Serre functors for quasicoherent and ind-coherent sheaves

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Abstract

We develop some tools to compute the Serre functor on quasicoherent and ind-coherent sheaves on fairly general algebraic stacks. This gives new conceptual proofs of some results of Beraldo.

Contents

1 Introduction

Fix a field k, with $\mathcal{D}(k)$ the derived ∞-category of k-vector spaces. Let $\mathcal{C}$ be a compactly generated k-linear presentable stable ∞-category, so $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^{\omega})$. For brevity we call any such $\mathcal{C}$ a *good* k-linear category. For any $A, B \in \mathcal{C}$, we write $R\mathrm{Hom}(A, B) \in \mathcal{D}(k)$ for the usual k-linear mapping spectrum.

Any good k-linear category admits a canonical functor $S = S^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, the *Serre functor* of $\mathcal{C}$, characterized by preservation of colimits together with the requirement that

$$R\mathrm{Hom}(A, S(B)) = R\mathrm{Hom}(B, A)^*$$

for all compact objects B and all objects A. Here $(-)^*$ denotes the naive dual in $\mathcal{D}(k)$. Most classically, if X is a projective variety over k and $\mathcal{C} = \mathrm{QCoh}(X)$ is the (unbounded) quasicoherent derived category, then $S(-) = - \otimes \omega_X$ is just the functor of tensor product with the normalized dualizing complex. However, the Serre functor exists much more generally, and is "a very useful piece of abstract nonsense" in the memorable phrasing of Beraldo [Ber21b].

In [Ber21b], Beraldo computed Serre functors in several situations, with somewhat surprising and varied outcomes which we now recall. For these two examples we assume k is algebraically closed of characteristic zero, and we fix a split reductive group G/k.

1. Let $X = \mathcal{N}/G$ be the equivariant nilpotent cone of G, and let $j : X^\circ \to X$ be the complement of the zero orbit. Then Beraldo computed the Serre functor of $\mathrm{QCoh}(X)$ as

$$S(-) = - \otimes \mathrm{cone}(\mathcal{O}_X \to j_*\mathcal{O}_{X^\circ})[\dim\mathcal{N} - 1].$$

   Note that the auxiliary sheaf $\mathrm{cone}(\mathcal{O}_X \to j_*\mathcal{O}_{X^\circ})[\dim\mathcal{N} - 1]$ appearing here is not coherent, but it is still extremely nice: it is concentrated in degree zero for the standard t-structure, it is set-theoretically supported at the zero orbit, and it has finite tor-dimension.

2. Set $X = (0 \times^{\mathbf{R}}_{\mathfrak{g}} 0)/G$ and let $\mathcal{C} = \mathrm{IndCoh}_{\mathcal{N}/G}(X)$. This is exactly the category appearing on the spectral side of the derived Satake equivalence. Beraldo proved that the Serre functor of this category is $\Xi_{0\to\mathcal{N}/G}\Psi_{\mathcal{N}/G\to 0}[-\dim G]$, where

$$\Xi_{0\to\mathcal{N}/G} : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}_{\mathcal{N}/G}(X) : \Psi_{\mathcal{N}/G\to 0}$$

   are the usual adjoint pair of functors recalled below.

In this note we develop some general methods for computing Serre functors on quasicoherent and ind-coherent sheaves. As a byproduct, we give new conceptual proofs of Beraldo's results, without any explicit computations.

Fix a field k of characteristic zero. All our results will be in the setting of QCA stacks locally almost of finite type over k. In this generality, Drinfeld-Gaitsgory proved that the category $\mathrm{IndCoh}(X)$ is compactly generated by $\mathrm{Coh}(X)$, and that $\mathrm{QCoh}(X)$ is dualizable [DG13]. Moreover, the dualizable and compact objects in $\mathrm{QCoh}(X)$ agree, and are given by the usual perfect complexes $\mathrm{Perf}(X)$. If $\mathrm{QCoh}(X)$ is compactly generated, we say X is *perfect*. This is a mild condition, and holds e.g. for all quotient stacks Y/G where Y is a quasiprojective k-scheme and G is a linear algebraic group [BZFN10].

In general, there is a natural quotient functor $\Psi : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$, which admits a fully faithful left adjoint Ξ if X is eventually coconnective. If X is quasismooth and $V \subset \mathrm{Sing}(X)$ is a closed conical subset, we have a more general pair of adjoint functors

$$\Xi_{0\to V} : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}_V(X) : \Psi_{V\to 0}$$

which recover the above functors when $V = \mathrm{Sing}(X)$.

Our main result is the following theorem.

Theorem 1.1. *Let X be a perfect QCA stack.*

i. *The Serre functor of $\mathrm{QCoh}(X)$ is $\mathcal{F} \mapsto \mathcal{F} \otimes \delta_X$, where $\delta_X \in \mathrm{QCoh}(X)$ represents the contravariant functor*

$$\begin{aligned} \mathrm{QCoh}(X) &\to \mathcal{D}(k) \\ \mathcal{G} &\mapsto R\Gamma(X, \mathcal{G})^*. \end{aligned}$$

ii. *Suppose moreover that X is Gorenstein. Then the Serre functor of $\mathrm{IndCoh}(X)$ is $\mathcal{F} \mapsto \Xi(\Psi(\mathcal{F}) \otimes \delta_X)$, with $\delta_X \in \mathrm{QCoh}(X)$ as in i.*

iii. *Suppose moreover that X is quasismooth, and $V \subset \mathrm{Sing}(X)$ is a closed conical subset such that $\mathrm{IndCoh}_V(X)$ is compactly generated. Then the Serre functor of $\mathrm{IndCoh}_V(X)$ is $\mathcal{F} \mapsto \Xi_{0\to V}(\Psi_{V\to 0}(\mathcal{F}) \otimes \delta_X)$.*

Here i. is an easy piece of formal nonsense, and in fact a version of i. holds for any *locally rigid* symmetric monoidal good k-linear category. Moreover, iii. is an easy formal consequence of ii. On the other hand, ii. is not formal at all, and we will comment on its proof below.

This theorem reduces the problem of understanding the Serre functor on quasicoherent or ind-coherent sheaves to understanding the sheaf $\delta_X \in \mathrm{QCoh}(X)$. In simple cases, this can be computed by hand. For instance, if $X = \mathrm{Spec}A$ is a (derived) affine scheme, then δ_X is (the quasicoherent sheaf associated with) the naive dual $R\mathrm{Hom}_k(A, k)$. In conjunction with Theorem 1.1.iii, this immediately recovers Beraldo's second result, noting that $\tilde{X} = 0 \times^{\mathbf{R}}_{\mathfrak{g}} 0$ is affine with $R\mathrm{Hom}_k(\mathcal{O}(\tilde{X}), k) = \mathcal{O}(\tilde{X})[-\mathrm{dim}G]$.

Moreover, for certain quotient stacks we can give a very clean description of δ_X. To explain this, assume k is algebraically closed, and let $X = (\mathrm{Spec}A)/G$ be a quotient stack where A is a classical finite type k-algebra and G is linearly reductive. It is not difficult to prove that the following conditions are equivalent:

i. A^G is an Artinian local k-algebra.

ii. G has a unique closed orbit on $\mathrm{Spec}A$.

iii. There is a presentation $X \cong (\mathrm{Spec}B)/H$ where H is linearly reductive and has a unique closed orbit on $\mathrm{Spec}B$ which is moreover a single point x.

The equivalence of i. and ii. follows from the theory of good moduli spaces [Alp13], and the equivalence of ii. and iii. can be deduced from Luna's etale slice theorem. We call a presentation as in iii. a good presentation.

Theorem 1.2. *Let $X = (\mathrm{Spec}A)/G$ be a good presentation of a quotient stack as above. Let $x \in \mathrm{Spec}A$ be the singleton closed orbit, with $\mathfrak{m} \subset A$ the associated maximal ideal. Let $E = \mathrm{colim}_n \mathrm{Hom}_k(A/\mathfrak{m}^n, k)$ be the injective hull of the residue field $A/\mathfrak{m} \cong k$, with its natural structure as a G-equivariant A-module. Then $\delta_X \cong \widetilde{E}$ is the associated quasicoherent sheaf on X.*

When X is the equivariant nilpotent cone, it is a simple exercise to identify $\tilde{E}$ with the sheaf $\mathrm{cone}(\mathcal{O}_X \to j_*\mathcal{O}_{X^\circ})[\mathrm{dim}\mathcal{N} - 1]$ exhibited by Beraldo. As such, this immediately recovers Beraldo's first result.

