Notes on relative perverse t-structures

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Warning

These are (no longer) private notes written by DH in October 2020, lightly edited and posted publicly in April 2022 at the request of V. Drinfeld. The main content of these notes is an alternative proof of the main theorem in the paper [HS], namely the existence of a relative perverse t-structure, at least in the case of torsion coefficients. See Theorem 0.13 below. The results proved in [HS] are more general, but the strategy here is rather different and much more classical. All errors and misconceptions here are the sole responsibility of DH.

Setup

We say a scheme S is *nice* if it is excellent, Noetherian and finite-dimensional. Any such scheme admits a dimension function $\delta_S : |S| \to \mathbf{Z}$ Zariski-locally, and globally under some mild conditions. In particular:

- If S is normal (e.g. regular), then $s \mapsto -\dim \mathcal{O}_{S,s}$ is a dimension function.
- If all irreducible components of S are equicodimensional, then $s \mapsto \dim \overline{\{s\}}$ is a dimension function. This holds e.g. for schemes of finite type over $\mathbf{Z}$, over a field, or over a K-affinoid ring.

To avoid certain circumlocutions, we will say that a nice scheme S is *really nice* if it admits a dimension function and every connected component of S is irreducible. The normalization of any nice scheme is really nice, and every nice scheme admits a really nice dense open subscheme: the image of the normal locus or the regular locus in S^{red} does the job.

If S is a nice scheme equipped with a dimension function δ_S, any locally finite type S-scheme $f : X \to S$ inherits a canonical dimension function from δ_S by setting

$$\delta(x) = \delta_S(f(x)) + \mathrm{tr.deg}\, k(x)/k(f(x)).$$

We also think of these as functions on geometric points $\overline{x}$ lying over points $x \in X$ in the obvious way.

Set $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for some n invertible on S. Unless noted otherwise, all derived categories will be derived categories of étale sheaves of Λ-modules, and we write $D(X) := D(X_{\acute{e}t}, \Lambda)$ for any scheme X, similarly for decorated versions etc.[1] For any pair (S, δ_S) with S nice, the results

[1] Essentially all statements and proofs below also hold with $\mathbf{Q}_\ell$-coefficients on nice schemes, with only minor changes. The most significant point is that one must decide on the "correct" definition of universal local acyclicity for objects in $D_c^b(X, \mathbf{Q}_\ell)$, such that all the usual properties of ULA sheaves still hold. The correct definition seems

in [ILO] give a dualizing complex $\omega_S \in D^b_{ctf}(S)$ uniquely determined by the requirement that $R\Gamma_{\{\overline{s}\}}(S_{\overline{s}}, \omega_S) \simeq \Lambda[2\delta_S(\overline{s})](\delta_S(\overline{s}))$ for all geometric points $\overline{s}$ lying over all points $s \in S$, $S_{\overline{s}}$ denoting strict Henselization. Similarly for any finite type X/S, with $\omega_X = Rf^!\omega_S$ as usual.

For S nice and equipped with a dimension function, any finite type X/S gets a natural perverse t-structure on $D(X)$ by defining ${}^{\mathfrak{p}}D^{\leq 0}(X)$ as the full subcategory of objects such that $\mathcal{H}^i(\mathcal{F}_{\overline{x}}) = 0$ for all $i > -\delta(\overline{x})$ and all $\overline{x} \to X$. The truncation functors for this t-structure preserve $D^b_c(X)$, and on D^b_c the perverse t-structure is self-dual for $\mathbf{D}_X := R\mathscr{H}\mathrm{om}(-, \omega_X)$. [2] If $S = \operatorname{Spec} k$ is a field and $\delta_S \equiv 0$ then this is the classical (middle-)perverse t-structure as considered in [BBDG], which we notate ${}^pD^{\leq 0}(X)$; we will *only* use this notation for schemes of finite type over a field.

Main results

Here is the central definition.

Definition 0.1. For any scheme S and any finite type morphism $f : X \to S$, define ${}^{p/S}D^{\leq 0}(X)$ as the full subcategory of $D(X)$ spanned by objects $\mathcal{F}$ such that $\mathcal{F}|X_{\overline{s}} \in {}^pD^{\leq 0}(X_{\overline{s}})$ for all geometric points $\overline{s} \to S$.

