ for some $\mathfrak{m}$ by Proposition A.1. $\square$

Proposition A.3. *For M any finite R-module, the equality*

$$\mathrm{projdim}_R(M) = \mathrm{sup}_{\mathfrak{m}\in \mathrm{mSupp}(M)}\mathrm{sup}\left\{i|\mathrm{Tor}_i^R(M, R/\mathfrak{m}) \neq 0\right\}$$

holds. If furthermore $\mathrm{projdim}_R(M) < \infty$ *then the equality*

$$\mathrm{projdim}_R(M) = \sup\left\{i|\mathrm{Ext}^i_R(M, R) \neq 0\right\}$$

holds as well.

Proof. The module $\mathrm{Tor}_i^R(M, R/\mathfrak{m})$ is a finite-dimensional $R/\mathfrak{m}$-vector space, so localization at $\mathfrak{m}$ leaves it unchanged, yielding

$$\begin{aligned} \mathrm{Tor}_i^R(M, R/\mathfrak{m}) &\simeq \mathrm{Tor}_i^R(M, R/\mathfrak{m})_{\mathfrak{m}} \\ &\simeq \mathrm{Tor}_i^{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}/\mathfrak{m}). \end{aligned}$$

Since the equality $\mathrm{projdim}_S(N) = \sup\left\{i|\mathrm{Tor}_i^S(N, S/\mathfrak{m}_S) \neq 0\right\}$ holds for any local ring S and any finite S-module N (see e.g. Lemma 19.1.ii of (Mat89)), the first claim now follows from Proposition A.2.

For the second claim, we first note that if S is a local ring and N is a finite S-module with $\mathrm{projdim}_S(N) < \infty$, then $\mathrm{projdim}_S(N) = \sup\{i|\mathrm{Ext}^i_S(N, S) \neq 0\}$ by Lemma 19.1.iii of (Mat89). Hence by Proposition A.2 we have

$$\begin{aligned} \mathrm{projdim}_R(M) &= \mathrm{sup}_{\mathfrak{m}\in \mathrm{mSupp}(M)}\mathrm{sup}\left\{i|\mathrm{Ext}^i_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}) \neq 0\right\} \\ &= \sup\{i|\mathrm{Ext}^i_R(M, R)_{\mathfrak{m}} \neq 0 \,\text{for some}\, \mathfrak{m}\} \\ &= \sup\{i|\mathrm{Ext}^i_R(M, R) \neq 0\}, \end{aligned}$$

as desired. □

Proposition A.4. *If R is a Cohen-Macaulay ring, M is a finite R-module of finite projective dimension, and $\mathfrak{p}$ is an associated prime of M, then* $\mathrm{ht}\mathfrak{p} = \mathrm{projdim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. *In particular,* $\mathrm{ht}\mathfrak{p} \leq \mathrm{projdim}_R(M)$.

Proof. Supposing $\mathfrak{p}$ is an associated prime of M, there is an injection $R/\mathfrak{p} \hookrightarrow M$; this localizes to an injection $R_\mathfrak{p}/\mathfrak{p} \hookrightarrow M_\mathfrak{p}$, so $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$. Now we compute

$$\begin{aligned} \mathrm{ht}\mathfrak{p} &= \dim(R_\mathfrak{p}) \\ &= \mathrm{depth}_{R_\mathfrak{p}}(R_\mathfrak{p}) \text{ (by the CM assumption)} \\ &= \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \text{ (by the Auslander} - \text{Buchsbaum formula)} \\ &= \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}), \end{aligned}$$

whence the result. □

Now we single out an especially nice class of modules, which are equidimensional in essentially every sense of the word. Recall the *grade* of a module M, written $\mathrm{grade}_R(M)$, is the $\mathrm{ann}_R(M)$-depth of R; by Theorems 16.6 and 16.7 of (Mat89),

$$\mathrm{grade}_R(M) = \inf\{i | \mathrm{Ext}^i_R(M, R) \neq 0\},$$

so quite generally $\mathrm{grade}_R(M) \leq \mathrm{projdim}_R(M)$.

Definition A.5. *A finite R-module M is* perfect *if* $\mathrm{grade}_R(M) = \mathrm{projdim}_R(M) < \infty$.

Proposition A.6. *Let R be a Noetherian ring, and let M be a perfect R-module, with* $\mathrm{grade}_R(M) = \mathrm{projdim}_R(M) = d$. *Then for any $\mathfrak{p} \in \mathrm{Supp}(M)$ we have* $\mathrm{grade}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = d$. *If furthermore R is Cohen-Macaulay, then M is Cohen-Macaulay as well, and every associated prime of M has height d.*

Proof. The grade of a module can only increase under localization (as evidenced by the Ext definition above), while the projective dimension can only decrease; on the other hand, $\mathrm{grade}_R(M) \leq \mathrm{projdim}_R(M)$ for any finite module over any Noetherian ring. This proves the first claim.