Let us mention some ingredients in the proof of Theorem 1.1.ii. One key idea is that if $\mathcal{C}$ is a good category which is equipped with a self-duality $\mathbf{D}_{\mathrm{can}} : \mathcal{C}^\omega \xrightarrow{\sim} \mathcal{C}^\omega$ on its compact objects, we can extract a second duality functor $\mathbf{D}_{\mathrm{ex}} : \mathcal{C} \to \mathcal{C}$ characterized by continuity together with the requirement that $S = \mathbf{D}_{\mathrm{ex}} \circ \mathbf{D}_{\mathrm{can}}$ on compact objects. Here "ex" stands for exotic, extra, exciting. In the particular cases of IndCoh and QCoh we give this functor its own notation:

- Let X be a QCA stack. For $\mathrm{IndCoh}(X)$ equipped with the usual Grothendieck-Serre self-duality $\mathbf{D}_{\mathrm{GS}}$ on $\mathrm{Coh}(X)$, we write $\mathbf{D}_{\mathrm{adm}}$ for the resulting exotic duality functor. This is the functor of *admissible duality* defined and studied in [HM26].
- Let X be a perfect QCA stack. For $\mathrm{QCoh}(X)$ equipped with the usual naive self-duality $\mathbf{D}_{\mathrm{nv}}$ on $\mathrm{Perf}(X)$, we write $\mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}$ for the resulting exotic duality functor.

We will need some highly non-obvious facts about these two exotic dualities. The first ingredient is the following surprising theorem, which was already proved in [HM26].

Theorem 1.3. *Let X be a Gorenstein QCA stack. Then for any $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathbf{D}_{\mathrm{adm}}\mathcal{F}$ is in the essential image of the functor Ξ.*

The other ingredient is the following theorem, which is new here but whose proof crucially uses some results in [HM26].

Theorem 1.4. *Let X be a perfect QCA stack which is eventually coconnective. Then $\mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}$ is t-bounded: there are integers a, b such that $\mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}\mathcal{F} \in \mathrm{QCoh}^{[a,b]}(X)$ for all $\mathcal{F} \in \mathrm{QCoh}^{\heartsuit}(X)$.*

With these results in hand, Theorem 1.1.ii follows from some additional arguments exploiting the t-structure on IndCoh.

As a corollary of Theorem 1.1.ii, we get the following formula for the admissible duality functor.

Corollary 1.5. *Let X be a QCA stack which is perfect and Gorenstein. Then for any $\mathcal{F} \in \mathrm{Coh}(X)$, we have $\mathbf{D}_{\mathrm{adm}}\mathcal{F} = \Xi(\mathbf{D}_{\mathrm{nv}}\Psi\mathcal{F} \otimes \delta_X)$.*

We end with a brief comment on our motivations. Our main interest is not primarily in the Serre functor, but in the exotic duality $\mathbf{D}_{\mathrm{adm}}$ on IndCoh, as this duality plays a fundamental role in the categorical local Langlands correspondence. More precisely, if $G/\mathbf{Q}_p$ is a quasisplit group and (B, ψ) is a Whittaker datum for G, we expect [HM26] a canonical equivalence of categories

$$\mathbf{L}_\psi : D(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Par}_G)$$

such that $c^*\mathbf{D}_{\mathrm{adm}}\mathbf{L}_\psi \simeq \mathbf{L}_{\psi^{-1}}\mathbf{D}_{\mathrm{Verd}}$, where c is the Chevalley involution. In other words, the categorical local Langlands equivalence intertwines Verdier duality on the automorphic side with (Chevalley-twisted) admissible duality on the spectral side. This motivated us to develop some tools for computing admissible duals. The results of this paper will be used in the forthcoming NUS PhD thesis of Jhan-Cyuan Syu.

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2 Proofs

A covariant functor is *continuous* if it preserves colimits. Since I don't like opposite categories, I will say a contravariant functor F is continuous if $F(\mathrm{colim}_i A_i) = \lim_i F(A_i)$ for all colimit diagrams.

2.1 General formalism

In this section we establish some basic formalism. Some of these ideas were independently observed by Zhu [Zhu25], and we also follow his use of the word "admissible" below.

Fix a field k, with $\mathcal{D}(k)$ the derived ∞-category of k-vector spaces. Let $\mathcal{C}$ be a compactly generated k-linear presentable stable ∞-category, so $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^\omega)$. As in the introduction, we call such $\mathcal{C}$ *good*. By a standard result, $\mathcal{C}$ has all (small) limits. Any such $\mathcal{C}$ admits a unique *Serre functor* $S = S^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, characterized by continuity together with the requirement that

$$R\mathrm{Hom}(A, S(B)) = R\mathrm{Hom}(B, A)^*$$

for all compact objects B and all objects A, where $(-)^*$ denotes the naive dual in $\mathcal{D}(k)$. Indeed, for fixed compact B, the right-hand side transforms colimits in A into limits in $\mathcal{D}(k)$, and therefore is representable. The representing object is $S(B)$ by definition. In general, S is conservative on compact objects, and fully faithful on compact objects if $\mathcal{C}$ is *proper*, i.e. if $R\mathrm{Hom}(A, B) \in \mathrm{Perf}(k)$

for all $A, B \in \mathcal{C}^\omega$. In the literature S is often defined only for proper $\mathcal{C}$, but this restriction is unnecessary.

In this level of generality, there is not much more to say. However, it is often the case that $\mathcal{C}$ comes with a canonical self-duality on compact objects, i.e. a contravariant self-equivalence $\mathbf{D}_{\text{can}} : \mathcal{C}^\omega \xrightarrow{\sim} \mathcal{C}^\omega$ such that $\mathbf{D}_{\text{can}}^2 \cong \text{id}$ ("biduality"). We can and do extend $\mathbf{D}_{\text{can}}$ to a continuous contravariant endofunctor $\mathbf{D}_{\text{can}} : \mathcal{C} \to \mathcal{C}$, i.e. we declare that the value of $\mathbf{D}_{\text{can}}$ on any ind-system $\text{colim}_i A_i$ of compact objects is $\lim_i \mathbf{D}_{\text{can}} A_i$; of course, this extended functor typically won't be an equivalence.

Given such a pair $(\mathcal{C}, \mathbf{D}_{\text{can}})$, we can extract a *second* contravariant endofunctor $\mathbf{D}_{\text{ex}} : \mathcal{C} \to \mathcal{C}$, uniquely characterized by continuity together with the requirement that $\mathbf{D}_{\text{ex}}(B) = (S \circ \mathbf{D}_{\text{can}})(B)$ for compact B. Here "ex" stands for exotic, exceptional, extra, exciting. Then it is easy to see that $S = \mathbf{D}_{\text{ex}} \circ \mathbf{D}_{\text{can}}$ on compact objects, using biduality for $\mathbf{D}_{\text{can}}$.

The functor $\mathbf{D}_{\text{ex}}$ has its own good properties, and induces a duality on a distinguished subcategory of $\mathcal{C}$. Say an object $A \in \mathcal{C}$ is *admissible* if $R\text{Hom}(B, A) \in \text{Perf}(k)$ for all compact B. These form a subcategory $\mathcal{C}^{\text{adm}}$, stable under finite (co)limits. Note that $\mathcal{C}$ is proper iff $\mathcal{C}^\omega \subset \mathcal{C}^{\text{adm}}$.

Proposition 2.1. *Let $(\mathcal{C}, \mathbf{D}_{\text{can}})$ be as above.*

1. *For compact A and arbitrary B, the formula*

$$R\text{Hom}(A, \mathbf{D}_{\text{ex}} B) = R\text{Hom}(\mathbf{D}_{\text{can}} A, B)^*$$

holds.