This definition is unchanged by instead taking fibers over all algebraic geometric points $\overline{s} = \operatorname{Spec} \overline{k(s)} \to S$, or over all separable closures $s^{\mathrm{sep}} \to S$ of points $s \in S$, or just over all points $\operatorname{Spec} k(s) \to S$. It is clear that ${}^{p/S}D^{\leq 0}(X)$ is preserved under extensions and (after upgrading to derived ∞-categories) under filtered colimits, and is set-theoretically reasonable, so ${}^{p/S}D^{\leq 0}(X)$ defines the connective part of a t-structure on $D(X)$ by [Lur16, Proposition 1.4.4.11]. We denote the coconnective part by ${}^{p/S}D^{\geq 0}(X)$, and we call this the *relative perverse t-structure* on X / on $D(X)$ (relative to S). We write ${}^{p/S}\tau^{\leq n}$ and ${}^{p/S}\tau^{\geq n}$ for the truncation functors for this t-structure.

This t-structure interpolates between two well-known cases: if $f : X \to X$ is the identity map it reduces to the standard t-structure on $D(X)$, while if $S = \operatorname{Spec} k$ is a point it gives the usual perverse t-structure on $D(X)$. We first establish some easy formal properties, which hold over any base scheme S.

Lemma 0.2. *Let $f : X \to S$ be a finite type map of arbitrary schemes.*

1. Let $j : U \to X$ resp. $i : Z \to X$ be an étale map (resp. a finite map). Then we have the following t-exactness properties with respect to the relative perverse t-structures: i^* and $j_!$ are right t-exact, i_* and j^* are t-exact, and Rj_* and $Ri^!$ are left t-exact.

2. Let $U \xrightarrow{j} X \xleftarrow{i} Z$ be an open-closed decomposition. Then the relative perverse t-structure on $D(X)$ is glued by recollement from the relative perverse t-structures on $D(U)$ and $D(Z)$.

3. If $g : S \to S'$ is a monomorphism or a quasifinite morphism, then ${}^{p/S}D^{\leq 0}(X) = {}^{p/S'}D^{\leq 0}(X)$, and similarly on coconnective parts.

4. If $g : T \to S$ is any morphism, with base change $\tilde{g} : X_T \to X$, then $\tilde{g}^*$ is right t-exact and $R\tilde{g}_*$ is left t-exact.

We will freely use these results without comment.

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to be that an object $\mathcal{F} \in D^b_c(X, \mathbf{Q}_\ell)$ is f-ULA (for f a finite type morphism of nice schemes) if it is dualizable in the category $\mathcal{C}_{S,\mathbf{Q}_\ell}$, in the notation and terminology of [LZ20]. Crucially, Deligne's generic ULA theorem still holds with this definition. (Sketch. The map $\alpha : \mathbf{D}_{X/S}\mathcal{F} \boxtimes_S \mathcal{F} \to R\mathscr{H}\mathrm{om}(\mathrm{pr}_1^*\mathcal{F}, R\mathrm{pr}_2^!\mathcal{F})$ is an isomorphism over all generic points of S by regular base change and some standard Kunneth formulas (ILO, SGA5), and the source and target of α are bounded and constructible, so then α is an isomorphism over a dense open subset of S.)

[2] Note that the perverse t-structure on $D(X)$ depends on the chosen dimension function for S; however, a different choice of dimension function will only change this t-structure by a shift which is locally constant on S.

Proof. 1. The right t-exactness of i^*, i_*, $j_!$ and j^* is trivial, and the remaining claims follow formally by adjunction: i_* and j^* are also left t-exact because they have right t-exact left adjoints, and then Rj_* and $Ri^!$ are left t-exact by the same reasoning.

2. By part 1., $j_!j^*$ carries ${}^{p/S}D^{\leq 0}(X)$ into itself, i.e. is right t-exact, so Proposition 1.4.12 in [BBDG] shows that the relative perverse t-structure on X is obtained by gluing from suitable t-structures on $D(U)$ and $D(Z)$. Moreover, the proof of that proposition identifies the left halves of these t-structures with $j^{*p/S}D^{\leq 0}(X)$ and $i^{*p/S}D^{\leq 0}(X)$. These are full subcategories of ${}^{p/S}D^{\leq 0}(U)$ and ${}^{p/S}D^{\leq 0}(Z)$, and an easy exercise (using part 1.) shows that they actually give everything in the latter left halves.

3. A geometric fiber of $g \circ f$ is a finite disjoint union of geometric fibers of f.

4. The fact that $\tilde{g}^*$ carries ${}^{p/S}D^{\leq 0}(X)$ into ${}^{p/T}D^{\leq 0}(X_T)$ is clear from the definitions, and then the left t-exactness of $R\tilde{g}_*$ follows by adjunction. □

The next order of business is to check that for any finite type map of nice schemes $f : X \to S$, the relative perverse truncation functors preserve bounded constructible complexes.