For the second claim, Theorems 16.6 and 17.4.i of (Mat89) combine to yield the formula

$$\dim(M_\mathfrak{p}) + \mathrm{grade}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim(R_\mathfrak{p})$$

for any $\mathfrak{p} \in \mathrm{Supp}(M)$. The Auslander-Buchsbaum formula reads

$$\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(R_\mathfrak{p}).$$

But $\dim(R_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(R_\mathfrak{p})$ by the Cohen-Macaulay assumption, and $\mathrm{grade}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p})$ by the first claim. Hence $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim(M_\mathfrak{p})$ as desired. The assertion regarding associated primes is immediate from the first claim and Proposition A.4. □

Bibliography

[AS08] Avner Ash and Glenn Stevens, *p-adic deformations of arithmetic cohomology*, preprint.

[Bel12] Joël Bellaïche, *Critical p-adic L-functions*, Invent. Math. **189** (2012), no. 1, 1–60. MR 2929082

[Ber11] Laurent Berger, *Trianguline representations*, Bull. Lond. Math. Soc. **43** (2011), no. 4, 619–635. MR 2820149 (2012h:11079)

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261, Springer-Verlag, Berlin, 1984, A systematic approach to rigid analytic geometry. MR 746961 (86b:32031)

[BS73] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault. MR 0387495 (52 #8337)

[BS07] Christophe Breuil and Peter Schneider, *First steps towards p-adic Langlands functoriality*, J. Reine Angew. Math. **610** (2007), 149–180. MR 2359853 (2009f:11147)

[Buz04] Kevin Buzzard, *On p-adic families of automorphic forms*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 23–44. MR 2058640 (2005g:11069)

[Buz07] ———, *Eigenvarieties*, L-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, pp. 59–120. MR 2392353 (2010g:11076)

[BW00] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. MR 1721403 (2000j:22015)

[BZ77] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR 0579172 (58 #28310)

[Che05] Gaëtan Chenevier, *Une correspondance de Jacquet-Langlands p-adique*, Duke Math. J. **126** (2005), no. 1, 161–194. MR 2111512 (2006f:11144)

[Che08] ______, *Quelques courbes de Hecke se plongent dans l'espace de Colmez*, J. Number Theory **128** (2008), no. 8, 2430–2449. MR 2394829 (2010g:11092)

[Clo90] Laurent Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159. MR 1044819 (91k:11042)

[CM98] R. Coleman and B. Mazur, *The eigencurve*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 1–113. MR 1696469 (2000m:11039)

[Col96] Robert F. Coleman, *Classical and overconvergent modular forms*, Invent. Math. **124** (1996), no. 1-3, 215–241. MR 1369416 (97d:11090a)

[Col97] ______, *p-adic Banach spaces and families of modular forms*, Invent. Math. **127** (1997), no. 3, 417–479. MR 1431135 (98b:11047)

[Con99] Brian Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 2, 473–541. MR 1697371 (2001c:14045)

[Con08] ______, *Several approaches to non-Archimedean geometry*, p-adic geometry, Univ. Lecture Ser., vol. 45, Amer. Math. Soc., Providence, RI, 2008, pp. 9–63. MR 2482345 (2011a:14047)

[dJ95] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 82, 5–96 (1996). MR 1383213 (97f:14047)

[Fra98] Jens Franke, *Harmonic analysis in weighted L_2-spaces*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 2, 181–279. MR 1603257 (2000f:11065)

[FS98] Jens Franke and Joachim Schwermer, *A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups*, Math. Ann. **311** (1998), no. 4, 765–790. MR 1637980 (99k:11077)

[Han12a] David Hansen, *Analytic p-adic L-functions for* GL_2, in preparation.

[Han12b] ———, *Universal coeffieicnts for overconvergent cohomology and the geometry of eigenvarieties*, preprint,.

[Hel12] Eugen Hellmann, *Families of trianguline representations and finite slope spaces*, preprint, 2012.

[Hid86] Haruzo Hida, *Galois representations into* $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ *attached to ordinary cusp forms*, Invent. Math. **85** (1986), no. 3, 545–613. MR 848685 (87k:11049)

[IM65] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 5–48. MR 0185016 (32 #2486)

[LS04] Jian-Shu Li and Joachim Schwermer, *On the Eisenstein cohomology of arithmetic groups*, Duke Math. J. **123** (2004), no. 1, 141–169. MR 2060025 (2005h:11108)

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461 (90i:13001)

[Ram00] Dinakar Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* SL(2), Ann. of Math. (2) **152** (2000), no. 1, 45–111. MR 1792292 (2001g:11077)

[Shi94] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994, Reprint of the 1971 original, Kanô Memorial Lectures, 1. MR 1291394 (95e:11048)

[Ste94] Glenn Stevens, *Rigid analytic modular symbols*, unpublished, 1994.

[Ste00] ______, *Families of overconvergent modular symbols*, unpublished notes for a course at Institute Henri Poicare, 2000.

[Taï12] Olivier Taïbi, *Eigenvarieties for classical groups and complex conjugations in galois representations*, preprint, 2012.

[Urb11] Eric Urban, *Eigenvarieties for reductive groups*, Ann. of Math. (2) **174** (2011), no. 3, 1685–1784. MR 2846490

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)