2. *The functor $\mathbf{D}_{\text{ex}}$ is conservative on $\mathcal{C}$.*

3. *An object $A \in \mathcal{C}$ is admissible if and only if $\mathbf{D}_{\text{ex}} A$ is admissible, and $\mathbf{D}_{\text{ex}}$ restricts to a contravariant self-equivalence $\mathcal{C}^{\text{adm}} \xrightarrow{\sim} \mathcal{C}^{\text{adm}}$ satisfying biduality.*

4. *For all $A, B \in \mathcal{C}$ we have*

$$R\text{Hom}(A, \mathbf{D}_{\text{ex}} B) = R\text{Hom}(B, \mathbf{D}_{\text{ex}} A).$$

The formula in 1. is extremely useful. We call it the *duality exchange* formula. Note that this formula can also be taken as the definition of $\mathbf{D}_{\text{ex}}$, using the adjoint functor theorem again. Note also that 4. holds for arbitrary A, B.

Proof. 1. Both sides convert colimits in B into limits, so we can assume B is compact. Then

$$\begin{aligned} R\text{Hom}(A, \mathbf{D}_{\text{ex}} B) &= R\text{Hom}(A, S(\mathbf{D}_{\text{can}} B)) \\ &= R\text{Hom}(\mathbf{D}_{\text{can}} B, A)^* \\ &= R\text{Hom}(\mathbf{D}_{\text{can}} A, B)^* \end{aligned}$$

where in the first line we used that $\mathbf{D}_{\text{ex}} = S \circ \mathbf{D}_{\text{can}}$ on compacts, in the second line we used the defining equation of the Serre functor, and in the the third line we used the duality and involutivity of $\mathbf{D}_{\text{can}}$ on compacts.

2. Immediate from 1., using the involutivity of $\mathbf{D}_{\text{can}}$ on compacts together with the conservativity of $(-)^*$ on $\mathcal{D}(k)$.

3. If $\mathbf{D}_{\text{ex}} A$ is admissible, then $R\text{Hom}(B, \mathbf{D}_{\text{ex}} A) = R\text{Hom}(\mathbf{D}_{\text{can}} B, A)^*$ is perfect for all compact B, so $R\text{Hom}(\mathbf{D}_{\text{can}} B, A)$ is already perfect. Therefore A is admissible by involutivity of $\mathbf{D}_{\text{can}}$ on compacts. The other claims are also easy consequences of the duality exchange formula.

4. Both sides define continuous contravariant functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}(k)$, so we can assume A, B are both compact. Then

$$\begin{aligned} R\mathrm{Hom}(A, \mathbf{D}_{\mathrm{ex}} B) &= R\mathrm{Hom}(\mathbf{D}_{\mathrm{can}} A, B)^* \\ &= R\mathrm{Hom}(\mathbf{D}_{\mathrm{can}} B, A)^* \\ &= R\mathrm{Hom}(B, \mathbf{D}_{\mathrm{ex}} A) \end{aligned}$$

where in the first and third lines we used duality exchange, and in the second line we used the duality and involutivity of $\mathbf{D}_{\mathrm{can}}$ on compacts. □

So much for general formalism. The fun is in the examples.

Exercise 2.2. i. Check that $R\mathrm{Hom}(S(A), S(B)) = R\mathrm{Hom}(A, B)$ if B is compact and admissible.

ii. Check that if $\mathcal{C}, \mathcal{D}$ are categories with Serre functors and $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and continuous with continuous right adjoint F^R, then $S^{\mathcal{C}} = F^R \circ S^{\mathcal{D}} \circ F$.

2.2 The locally rigid case

Keep the setup as in the previous section. Suppose now that $\mathcal{C}$ is also presentably symmetric monoidal and *locally rigid* [Ram24]. Then compact objects of $\mathcal{C}$ are dualizable with compact duals, so we indeed have a canonical duality $\mathbf{D}_{\mathrm{can}}$ in the picture: on compact objects it is the functor sending A to $A^\vee$. Therefore the above paradigm applies, and we get an exotic duality functor $\mathbf{D}_{\mathrm{ex}}$.

Proposition 2.3. *In this situation, $S(A \otimes B) = A \otimes S(B)$ for all $A, B \in \mathcal{C}$. In particular, $S(A) = A \otimes S(\mathbf{1}_{\mathcal{C}})$.*

Proof. Both sides commute with colimits in A and B, so we can assume A and B are compact, whence also $A \otimes B$ is compact. Then for any compact C, we compute

$$\begin{aligned} R\mathrm{Hom}(C, S(A \otimes B)) &= R\mathrm{Hom}(A \otimes B, C)^* \\ &= R\mathrm{Hom}(B, A^\vee \otimes C)^* \\ &= R\mathrm{Hom}(A^\vee \otimes C, S(B)) \\ &= R\mathrm{Hom}(C, A \otimes S(B)). \end{aligned}$$

Here we used the defining equation of the Serre functor twice, the dualiziblity of compact objects, the preservation of compactness under tensor product, and the fact that $A \otimes -$ is both left and right adjoint to $A^\vee \otimes -$. □

If $\mathbf{1}_{\mathcal{C}}$ is compact, i.e. if $\mathcal{C}$ is *rigid* [GR17], then $S(\mathbf{1}_{\mathcal{C}}) = \mathbf{D}_{\mathrm{ex}} \mathbf{1}_{\mathcal{C}}$ represents the functor $A \mapsto R\mathrm{Hom}(\mathbf{1}_{\mathcal{C}}, A)^*$. In the general locally rigid setting, $S(\mathbf{1}_{\mathcal{C}})$ represents the unique continuous functor extension of the functor

$$\begin{aligned} \mathcal{C}^\omega &\to \mathcal{D}(k) \\ A &\mapsto R\mathrm{Hom}(\mathbf{1}_{\mathcal{C}}, A)^* \end{aligned}$$

from compact objects.

Precomposing with $\mathbf{D}_{\mathrm{can}}$, we get that

$$\mathbf{D}_{\mathrm{ex}}(-) = \mathbf{D}_{\mathrm{can}}(-) \otimes S(\mathbf{1}_{\mathcal{C}})$$

on compact objects. By general nonsense, this induces a canonical map

$$\mathbf{D}_{\mathrm{can}}(-) \otimes S(\mathbf{1}_{\mathcal{C}}) \to \mathbf{D}_{\mathrm{ex}}(-)$$

on all objects, which is typically not an isomorphism. In a few exceptional situations discussed below, this map is an isomorphism on (some!) non-compact objects.

2.3 Quasicoherent sheaves

Let X be a QCA (derived) algebraic stack over $\mathrm{Spec} k$.[1] Assume that $\mathrm{char} k = 0$, or that X is eventually coconnective and all stabilizers at geometric points are linearly reductive. By Drinfeld-Gaitsgory in the first case, and a folklore argument in the second case, the compact objects in $\mathrm{QCoh}(X)$ are exactly the perfect complexes. If $\mathrm{QCoh}(X)$ is compactly generated, i.e. X has enough perfect complexes, i.e. X is *perfect*, then $\mathrm{QCoh}(X)$ falls into the paradigm above, with compact objects $\mathrm{Perf}(X)$ and monoidal unit $\mathcal{O}_X$. Here of course $\mathbf{D}_{\mathrm{can}}$ is just the internal hom towards $\mathcal{O}_X$. But $\mathbf{D}_{\mathrm{ex}}$ is much weirder, and its behavior varies a lot depending on the specific nature of X.

Let us look at some examples.

Example 2.4. Let X be a smooth projective k-scheme. Then $S(\mathcal{O}_X) = \omega_X = \Omega_X^{\dim X}[\dim X]$ is the normalized dualizing sheaf, $S(A) = A \otimes \omega_X$ for all A, and

$$\mathbf{D}_{\mathrm{ex}} A = \mathbf{D}_{\mathrm{can}} A \otimes \omega_X$$

for *all* $A \in \mathrm{QCoh}(X)$. This is the most classical situation in which "Serre functors" were originally studied. In this situation, $\mathbf{D}_{\mathrm{ex}}$ isn't anything new: it is just normalized Grothendieck-Serre duality, i.e. internal hom towards ω_X.