Proposition 0.3. *Let S be a Noetherian scheme and let $f : X \to S$ be any finite type morphism. Then the function*

$$p : X \to \mathbf{Z}$$
$$x \mapsto -\text{tr.deg}k(x)/k(f(x))$$

is a weak perversity function in the sense of [G]. Moreover, if S is universally catenary, the conditions of [G, Theorem 8.2] hold: for all $x \in X$, there is a dense open subset $U \subset \overline{\{x\}}$ such that

$$p(y) \leq p(x) + 2\text{codim}(\overline{\{y\}}, \overline{\{x\}})$$

for all $y \in U$.

Recall that for a Noetherian scheme X, a function $p : X \to \mathbf{Z}$ is a weak perversity function if for every $x \in X$ and every $m \in \mathbf{Z}$, $p(y) \geq \min(p(x), m)$ for all y in some nonempty open $U \subset \overline{\{x\}}$. Any such function defines a t-structure on $D(X)$ by the results in [G]. By definition, the relative perverse t-structure is the t-structure associated with the weak perversity function $x \mapsto -\text{tr.deg}k(x)/k(f(x))$.

Proof. Fix a point $x \in X$ with scheme-theoretic closure $\overline{\{x\}}$. Let $Z \subset S$ be the scheme-theoretic closure of x in S, so $\overline{\{x\}} \to Z$ is a finite type map of integral Noetherian schemes with scheme-theoretically dense image. By Chevalley's theorem and generic flatness, there is a dense open $V \subset Z$ such that $U = \overline{\{x\}} \times_Z V \to V$ is flat and surjective. Note that U is a dense open subscheme of $\overline{\{x\}}$ and that x resp. $f(x)$ is the generic point of the integral scheme U resp. V. I claim that for all $y \in U$, $p(y) \geq p(x)$ and that if additionally S is universally catenary then also $p(y) \leq p(x)+2\text{codim}(\overline{\{y\}}, U)$. Since U is dense in $\overline{\{x\}}$ and $\text{codim}(\overline{\{y\}}, U) = \text{codim}(\overline{\{y\}}, \overline{\{x\}})$, these inequalities imply the desired results.

Rephrasing slightly, we need to show that if $f : Y \to T$ is a flat surjective finite type map of integral Noetherian schemes, then for any point $y \in Y$ we have an inequality

$$\text{tr.deg}k(y)/k(f(y)) \leq \text{tr.deg}k(\eta)/k(f(\eta)),$$

and if T is universally catenary then also

$$\text{tr.deg}k(\eta)/k(f(\eta)) \leq \text{tr.deg}k(y)/k(f(y)) + 2\text{codim}(\overline{\{y\}}, Y).$$

Here $\eta \in Y$ denotes the generic point. Since $f(\eta)$ is the generic point of T, the first inequality is immediate from constancy of fiber dimensions for finite type flat maps of irreducible Noetherian schemes. For the second inequality, note that equality holds for $y = \eta$, so it suffices to see that the function

$$\alpha(y) = \mathrm{tr.deg} k(y)/k(f(y)) + 2\mathrm{codim}(\overline{\{y\}}, Y)$$

is non-decreasing under specializations in Y. This reduces immediately to the case of immediate specializations. If $y \rightsquigarrow z$ is an immediate specialization in Y, then $2\mathrm{codim}(\overline{\{z\}}, Y) = 2\mathrm{codim}(\overline{\{y\}}, Y)+2$ by universal catenarity, so to conclude it's enough to observe that

$$\mathrm{tr.deg} k(z)/k(f(z)) \geq \mathrm{tr.deg} k(y)/k(f(y)) - 1.$$

For this last statement, there are two cases:

- $f(y) = f(z)$, in which case tr.deg drops by 1.
- $f(y) \rightsquigarrow f(z)$ is a nontrivial immediate specialization, in which case tr.deg does not change by Nagata's altitude formula. □

Corollary 0.4. *Let $f : X \to S$ be a finite type map of Noetherian schemes. Then ${}^{p/S}D^{\geq 0}(X)$ is stable under filtered colimits.*

If moreover S is nice, the truncation functors ${}^{p/S}\tau^{\geq n}$ and ${}^{p/S}\tau^{\leq n}$ preserve $D^b_c(X)$.

Proof. The first claim is a general property of the coconnective part of the t-structure associated with any weak perversity function.