Now let X be any projective k-scheme. Then $S(\mathcal{O}_X) = \omega_X$ is still the normalized dualizing complex, and $S(A) = A \otimes \omega_X$ for all A, but without further conditions the equation

$$\mathbf{D}_{\mathrm{ex}} A = \mathbf{D}_{\mathrm{can}} A \otimes \omega_X$$

is only valid for $A \in \mathrm{Perf}(X)$: for instance, if X is not Gorenstein, then ω_X is not a compact object in QCoh, and the right-hand expression need not be a continuous functor of A.

Exercise 2.5. For X a projective k-scheme, check that the following conditions are equivalent.

i. X is Gorenstein.
ii. ω_X has finite tor-amplitude.
iii. ω_X is a perfect complex.
iv. The equation $\mathbf{D}_{\mathrm{ex}} A = \mathbf{D}_{\mathrm{can}} A \otimes \omega_X$ remains valid for all $A \in \mathrm{Coh}(X)$.

As another example, let $X = \mathrm{Spec} R$ an affine k-scheme. For M an R-module with associated sheaf $\widetilde{M} \in \mathrm{QCoh}(X)$, $\mathbf{D}_{\mathrm{ex}}(\widetilde{M})$ is the quasicoherent sheaf associated with the R-module $\mathrm{Hom}_k(M, k)$. Note that we are really taking k-linear maps, so this is a pretty disgusting module unless M is very small. For affine schemes, there's not much more to say.

Now we turn to stacks, where things are much more interesting. We begin by recording some stability properties for exotic duals and admissible quasicoherent sheaves.

[1] For us, QCA means "quasicompact with affine diagonal". This is slightly more restrictive than the meaning in Drinfeld-Gaitsgory, so all their theorems apply.

Proposition 2.6. *Let $f : X \to Y$ be any map of perfect QCA stacks.*

i. *The pushfoward f_* preserves $\mathrm{QCoh}^{\mathrm{adm}}$, and $\mathbf{D}_{\mathrm{ex}} f_* = f_* \mathbf{D}_{\mathrm{ex}}$ on all sheaves.*

ii. *If f is schematic, proper and quasismooth, f^* preserves $\mathrm{QCoh}^{\mathrm{adm}}$ and $\mathbf{D}_{\mathrm{ex}}$ commutes with f^* up to an explicit invertible twist.*

Proof. Since f^* preserves perfect complexes, preservation of admissibility under f_* is formal. Moreover, for any $\mathcal{F} \in \mathrm{QCoh}(X)$ and $\mathcal{G} \in \mathrm{Perf}(Y)$, we compute

$$\begin{aligned} R\mathrm{Hom}(\mathcal{G}, \mathbf{D}_{\mathrm{ex}} f_* \mathcal{F}) &= R\mathrm{Hom}(\mathbf{D}_{\mathrm{can}} \mathcal{G}, f_* \mathcal{F})^* \\ &= R\mathrm{Hom}(f^* \mathbf{D}_{\mathrm{can}} \mathcal{G}, \mathcal{F})^* \\ &= R\mathrm{Hom}(\mathbf{D}_{\mathrm{can}} f^* \mathcal{G}, \mathcal{F})^* \\ &= R\mathrm{Hom}(f^* \mathcal{G}, \mathbf{D}_{\mathrm{ex}} \mathcal{F}) \\ &= R\mathrm{Hom}(\mathcal{G}, f_* \mathbf{D}_{\mathrm{ex}} \mathcal{F}) \end{aligned}$$

where we used the duality exchange formula in the first and fourth lines, along with basic adjunctions and the commutation of f^* with $\mathbf{D}_{\mathrm{can}}$. Since $\mathcal{G}$ is arbitrary, this gives i. by Yoneda. For ii. one argues similarly, noting that under the stated conditions on f, f_* preserves perfect complexes, commutes up to twist with canonical duality, and is left adjoint to an invertible twist of f^*. □

With this in hand, we can prove a very general t-boundedness result for $\mathbf{D}_{\mathrm{ex}}$. Note that the analogue of this result *fails* for $\mathbf{D}_{\mathrm{can}}$ in general.

Theorem 2.7. *Let X be a perfect QCA stack which is eventually coconnective. Then $\mathbf{D}_{\mathrm{ex}}$ is t-bounded: there are integers a, b such that $\mathbf{D}_{\mathrm{ex}} \mathcal{F} \in \mathrm{QCoh}^{[a,b]}(X)$ for all $\mathcal{F} \in \mathrm{QCoh}^{\heartsuit}(X)$.*

Proof. Let $f : U \to X$ be a smooth cover by an eventually coconnective derived affine scheme, and let $f_n : U_n \to X$ be the Cech nerve of f. Note that all U_n are affine and eventually coconnective. By a trivial computation, the theorem holds with any U_n in place of X.

Now, the sheaf $f_* \mathcal{O}_U$ is a descendable algebra object in $\mathrm{QCoh}(X)$ by [HM26, Proposition 2.7.8]. In particular, $\mathcal{O}_X$ can be obtained from the objects $(f_* \mathcal{O}_U)^{\otimes n} \cong f_{n*} \mathcal{O}_{U_n}$ by finitely many shifts, cones and retracts. Tensoring with any $\mathcal{F}$ and using a trivial projection formula, this shows that $\mathcal{F} \in \mathrm{QCoh}(X)$ can be obtained from $f_{n*} f_n^* \mathcal{F}$ by finitely many shifts, cones and retracts in a manner which is independent of $\mathcal{F}$. Dualizing, we see likewise that $\mathbf{D}_{\mathrm{ex}} \mathcal{F}$ can be obtained from finitely many of $\mathbf{D}_{\mathrm{ex}} f_{n*} f_n^* \mathcal{F}$ in a manner which is independent of $\mathcal{F}$. Now $\mathbf{D}_{\mathrm{ex}} f_{n*} f_n^* \mathcal{F} = f_{n*} \mathbf{D}_{\mathrm{ex}} f_n^* \mathcal{F}$ by Proposition 2.6.i, so to conclude it suffices to see that the functor $f_{n*} \mathbf{D}_{\mathrm{ex}} f_n^* -$ is t-bounded for every n. We then observe the following:

i. f_n^* is t-exact since f_n is smooth;

ii. $\mathbf{D}_{\mathrm{ex}}$ is t-bounded on $\mathrm{QCoh}(U_n)$ since U_n is affine and eventually coconnective;

iii. f_{n*} is t-bounded by [DG13, Corollary 1.4.5].

Putting these results together, we see that $f_{n*} \mathbf{D}_{\mathrm{ex}} f_n^* -$ is t-bounded for every n, as desired. □

For most applications, it suffices to consider quotient stacks of the form $X = (\mathrm{Spec} A)/G$ where A is a finite type classical k-algebra and G is linearly reductive. We restrict ourselves to this case for the moment. It is well-known that for such stacks, $\mathrm{QCoh}(X)$ is compactly generated, with an explicit set of compact generators given by vector bundles of the form $A \otimes V$ as V varies over irreducible representations of G. In this case we get a much stronger form of t-boundedness.

Theorem 2.8. *For a quotient stack $X = (\mathrm{Spec} A)/G$ as above, $\mathbf{D}_{\mathrm{ex}}$ is t-anti-exact for the standard t-structure on* QCoh. *More precisely, if $A \in \mathrm{QCoh}^{[a,b]}(X)$ then $\mathbf{D}_{\mathrm{ex}} A \in \mathrm{QCoh}^{[-b,-a]}(X)$. Moreover, $\mathbf{D}_{\mathrm{ex}} \mathcal{O}_X$ is an injective object in $\mathrm{QCoh}(X)^{\heartsuit}$.*

Proof. Consider the natural map $\pi : \mathrm{Spec}(A)/G \to BG$. For any irreducible representation V of G, let π^*V be the associated vector bundle on X. Then the collection of functors

$$\begin{aligned} \mathrm{QCoh}(X) &\to \mathcal{D}(k) \\ \mathcal{F} &\mapsto R\Gamma(X, \mathcal{F} \otimes \pi^* V) \end{aligned}$$

is a *t-exact* conservative family as V varies over $\mathrm{Irr}(G)$. On the other hand, it is clear from the definitions that

$$R\Gamma(X, \mathbf{D}_{\mathrm{ex}}\mathcal{F} \otimes \pi^* V) = R\Gamma(X, \mathcal{F} \otimes \pi^* V^\vee)^*$$

for all V, which easily gives the result since $(-)^* : \mathcal{D}(k) \to \mathcal{D}(k)$ is t-anti-exact.

The final injectivity claim is clear, since the argument so far shows that

$$\mathrm{Hom}(-, \mathbf{D}_{\mathrm{ex}}\mathcal{O}_X) = H^0(R\Gamma(X, -))^*$$

is an exact functor on $\mathrm{QCoh}(X)^\heartsuit$. □

Proposition 2.9. *Suppose* $X = (\mathrm{Spec}A)/G$ *is as above, and that* A^G *is an Artinian* k*-algebra. Then* $\mathrm{QCoh}(X)$ *is proper, and every irreducible* G*-representation appears in the coordinate ring* A *with finite multiplicity.*

One can check that A^G is Artinian iff G has finitely many closed orbits on $\mathrm{Spec}A$ after base change to an algebraic closure of k, using the fact that $X \to \mathrm{Spec}A^G$ is a good moduli space.

Proof. To see that $\mathrm{QCoh}(X)$ is proper, it suffices to show that $R\Gamma(X, \mathcal{F}) \in \mathrm{Perf}(k)$ for all $\mathcal{F} \in \mathrm{Perf}(X)$. Let $f : X \to \mathrm{Spec}\, A^G$ be the GIT quotient map. According to Alper, this is a good moduli space, so f_* is exact and preserves coherence. Then for any $\mathcal{F} \in \mathrm{Perf}(X)$ we may write

$$R\Gamma(X, \mathcal{F}) = R\Gamma(\mathrm{Spec}A^G, f_*\mathcal{F}),$$

and by result recalled in the previous sentence $f_*\mathcal{F}$ is a bounded complex whose cohomologies are finitely generated A^G-modules. It therefore suffices to see that A^G has finite length as a k-vector space. For this, note that A^G is Artinian by assumption, and moreover it is a finitely generated k-algebra (by the linear reductivity assumption), so it necessarily has finite length as a k-vector space.

For the final claim, let $\pi : X \to BG$ be the canonical map. For V any irreducible G-representation, it is clear by unwinding definitions that the multiplicity of V in A is given by

$$\dim_k \mathrm{Hom}_G(V, A) = \dim_k (A \otimes V^*)^G = \dim_k H^0(X, \pi^* V^*),$$

and this H^0 is a finite-dimensional k-vector space by the first part of the proof. □

Under some further conditions, we can describe $S(\mathcal{O}_X) = \mathbf{D}_{\mathrm{ex}}\mathcal{O}_X$ explicitly for such quotient stacks. For simplicity we restrict to k algebraically closed and consider quotient stacks $X = (\mathrm{Spec}A)/G$ as above satisfying any of the following equivalent conditions:

i. A^G is an Artinian local k-algebra.

ii. G has a unique closed orbit on $\mathrm{Spec}(A)$.

iii. There is a presentation $X \cong (\mathrm{Spec}B)/H$ where H is linearly reductive and has a unique closed orbit on $\mathrm{Spec}(B)$ which is moreover a single point x.

The equivalence of i. and ii. follows from the theory of good moduli spaces, and the equivalence of ii. and iii. is a consequence of Luna's etale slice theorem. We call a choice of presentation as in iii. a good presentation.

Theorem 2.10. *Let $X = (\mathrm{Spec}A)/G$ be a good presentation of a quotient stack as above. Let $x \in \mathrm{Spec}A$ be the singleton closed orbit, with $\mathfrak{m} \subset A$ the associated maximal ideal. Let $E = \mathrm{colim}_n \mathrm{Hom}_k(A/\mathfrak{m}^n, k)$ be the injective hull of the residue field $A/\mathfrak{m}$, with its natural structure as a G-equivariant A-module. Then $S(\mathcal{O}_X) = \widetilde{E}$ is the associated quasicoherent sheaf.*

When A is Gorenstein at $\mathfrak{m}$, we can rewrite this in a really pleasant way: letting $j : X^\circ \to X$ denote the complement of the unique closed orbit, $S(\mathcal{O}_X)$ is the unique nonzero cohomology sheaf of the complex $\mathrm{cone}(\mathcal{O}_X \to j_*\mathcal{O}_{X^\circ})$, and in fact

$$S(\mathcal{O}_X) = \mathrm{cone}(\mathcal{O}_X \to j_*\mathcal{O}_{X^\circ})[\dim A_{\mathfrak{m}} - 1].$$

In the particular case where k is of characteristic zero, G is split reductive, and $X = \mathcal{N}/G$ is the equivariant nilpotent cone, this formula for $S(\mathcal{O}_X)$ is due to Beraldo [Ber21b, Proposition 4.1.2], who proved it by a direct computation. Our argument below recovers his result without any explicit calculations at all.[2]

In the proof, we will need the following standard form of Matlis duality.

Lemma 2.11. *Let $(A, \mathfrak{m}, k_A)$ be a Noetherian local ring containing a field k such that the residue field k_A is of finite degree over k. Then the A-module $E := \mathrm{colim}_n \mathrm{Hom}_k(A/\mathfrak{m}^n, k)$ is an injective hull of the residue field k_A. Moreover, $\mathrm{Hom}_k(N, k) = \mathrm{Hom}_A(N, E)$ functorially in all finite length A-modules M.*

Now we begin the proof. Let E be as in the statement of the theorem, and let $\widetilde{E}$ be the associated quasicoherent sheaf on X. By the defining property of $\mathbf{D}_{\mathrm{ex}}\mathcal{O}_X$, we need to show that

$$R\Gamma(X, \mathcal{F})^* = R\mathrm{Hom}(\mathcal{F}, \widetilde{E})$$

for all $\mathcal{F} \in \mathrm{Perf}(X)$. Switching to equivariant A-modules, we need to prove the equivalent statement that

$$R\mathrm{Hom}_k(N^G, k) = R\mathrm{Hom}_A(N, E)^G$$

for all $N \in \mathrm{Perf}(X) = \mathrm{Perf}(A)^{BG}$. Note that by linear reductivity of G, we can rewrite the left side here as $R\mathrm{Hom}_k(N, k)^G$. If the total cohomology of N has finite length as an A-module, the desired result is clear because then N is automatically supported at the unique closed orbit and

$$R\mathrm{Hom}_k(N, k) = R\mathrm{Hom}_A(N, E)$$

before taking G-invariants, by the result on Matlis duality recalled above.

Now the key trick is as follows. The ideal $\mathfrak{m} \subset A$ is G-stable, so all quotients $A/\mathfrak{m}^n$ are G-equivariant. We claim that for any fixed N,

$$R\mathrm{Hom}_k(N, k)^G = R\mathrm{Hom}_k(N \otimes A/\mathfrak{m}^n, k)^G$$

for all $n \gg 0$, and also that

$$R\mathrm{Hom}_A(N, E)^G = R\mathrm{Hom}_A(N \otimes A/\mathfrak{m}^n, E)^G$$

for all $n \gg 0$. Since the total cohomology of $N \otimes A/\mathfrak{m}^n$ has finite length as an A-module for every n, this reduces us to the trivial case established in the previous paragraph.

[2] We "only" need that the nilpotent cone is affine and Gorenstein, with a closed point as the unique closed orbit. The Gorenstein property at least is not trivial.