For the second claim, we can work locally on S, so then S (and then also X) admits a dimension function and a dualizing complex. The result is then immediate from the previous proposition and [G, Theorem 8.2]. □

From now on, we assume unless stated otherwise that our base scheme S is nice, i.e. excellent Noetherian finite dimensional.

If S is a nice scheme equipped with a dimension function δ_S, let $\dim_S : |S| \to \mathbf{Z}$ be the locally constant function sending any point $s \in S$ to $\max_\eta \delta_S(\eta)$, where η runs over the generic points of the connected component of S containing s.[3] This function is "stupid" unless S has irreducible connected components, i.e. unless S is really nice. We will primarily use this function to renormalize the perverse t-structure on finite type S-schemes X: the category ${}^{\mathfrak{p}}D^{\leq 0}(X)[-\dim_S]$, when it makes sense, is independent of the choice of dimension function δ_S. In light of this, we will be somewhat sloppy about notating the dependence of things on δ_S in what follows; this should cause no confusion.

Proposition 0.5. *Let $f : X \to S$ be a finite type map of nice schemes. For any point $s \in S$, write $i_s : X_s = X \times_S s \to X$ for the inclusion of the fiber over s.*

1. *Suppose that S is equipped with a dimension function. If $\eta \to S$ is the inclusion of a generic point, then $i_\eta^*(-)[-\delta_S(\eta)] : D^b_c(X) \to D^b_c(X_\eta)$ carries ${}^{\mathfrak{p}}D^{\geq 0}(X)$ into ${}^{p}D^{\geq 0}(X_\eta)$ and ${}^{\mathfrak{p}}D^{\leq 0}(X)$ into ${}^{p}D^{\leq 0}(X_\eta)$, i.e. $i_\eta^*(-)[-\delta_S(\eta)]$ is perverse t-exact. If S is really nice, then $i_\eta^*(-)[-\dim_S]$ is perverse t-exact for all generic points $\eta \in S$.*
2. *Let $\mathcal{F} \in D^b_c(X)$ be any given object. Then there is a dense open really nice subscheme $S' \subset S$ such that for all $s \in S'$, $i_s^*(-)[-\dim_{S'}]$ commutes with all perverse truncations of $\mathcal{F}|X_{S'}$, i.e. such that*

$$i_s^*({}^{\mathfrak{p}}\tau^{\geq n}\mathcal{F}|X_{S'})[-\dim_{S'}] \cong {}^{p}\tau^{\geq n}(i_s^*\mathcal{F}[-\dim_{S'}])$$

for all $n \in \mathbf{Z}$.

[3]This function has nothing to do with the Krull dimension of S, unless the dimension function is the function $s \mapsto \dim \overline{\{s\}}$.

This result has the following attractive and useful consequences, which we'll use many times: if $f : X \to S$ is a finite type morphism of nice schemes and $\mathcal{F} \in D^b_c(X)$ is any given object, then there is a dense open really nice subscheme $S' \subset S$ such that for every generic point $\eta \in S$, the perverse cohomological amplitude of $i^*_\eta \mathcal{F}$ coincides with the perverse cohomological amplitude of $(\mathcal{F}|X_{S' \cap \overline{\{\eta\}}})[\dim_{S'}]$[4] and bounds the perverse cohomological amplitudes of $i^*_s \mathcal{F}$ for all $s \in S' \cap \overline{\{\eta\}}$. I will call this (and very minor variants) the *generic amplitude principle*.

Proof. 1. is standard and left to the reader.

For 2., we can assume S is reduced. By Deligne's generic ULA theorem, we can choose some dense open regular subscheme $S' \subset S$ such that ${}^{\mathfrak{p}}\mathcal{H}^n(\mathcal{F}|X_{S'})$ is ULA for the morphism $X_{S'} \to S'$ for all $n \in \mathbf{Z}$. This reduces us to showing that for any $s \in S'$, $i^*_s[-\dim_{S'}]$ is t-exact as a functor from $D^b_{c,f-\mathrm{ULA}}(X_{S'})$ to $D^b_c(X_s)$ where the source and target are equipped with the t-structures ${}^{\mathfrak{p}}D$ resp. ${}^{p}D$.[5]Factor i_s as $X_s \overset{u}{\to} X_T \overset{h}{\to} X_{S'}$ where T is the scheme-theoretic closure of s in S' and h is the base change of the closed immersion $T \to S'$. Replacing S' by the connected component containing T and discarding a nowhere-dense closed subset of T, we can assume that T is regular and irreducible, so $T \to S'$ is a regular immersion of some pure codimension c. Note that T inherits a dimension function from S', and that $\dim_T + c = \dim_{S'}|_T$. By this numerology and part 1. applied to u, we are reduced to proving that