It remains to verify this claim. Since $\mathrm{Perf}(A)^{BG}$ is generated under finite colimits and retracts by objects of the form $V \otimes A$ with V an irreducible G-representation, we can assume N is of this form and in degree zero. Then for the first part of the claim, we need to see that $R\mathrm{Hom}_k(V \otimes \mathfrak{m}^n, k)^G = 0$ for all $n \gg 0$. This is equivalent to $R\mathrm{Hom}_G(V^*, \mathfrak{m}^n) = 0$ for all $n \gg 0$. But now we win: as proved earlier, the assumptions on the G-action guarantee that any irreducible G-representation appears in A with finite multiplicity. Moreover, since G is linearly reductive, we have that $A \simeq \mathfrak{m}^n \oplus A/\mathfrak{m}^n$ as G-representations. Using that A is $\mathfrak{m}$-adically separated, we deduce that any given irreducible G-representation does not occur in $\mathfrak{m}^n$ for all sufficiently large n. This gives the desired vanishing for the first part of the claim. The second part is similar, using Matlis duality to show that any given irreducible G-representation does not occur in $\mathrm{Hom}_A(\mathfrak{m}^n, E) = E/E[\mathfrak{m}^n]$ for all large enough n. The point here is that

$$\begin{aligned} E &= \mathrm{colim}_n E[\mathfrak{m}^n] \\ &= \mathrm{colim}_n \mathrm{Hom}_A(A/\mathfrak{m}^n, E) \\ &= \mathrm{colim}_n \mathrm{Hom}_k(A/\mathfrak{m}^n, k) \end{aligned}$$

using Matlis duality again, so the multiplicity of any irreducible representation V in E equals the multiplicity of V^* in A and thus is finite, and all maps from V into E automatically factor over $E[\mathfrak{m}^n]$ for all sufficiently large n.

Corollary 2.12. *Keep the assumptions of Theorem 2.10, and assume moreover that X is Gorenstein. Then the equation*

$$\mathbf{D}_{\mathrm{ex}} A = \mathbf{D}_{\mathrm{can}} A \otimes S(\mathcal{O}_X)$$

remains valid for all $A \in \mathrm{Coh}(X)$.

Proof. For any $A \in \mathrm{Coh}(X)$ and any large n, we can pick $A_n \in \mathrm{Perf}(X)$ and a map $A_n \to A$ whose cone C_n is in degrees $\leq -n$. Then both sides of the above equation will transform C_n into something in degrees $\geq n - M$ for some M depending only on X. For the left-hand side this follows from Theorem 2.8, and for the right-hand side it follows from the fact that $\mathbf{D}_{\mathrm{can}}$ has uniformly bounded t-amplitude for Gorenstein stacks, together with the observation that $S(\mathcal{O}_X)$ has bounded tor-dimension, which (again using the Gorenstein condition) is an immediate consequence of the formula

$$S(\mathcal{O}_X) = \mathrm{cone}(\mathcal{O}_X \to j_* \mathcal{O}_{X^\circ})[\dim A_{\mathfrak{m}} - 1]$$

recorded earlier. Since both sides of the equation match on A_n, they match on A in all degrees $> n - M$. Now take $n \to \infty$. ∎

2.4 Ind-coherent sheaves

Fix k of characteristic zero, and let X be a QCA algebraic stack over k. Besides QCoh, we have another sheaf theory which is better behaved in many ways, namely Gaitsgory's category IndCoh of ind-coherent sheaves. Recall that $\mathrm{IndCoh}(X)$ comes with its own symmetric monoidal structure denoted $\otimes^!$, for which the monoidal unit is the dualizing complex ω_X. It is *not* rigid symmetric monoidal in general. It also comes with functors f_*^{IndCoh} and $f^{!\mathrm{IndCoh}}$ for $f : X \to Y$ any morphism of QCA stacks, which satisfy the abstract properties of a lower-! and upper-$*$ in Mann's language of six functor formalisms. There is also a continuous functor $\Psi : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ which is the obvious embedding on $\mathrm{Coh}(X)$, and which intertwines f_*^{IndCoh} with the usual QCoh push-foward f_*. This gives rise to a (continuous) functor of global sections by setting $R\Gamma^{\mathrm{IndCoh}}(X, \mathcal{F}) := R\Gamma(X, \Psi\mathcal{F}) \in \mathcal{D}(k)$.

For general prestacks this category still exists and is defined by a descent procedure, but for QCA stacks Drinfeld-Gaitsgory proved that $\mathrm{IndCoh}(X)$ is compactly generated with compact generators $\mathrm{Coh}(X)$, so the formula $\mathrm{IndCoh} = \mathrm{Ind}(\mathrm{Coh})$ is literally true. Moreover, there is a canonical involutive duality on $\mathrm{Coh}(X)$, namely Grothendieck-Serre duality which we denote $\mathbf{D}_{\mathrm{GS}}$. Therefore the whole paradigm of Section 2.1 applies and we get a Serre functor and an exotic duality, which in this case we call *admissible duality* and denote $\mathbf{D}_{\mathrm{adm}}$. Of course the duality exchange formula still applies, and in this new notation it reads $R\mathrm{Hom}(\mathcal{F}, \mathbf{D}_{\mathrm{adm}}\mathcal{G}) = R\mathrm{Hom}(\mathbf{D}_{\mathrm{GS}}\mathcal{F}, \mathcal{G})^*$ for all $\mathcal{F} \in \mathrm{Coh}(X)$ and $\mathcal{G} \in \mathrm{IndCoh}(X)$. We write $\mathrm{Adm}(X) \subset \mathrm{IndCoh}(X)$ for the subcategory of admissible ind-coherent sheaves.

Admissible duality satisfies another clean formula in this setting, which can also be taken as its definition: we have

$$R\mathrm{Hom}(\mathcal{F}, \mathbf{D}_{\mathrm{adm}}\mathcal{G}) = R\Gamma^{\mathrm{IndCoh}}(X, \mathcal{F} \otimes^! \mathcal{G})^*$$

for *all* ind-coherent sheaves $\mathcal{F}, \mathcal{G}$. After a trivial reduction to the case where $\mathcal{F}$ is coherent, this is a consequence of duality exchange plus the formula

$$R\mathrm{Hom}(\mathbf{D}_{\mathrm{GS}}\mathcal{F}, \mathcal{G}) = R\Gamma^{\mathrm{IndCoh}}(X, \mathcal{F} \otimes^! \mathcal{G})$$

which is [DG13, Proposition 4.4.4].

As usual, the category $\mathrm{Adm}(X)$ is hard to describe, beyond the tautological fact that if $\mathrm{IndCoh}(X)$ is proper then $\mathrm{Coh}(X) \subset \mathrm{Adm}(X)$.

Exercise 2.13. i. Let X be a separated (classical) k-scheme of finite type. Prove that $\mathrm{IndCoh}(X)$ is proper if and only if X is smooth and proper over k.

ii. Let $X = \mathcal{N}/G$ be the equivariant nilpotent cone associated with a split reductive G/k. Prove that $\mathrm{IndCoh}(X)$ is *not* proper.

We also record some basic functorialities; the proofs are formal manipulations and we omit them.

Proposition 2.14. *Let $f : X \to Y$ be any map of QCA stacks.*

1. *If f is eventually coconnective, the pushfoward f_*^{IndCoh} preserves* Adm*, and if f is Gorenstein then $\mathbf{D}_{\mathrm{adm}}$ commutes with f_*^{IndCoh} up to an invertible twist.*
2. *If f is schematic and proper, $f^{!\mathrm{IndCoh}}$ preserves* Adm *and $\mathbf{D}_{\mathrm{adm}}$ commutes with $f^{!\mathrm{IndCoh}}$.*

Under further conditions on X, we can say a lot more. Recall that if X is eventually coconnective, the functor Ψ has a fully faithful left adjoint $\Xi : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$. If X is perfect, this is just the ind-completion of the tautological functor $\mathrm{Perf}(X) \to \mathrm{Coh}(X)$. If X is also Gorenstein (which is stronger than being eventually coconnective, and holds in most applications), we can compare the dualities on QCoh and IndCoh. To avoid confusion, we will write $\mathbf{D}_{\mathrm{nv}}$ for the canonical self-duality on $\mathrm{Perf}(X)$ used in Section 2.3, and $\mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}$ for the associated exotic duality on $\mathrm{QCoh}(X)$ which we discussed at length in that section.