$$h^*[-c] : D^b_c(X_{S'}) \to D^b_c(X_T)$$

is perverse t-exact on ULA objects. Using Gabber's theorem on the cohomological dimension of affine morphisms and induction on c, it is an easy exercise to check that $Rh^![c]$ is right perverse t-exact, and hence by duality that $h^*[-c]$ is left perverse t-exact. By Lemma 0.6 below, $h^*\mathcal{G}[-c] \overset{\sim}{\to} Rh^!\mathcal{G}[c](c)$ for f-ULA objects $\mathcal{G} \in D^b_c(X_{S'})$, so the desired t-exactness follows. □

In the previous proof, we used the following crucial lemma, which is essentially due to Lu-Zheng.

Lemma 0.6. *Let $g : T \to S$ be an immersion of regular Noetherian schemes of codimension c. Let $f : X \to S$ be a locally finite type morphism, and let $\mathcal{F} \in D^b(X, \Lambda)$ be a complex which is f-locally acyclic. Then the Gysin map $\tilde{g}^*\mathcal{F}[-2c](-c) \to R\tilde{g}^!\mathcal{F}$ is an isomorphism, where $\tilde{g} : X_T \to X$ is the obvious pullback of g.*

Proof. This follows from Remark 4.7 and Theorem 6.8 in [LZ19]. In particular, an examination of the proof of [LZ19, Theorem 6.8] shows that the assumption of finite tor-dimension in loc. cit. is unnecessary under the stated hypotheses. □

Proposition 0.7. *Let $f : X \to S$ be a finite type map of nice schemes, and suppose S is equipped with a dimension function. Then ${}^{p/S}D^{\leq 0}(X) \subset {}^{\mathfrak{p}}D^{\leq 0}(X)[-\dim_S]$ and ${}^{\mathfrak{p}}D^{\geq 0}(X)[-\dim_S] \subset {}^{p/S}D^{\geq 0}(X)$.*

Moreover, if S is regular there is an inclusion

$${}^{\mathfrak{p}}D^{\leq 0}(X)[-\dim_S] \cap D^b_{c,f-\mathrm{ULA}}(X) \subset {}^{p/S}D^{\leq 0}(X).$$

[4]Use the fact that $f(\mathrm{supp}\, {}^{\mathfrak{p}}\mathcal{H}^n(\mathcal{F})) \subset S$ is constructible for each n.

[5]This is a slight abuse of terminology, since the $\mathfrak{p}$-truncation functors don't obviously preserve the ULA condition (although we will prove later that this actually holds). Of course what we really mean is that $i^*_s[-\dim_{S'}]$ carries $D^b_{c,f-\mathrm{ULA}}(X_{S'}) \cap {}^{\mathfrak{p}}D^{\leq n}(X_{S'})$ into ${}^{p}D^{\leq n}(X_s)$, and likewise on coconnective parts.

Proof. For the first claim, the second inclusion follows formally from the first by taking right orthogonals. Suppose $\mathcal{F} \in {}^{p/S}D^{\leq 0}(X)$ is given, so $\mathcal{H}^n(\mathcal{F}_{\overline{x}}) = 0$ for all $n > -\text{tr.deg}k(x)/k(f(x))$. Then $\mathcal{H}^n(\mathcal{F}_{\overline{x}}[\dim_S]) = 0$ for all

$$n > -\text{tr.deg}k(x)/k(f(x)) - \dim_S(f(x)).$$

Since

$$\begin{aligned}\delta(x) &= \text{tr.deg}k(x)/k(f(x)) + \delta_S(f(x)) \\ &\leq \text{tr.deg}k(x)/k(f(x)) + \dim_S(f(x))\end{aligned}$$

by the definition of $\dim_S$, we get that also $\mathcal{H}^n(\mathcal{F}_{\overline{x}}[\dim_S]) = 0$ for all $n > -\delta(x)$, so then $\mathcal{F}[\dim_S] \in {}^{\mathfrak{p}}D^{\leq 0}(X)$ by definition.

The second claim follows by arguing as in the proof of part 2. of the previous proposition. □

The next goal is Theorem 0.13, which gives a concrete description of the objects in ${}^{p/S}D^{\geq 0}(X)$. In preparation for this, we first show that generically on S, a given object in D^b_c is contained in ${}^{p/S}D^{\geq 0}$ if and only if its pullback to every geometric fiber of $X \to S$ lies in ${}^{p}D^{\geq 0}$. The next two propositions make this statement precise.

Proposition 0.8. *Let $f : X \to S$ be a finite type map of nice schemes, and let $\mathcal{F} \in D^b_c(X)$ be given. If $\mathcal{F}|X_{\overline{s}} \in {}^{p}D^{\geq 0}(X_{\overline{s}})$ for all geometric points $\overline{s} \to S$, there is a dense open $S' \subset S$ such that $\mathcal{F}|X_{S'} \in {}^{p/S}D^{\geq 0}(X_{S'})$.*

Proof. The assumption applied at the generic points of S together with the generic amplitude principle implies that $(\mathcal{F}|X_{S'})[\dim_{S'}] \in {}^{\mathfrak{p}}D^{\geq 0}(X_{S'})$ for some dense open really nice $S' \subset S$. The result now follows from the previous proposition. □

Proposition 0.9. *Let $f : X \to S$ be a finite type map of nice schemes, and let $\mathcal{F} \in D^b_c(X) \cap {}^{p/S}D^{\geq 0}(X)$ be given. Then there is a dense open subset $S' \subset S$ such that $\mathcal{F}|X_{\overline{s}} \in {}^{p}D^{\geq 0}(X_{\overline{s}})$ for all geometric points $\overline{s} \to S'$.*

Proof. We can assume S reduced. By the generic amplitude principle, it's enough to show that $i^*_\eta\mathcal{F} \in {}^{p}D^{\geq 0}(X_\eta)$ for all generic points $\eta \in S$. To verify this, it suffices to check that $R\mathscr{H}\text{om}(\mathcal{G}, i^*_\eta\mathcal{F}) \in D^{\geq 1}(X_\eta)$ for all $\mathcal{G} \in {}^{p}D^{\leq -1}(X_\eta) \cap D^b_c$. We can assume that $\mathcal{G}$ is a shifted perverse sheaf, and then (filtering $\mathcal{G}$) that $\mathcal{G}$ is a shifted IC sheaf. By the definition of IC sheaves, any such $\mathcal{G}$ spreads out to some $\mathcal{G}' \in D^b_c(X_U)$ over a small open neighborhood $j : U \to S$ of η. By the generic amplitude principle and Deligne's generic ULA theorem, we can assume (after possibly shrinking U further) that U is regular and connected, $\mathcal{G}' \in {}^{\mathfrak{p}}D^{\leq -1}(X_U)[-\dim_U]$, and that $\mathcal{G}'$ is ULA for the morphism $X_U \to U$. By Proposition 0.7, this implies that $\mathcal{G}' \in {}^{p/S}D^{\leq -1}(X_U)$.

Since $X_\eta \to X_U$ is a regular morphism, $R\mathscr{H}\text{om}(\mathcal{G}, i^*_\eta\mathcal{F})$ is the pullback of $R\mathscr{H}\text{om}(\mathcal{G}', \mathcal{F}|X_U)$, so it's enough to show that $R\mathscr{H}\text{om}(\mathcal{G}', \mathcal{F}|X_U) \in D^{\geq 1}(X_U)$. Since $\mathcal{G}' \in {}^{p/S}D^{\leq -1}(X_U)$ and $\mathcal{F}|X_U \in {}^{p/S}D^{\geq 0}(X_U)$, so this follows from the subsequent lemma. □

Lemma 0.10. *Let $f : X \to S$ be a finite type morphism of arbitrary schemes, and let $\mathcal{G} \in {}^{p/S}D^{\leq -1}(X)$ and $\mathcal{F} \in {}^{p/S}D^{\geq 0}(X)$ be any objects. Then $R\mathscr{H}\text{om}(\mathcal{G}, \mathcal{F}) \in D^{\geq 1}(X)$.*

Proof. For any $n \geq 0$ and any étale map $j : U \to X$, we have

$$\begin{aligned}H^{-n}(R\Gamma(U, j^*R\mathscr{H}\text{om}(\mathcal{G}, \mathcal{F}))) &= \text{Hom}_{D(U)}(j^*\mathcal{G}, j^*\mathcal{F}[-n]) \\ &= \text{Hom}_{D(U)}(X, Y)\end{aligned}$$

for some $X \in {}^{p/S}D^{\leq -1}(U)$ and $Y \in {}^{p/S}D^{\geq n}(U)$, using that j^* is relative perverse t-exact. Then $\mathrm{Hom}_{D(U)}(X, Y) = 0$, and the result follows. □

We now have the following result, which is the key technical ingredient in the characterization of $D^b_c \cap {}^{p/S}D^{\geq 0}$ proved below.