Proposition 2.15. *If X is a QCA stack which is eventually coconnective and perfect, then $\mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}\mathcal{O}_X \cong \Psi(\mathbf{D}_{\mathrm{adm}}\omega_X)$.*

If moreover X is Gorenstein, then

$$\Psi\mathbf{D}_{\mathrm{adm}}\mathcal{F} = \omega_X \otimes \mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}\Psi\mathcal{F}$$

for all $\mathcal{F} \in \mathrm{IndCoh}(X)$. Moreover, Ψ sends admissible ind-coherent sheaves towards admissible quasicoherent sheaves.

Note that if X is Gorenstein, ω_X is perfect, and in fact invertible.

Proof. The first part follows formally from the general nonsense identity $S^{\mathrm{QCoh}} = \Psi \circ S^{\mathrm{IndCoh}} \circ \Xi$, by evaluating it on $\mathcal{O}_X$.

For the second part, pick any $\mathcal{G} \in \mathrm{Perf}(X)$. It is easy to see that

$$\begin{aligned}\mathbf{D}_{\mathrm{GS}}\Xi\mathcal{G} &= \Xi(\omega_X \otimes \mathbf{D}_{\mathrm{nv}}\mathcal{G}) \\ &= \Xi\mathbf{D}_{\mathrm{nv}}(\mathcal{G} \otimes \omega_X^{-1}),\end{aligned}$$

using the invertibilty of ω_X in the second line. We then compute

$$\begin{aligned}R\mathrm{Hom}(\mathcal{G}, \Psi\mathbf{D}_{\mathrm{adm}}\mathcal{F}) &= R\mathrm{Hom}(\Xi\mathcal{G}, \mathbf{D}_{\mathrm{adm}}\mathcal{F}) \\ &= R\mathrm{Hom}(\mathbf{D}_{\mathrm{GS}}\Xi\mathcal{G}, \mathcal{F})^* \\ &= R\mathrm{Hom}(\Xi(\mathbf{D}_{\mathrm{nv}}(\mathcal{G} \otimes \omega_X^{-1})), \mathcal{F})^* \\ &= R\mathrm{Hom}(\mathbf{D}_{\mathrm{nv}}(\mathcal{G} \otimes \omega_X^{-1}), \Psi\mathcal{F})^* \\ &= R\mathrm{Hom}(\mathcal{G} \otimes \omega_X^{-1}, \mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}\Psi\mathcal{F}) \\ &= R\mathrm{Hom}(\mathcal{G}, \omega_X \otimes \mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}\Psi\mathcal{F})\end{aligned}$$

where we used duality exchange several times, along with the invertibility of ω_X and various adjunctions. Since $\mathcal{G}$ is arbitrary, the desired formula now follows by Yoneda.

The final claim is clear by the Ξ/Ψ adjunction and the inclusion $\Xi(\mathrm{Perf}) \subset \mathrm{Coh}$. □

This formula is especially useful in combination with the following surprising theorem, which is proved in [HM26].

Theorem 2.16. *Let X be a Gorenstein QCA stack. Then for any $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathbf{D}_{\mathrm{adm}}\mathcal{F}$ lies in the essential image of Ξ.*

Since Ξ is fully faithful, this is equivalent the claim that the counit induces a natural isomorphism $\Xi\Psi\mathbf{D}_{\mathrm{adm}}\mathcal{F} \xrightarrow{\sim} \mathbf{D}_{\mathrm{adm}}\mathcal{F}$ for any coherent $\mathcal{F}$. Combining this with the previous theorem, we can rewrite $\mathbf{D}_{\mathrm{adm}}\mathcal{F}$ further as

$$\mathbf{D}_{\mathrm{adm}}\mathcal{F} \cong \Xi\left(\omega_X \otimes \mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}\Psi\mathcal{F}\right).$$

In many cases of interest, $\mathcal{F}$ arises as an explicit pushfoward from a much smaller stack. Since Ψ and $\mathbf{D}_{\mathrm{adm}}^{\mathrm{QCoh}}$ interact very cleanly with pushfowards, this often reduces the computation of $\mathbf{D}_{\mathrm{adm}}\mathcal{F}$ to a manageable computation on an auxiliary stack.

The preceding theorem looks extremely weird on first glance. Let us discuss a simple but already very nontrivial example. Consider the stack

$$X = (\mathrm{Spec}k[x,y]/(xy))/\mathbf{G}_m$$

where $t \in \mathbf{G}_m$ acts as $t \cdot (x,y) = (x, ty)$. There is a natural map $\alpha : X \to \mathbf{A}^1 = \mathrm{Spec}k[x]$ which is a good moduli space. The fiber at any closed point away from 0 is just a $B\mathbf{G}_m$, while the fiber at 0 is a closed substack $Z \cong \mathbf{A}^1/\mathbf{G}_m \overset{i}{\hookrightarrow} X$. We will sketch a computational proof that for any $\mathcal{F} \in \mathrm{Coh}(X)$ which is *also* admissible, $\mathbf{D}_{\mathrm{adm}}\mathcal{F}$ lies in the image of Ξ.[3]

[3] This was our original form of Theorem 2.16, which we discovered experimentally after looking at exactly this example. Sam Raskin suggested to us later that admissibility is not actually needed.

It is easy to see that if $\mathcal{F} \in \mathrm{Coh}(X)$ is admissible, then $\mathcal{F}$ is supported on finitely many closed fibers of the map α. The summands supported on the fibers away from 0 are uninteresting since those fibers are smooth, so we restrict our attention to the subcategory $\mathrm{Coh}(X)_Z$ of objects set-theoretically supported on the interesting fiber. By a direct calculation, first done independently by Bertoloni Meli and Koshikawa, one can show that *all* objects of $\mathrm{Coh}(X)_Z$ are admissible. This is *not* obvious because the stack X is singular! Nevertheless, it is true. Moreover, with some effort one can show that the kernel of $\Psi : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ is freely generated by a single sheaf $\mathcal{F}^+$, which moreover is its own admissible dual up to a shift.[4]

Now take any $\mathcal{F} \in \mathrm{Coh}(X)_Z$. The theorem claims that $\mathbf{D}_{\mathrm{adm}}\mathcal{F} \in \mathrm{im}\Xi$. Since the kernel of Ψ is freely generated by $\mathcal{F}^+$, this claim is equivalent to the statement that $R\mathrm{Hom}(\mathbf{D}_{\mathrm{adm}}\mathcal{F}, \mathcal{F}^+) = 0$. Now, using the fact that $\mathcal{F}^+ = \mathbf{D}_{\mathrm{adm}}\mathcal{F}^+[m]$ for some (irrelevant) m, we can rewrite this $R\mathrm{Hom}$ as $R\mathrm{Hom}(\mathcal{F}^+, \mathbf{D}^2_{\mathrm{adm}}\mathcal{F}[m])$. But $\mathcal{F}$ is admissible, so biduality lets us rewrite this as $R\mathrm{Hom}(\mathcal{F}^+, \mathcal{F}[m])$. But now we win, since

$$R\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = R\mathrm{Hom}(\Psi\mathcal{F}_1, \Psi\mathcal{F}_2)$$

for all $\mathcal{F}_1 \in \mathrm{IndCoh}$ and all $\mathcal{F}_2 \in \mathrm{Coh}$, so

$$\begin{aligned} R\mathrm{Hom}(\mathcal{F}^+, \mathcal{F}[m]) &= R\mathrm{Hom}(\Psi\mathcal{F}^+, \Psi\mathcal{F}[m]) \\ &= 0 \end{aligned}$$

because $\Psi\mathcal{F}^+ = 0$.

Pushing these ideas further, we compute the Serre functor on all of IndCoh.

Theorem 2.17. *Let X be a QCA stack which is Gorenstein and perfect. Then the Serre functor on* $\mathrm{IndCoh}(X)$ *is given explicitly as* $\Xi(\Psi(-) \otimes \mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathcal{O}_X)$.

If moreover X is quasismooth and $V \subset \mathrm{Sing}(X)$ is any closed conical subset, the Serre functor on $\mathrm{IndCoh}_V(X)$ *is given explicitly as* $\Xi_{0\to V}(\Psi_{V\to 0}(-) \otimes \mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathcal{O}_X)$.