Theorem 0.11. *Let S be a nice scheme. Let $f : X \to S$ be a finite type map, and let $\mathcal{F} \in D^b_c(X)$ be any object. Then the following are equivalent.*

1) *There is a dense open $S' \subset S$ such that $\mathcal{F}|X_{S'} \in {}^{p/S}D^{\geq 0}(X_{S'})$.*

2) *There is a really nice dense open $S'' \subset S$ with S''^{red} normal such that $\mathcal{F}|X_{S''} \in {}^{\mathfrak{p}}D^{\geq 0}(X_{S''})[-\dim_{S''}]$.*

3) *There is a dense open $S''' \subset S$ such that $\mathcal{F}|X_{\overline{s}} \in {}^{p}D^{\geq 0}(X_{\overline{s}})$ for all geometric points $\overline{s} \to S'''$.*

Moreover, if any of these conditions holds, there is a dense open immersion $j : U \to S$ with closed complement $i : Z \to S$ such that $\tilde{i}^* R\tilde{j}_*\tilde{j}^*\mathcal{F} \in {}^{p/S}D^{\geq 0}(X_Z)$, where $\tilde{i} : X_Z \to X$ and $\tilde{j} : X_U \to X$ are the obvious base changes.

Note that the appropriate dense open subsets in 1)-3) and in the conclusion may differ. However, the proof will show that in passing from 2) to 1) we can take $S' = S''$, and in passing to the conclusion we can take $U = S''$. The argument also shows that we can replace each of S', S'', S''', or U with any dense open subset thereof without affecting the truth of 1), 2), 3), or the conclusion (respectively). In particular, over the dense open $S'' \cap S'''$, conditions 1)-3) and the conclusion all hold simultaneously.

Proof. 2) implies 1). This is the first half of Proposition 0.7.

1) implies 3). This is Proposition 0.9.

3) implies 2). This is the generic amplitude principle, combined with the density of the normal locus in S^{red}.

Finally, we prove that 2) implies the conclusion, with $U = S''$ (or any dense open subset thereof). First, we can assume S reduced. We can then replace S by its normalization; since this doesn't change U, the justification for this is an easier variant of the proper base change argument in the next paragraph, and I omit it. We can now work one irreducible component of S at a time, so we can assume that S is irreducible and admits a dimension function.

Next, let $b : \check{S} \to S$ be the blowup of Z in S, so $\check{S}$ is irreducible and admits a dimension function, and $\check{Z} = b^{-1}(Z) \subset \tilde{S}$ is an effective Cartier divisor. In particular, $U = b^{-1}(U) \to \check{S}$ is an affine open immersion. We get an obvious diagram of base changes

$$\begin{array}{ccccc} X_{\check{Z}} & \xrightarrow{\check{i}} & X_{\check{S}} & \xleftarrow{\check{j}} & X_U \\ \downarrow h & & \downarrow g & \swarrow \tilde{j} & \downarrow \wr \\ X_Z & \xrightarrow{\tilde{i}} & X & \xleftarrow{\tilde{j}} & X_U \end{array}$$

with $\tilde{j}$ affine. By proper base change, there is an isomorphism

$$\tilde{i}^* R\tilde{j}_*\tilde{j}^*\mathcal{F} \cong \tilde{i}^* Rg_* R\check{j}_*\tilde{j}^*\mathcal{F} \cong Rh_*\check{i}^* R\check{j}_*\tilde{j}^*\mathcal{F}.$$

Since h is obtained via base change from the morphism $\check{Z} \to Z$, Rh_* is left t-exact for the relative perverse t-structures, so it suffices to show that

$$\check{i}^* R\check{j}_*\tilde{j}^*\mathcal{F} \in {}^{p/\check{Z}}D^{\geq 0}(X_{\check{Z}}) = {}^{p/\check{S}}D^{\geq 0}(X_{\check{Z}}).$$

In other words, we have reduced the general case of "2) implies the conclusion" to the special case where S is irreducible and admits a dimension function, and U is the complement of an effective Cartier divisor Z. For the remainder of the proof, we put ourselves in that case (and revert to our usual notation).