Proof. Since $S^{\mathrm{IndCoh}} = \mathbf{D}_{\mathrm{adm}}\mathbf{D}_{\mathrm{GS}}$ on $\mathrm{Coh}(X)$, Theorem 2.16 shows that

$$S^{\mathrm{IndCoh}} = \Xi\Psi S^{\mathrm{IndCoh}}$$

after restricting to $\mathrm{Coh}(X)$, and then also on $\mathrm{IndCoh}(X) = \mathrm{Ind}(\mathrm{Coh}(X))$ since all functors are compatible with filtered colimits. Now for any $\mathcal{F} \in \mathrm{Coh}(X)$ we can rewrite

$$\begin{aligned} \Psi S^{\mathrm{IndCoh}}\mathcal{F} &= \Psi\mathbf{D}_{\mathrm{adm}}\mathbf{D}_{\mathrm{GS}}\mathcal{F} \\ &= \omega_X \otimes \mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\Psi\mathbf{D}_{\mathrm{GS}}\mathcal{F} \\ &= \omega_X \otimes \mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}(\omega_X \otimes \mathbf{D}_{\mathrm{nv}}\Psi\mathcal{F}) \\ &= \mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathbf{D}_{\mathrm{nv}}\Psi\mathcal{F} \end{aligned}$$

where the second line follows from Proposition 2.15 and the final line follows from the invertibility of ω_X. Now, this final composite has uniformly bounded t-amplitude as a functor $\mathrm{Coh} \to \mathrm{QCoh}$, using the Gorenstein condition to control the amplitude of $\mathbf{D}_{\mathrm{nv}}$ and Theorem 2.7 to control the amplitude of $\mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}$. This calculation shows that S^{IndCoh} carries $\mathrm{Coh}^{\leq n}$ into $\mathrm{IndCoh}^{\leq n+N}$ for some fixed N,

[4] There are some obvious coherent sheaves $\mathcal{L}_n \in \mathrm{Coh}(X)^{\heartsuit}_Z$ arising from the usual weight n line bundle on Z. These come with canonical injective maps $\mathcal{L}_n \hookrightarrow \mathcal{L}_{n-1}$, and less obviously there are also canonical nonzero maps $\mathcal{L}_n \to \mathcal{L}_{n+1}[2]$ in the derived category. The sheaf $\mathcal{F}^+$ is defined as "colim"$_{i\to\infty}\mathcal{L}_i[2i]$. It is immediate from this definition that $\Psi\mathcal{F}^+ = 0$.

so passing to colimits and taking $n \to -\infty$ we see that S^{IndCoh} takes $\mathrm{IndCoh}^{\leq -\infty}$ into $\mathrm{IndCoh}^{\leq -\infty}$. But we already proved that it takes values in QCoh $\subset$ IndCoh, and QCoh $\cap\, \mathrm{IndCoh}^{\leq -\infty} = 0$, so S^{IndCoh} annihilates $\mathrm{IndCoh}^{\leq -\infty}$. Said differently, $S^{\mathrm{IndCoh}} = S^{\mathrm{IndCoh}}\Xi\Psi$, so combining this with the first inset equation we get

$$\begin{aligned} S^{\mathrm{IndCoh}} &= \Xi\Psi S^{\mathrm{IndCoh}}\Xi\Psi \\ &= \Xi S^{\mathrm{QCoh}}\Psi \end{aligned}$$

where the second line is general nonsense. Since $S^{\mathrm{QCoh}}(-) = (-) \otimes \mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathcal{O}_X$, this gives the result.

The final part is formal, by writing

$$\begin{aligned} S^{\mathrm{IndCoh}_V} &= \Psi_{\mathrm{all}\to V} S^{\mathrm{IndCoh}}\Xi_{V\to\mathrm{all}} \\ &= \Psi_{\mathrm{all}\to V}\Xi_{0\to\mathrm{all}} S^{\mathrm{QCoh}}\Psi_{\mathrm{all}\to 0}\Xi_{V\to\mathrm{all}} \\ &= \Xi_{0\to V} S^{\mathrm{QCoh}}\Psi_{V\to 0} \end{aligned}$$

where the first and third lines are straightforward, and the second line follows from the first part of the theorem. ☐

Example 2.18. Suppose k is algebraically closed with G/k split reductive. In [Ber21b, Theorem 4.3.11], Beraldo computed the Serre functor of the spectral derived Satake category $\mathrm{IndCoh}_{\mathcal{N}/G}(0 \times^{\mathbf{R}}_{\mathfrak{g}} 0/G)$. Through hard effort, he showed that it is $\Xi_{0\to\mathcal{N}/G}\Psi_{\mathcal{N}/G\to 0}[-\mathrm{dim}G]$. From our perspective this is nearly trivial, and the only thing one needs to actually compute is $\mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathcal{O}_X$ where $X = 0 \times^{\mathbf{R}}_{\mathfrak{g}} 0$. But it is straightforward to see that $\mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathcal{O}_X = \mathcal{O}_X[-\mathrm{dim}G]$: this is an immediate consequence of the fact that $R\mathrm{Hom}_k(\mathcal{O}(X), k) = \mathcal{O}(X)[-\mathrm{dim}G]$

Note that for X QCA perfect and eventually coconnective, $\mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathcal{O}_X = \Psi(\mathbf{D}_{\mathrm{adm}}\omega_X)$ as noted earlier, and if X is also Gorenstein then additionally $\mathbf{D}_{\mathrm{adm}}\omega_X \cong \Xi\mathbf{D}^{\mathrm{QCoh}}_{\mathrm{adm}}\mathcal{O}_X$. As discussed in the introduction, this puts the sheaf

$$\delta_X := S^{\mathrm{QCoh}}(\mathcal{O}_X) = S^{\mathrm{IndCoh}}(\mathcal{O}_X) = \mathbf{D}_{\mathrm{adm}}\omega_X$$

at center stage. For affine quotient stacks, one can compute all the !-stalks of this sheaf.

Theorem 2.19. *Assume k is algebraically closed, and consider a quotient stack $X = (\mathrm{Spec}A)/G$ where A is a classical finite type k-algebra and G is linearly reductive. Let $x \in X$ be any k-point, with residual gerbe $i_x : B\mathcal{G}_x \hookrightarrow X$. Then*

$$i_x^{!\mathrm{IndCoh}}\delta_X = \begin{cases} \mathcal{O}_{B\mathcal{G}_x} \text{ if } x \text{ is closed} \\ 0 \text{ if not.} \end{cases}$$

Proof. For brevity we write $i^! = i^{!\mathrm{IndCoh}}$. Let V be the scheme-theoretic closure of x in X. We can factor i_x as $B\mathcal{G}_x \xrightarrow{j} V \xrightarrow{i} X$ where j is an open immersion and i is a closed immersion. Then $i_x^! = j^* i^!$ by construction, so then

$$\begin{aligned} i_x^!\delta_X &= j^* i^! \mathbf{D}_{\mathrm{adm}}\omega_X \\ &= j^*\mathbf{D}_{\mathrm{adm}} i^!\omega_X \\ &= j^*\mathbf{D}_{\mathrm{adm}}\omega_V \\ &= j^*\delta_V \end{aligned}$$

where in the second line we used the commutation of admissible duality with !-pullback along proper morphisms stated above. If x is a closed point, then j is the identity morphism and it is easy to calculate that $\delta_{BH} = \mathcal{O}_{BH}$ for any linearly reductive H. This gives the claim when x is closed. If x is not closed, V is a quotient stack satisfying the conditions of Theorem 2.10, so δ_V is set-theoretically supported at the closed orbit of V, which is disjoint from x. Therefore $j^*\delta_V = 0$ as claimed. $\square$

It is not hard to extend the previous theorem to any classical QCA stack admitting a good moduli space. This applies in particular to the stack $\mathrm{LS}_{\check{G}}$ of $\check{G}$-local systems on a projective curve. Comparing the previous result to [Ber21a, Theorem D'] strongly suggests some direct relationship between Beraldo's Steinberg sheaf $\underline{\mathrm{St}}_{\check{G}}$ [Ber21a] and the sheaf $\delta_{\mathrm{LS}_{\check{G}}}$. It would be very interesting to understand this more explicitly.

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