Note that Z and U inherit dimension functions from S, and that $\dim_Z < \dim_S$ by the density of U, so trivially we get an inclusion

$$^{\mathfrak{p}}D^{\geq 0}(X_Z)[-\dim_S + 1] \subseteq {}^{\mathfrak{p}}D^{\geq 0}(X_Z)[-\dim_Z].$$

By Proposition 0.7, there is also an inclusion $^{\mathfrak{p}}D^{\geq 0}(X_Z)[-\dim_Z] \subseteq {}^{p/S}D^{\geq 0}(X_Z)$, so we get a composite inclusion

$$^{\mathfrak{p}}D^{\geq 0}(X_Z)[-\dim_S + 1] \subseteq {}^{p/S}D^{\geq 0}(X_Z).$$

Therefore, starting with any $\mathcal{F}$ as in 2) (and with $S \smallsetminus U$ an effective Cartier divisor), the conclusion of the theorem follows if we can prove that

$$\tilde{i}^* R\tilde{j}_* : D^b_c(X_U) \to D^b_c(X_Z)$$

carries $^{\mathfrak{p}}D^{\geq 0}(X_U)[-\dim_S]$ into $^{\mathfrak{p}}D^{\geq 0}(X_Z)[-\dim_S + 1]$, or equivalently that it carries $^{\mathfrak{p}}D^{\geq 0}(X_U)$ into $^{\mathfrak{p}}D^{\geq -1}(X_Z)$. Since $X_Z \subset X$ is an effective Cartier divisor, this follows from the next lemma. □

Lemma 0.12. *Let X be a nice scheme equipped with a dimension function. Let $j : U \to X$ be an open immersion which is the complement of an effective Cartier divisor $i : Z \to X$. Then $i^* Rj_* : D^b_c(U) \to D^b_c(Z)$ has perverse cohomological amplitude $[-1, 0]$, i.e. it is perverse right t-exact and carries $^{\mathfrak{p}}D^{\geq 0}(U)$ into $^{\mathfrak{p}}D^{\geq -1}(Z)$.*

Proof. By Gabber's results on biduality and his Artin-Grothendieck theorem for affine morphisms of nice schemes, Rj_* and $j_!$ are perverse t-exact. Then i^* has perverse cohomological amplitude $[-1, 0]$ by looking at the triangle $j_! j^* \to \mathrm{id} \to i_* i^* \to$ where the other functors are perverse t-exact. □

Remark. The numerology in this lemma might seem inconsistent with the more familiar result that if S is a Henselian DVR with generic and special point $\eta, s \in S$ (equipped with the canonical dimension function sending s to 0) and $f : X \to S$ is of finite type, then the nearby cycles $R\psi : D^b_c(X_\eta) \to D^b_c(X_{\overline{s}})$ are perverse t-exact and $\tilde{i}^* R\tilde{j}_* : D^b_c(X_\eta) \to D^b_c(X_s)$ has perverse cohomological amplitude $[0, 1]$. The thing to remember is that a shift intervenes when gluing $^{\mathfrak{p}}D^{\leq 0}(X)$ from data on the special and generic fibers: $^{\mathfrak{p}}D^{\leq 0}(X)$ is glued from $^{p}D^{\leq 0}(X_s)$ and $^{p}D^{\leq -1}(X_\eta)$. In other words, $^{\mathfrak{p}}D^{\geq 0}(X_\eta) = {}^{p}D^{\geq -1}(X_\eta)$, so the lemma is consistent with this result after accounting for this shift.

Granted this result, we get the following characterization of bounded constructible objects in $^{p/S}D^{\geq 0}(X)$.

Theorem 0.13. *Let $f : X \to S$ be a finite type map of nice schemes, and let $\mathcal{F} \in D(X)$ be any object. Then $\mathcal{F} \in {}^{p/S}D^{\geq 0}(X)$ if and only if $\mathcal{F}|X_{\overline{s}} \in {}^{p}D^{\geq 0}(X_{\overline{s}})$ for all geometric points $\overline{s} \to S$.*

Proof. We first prove the theorem for objects $\mathcal{F} \in D^b_c(X)$, by simultaneous induction on $\dim S$. When $\dim S = 0$ the result is clear. Suppose the result is known for all base schemes of dimension $\leq n - 1$. Let S be a nice scheme of dimension n.

"Only if." Suppose given $\mathcal{F} \in {}^{p/S}D^{\geq 0}(X) \cap D^b_c$. By Theorem 0.11, we can choose a dense open $j : U \to S$ with closed complement $i : Z \to S$ such that $\tilde{j}^* \mathcal{F}$ is fiberwise perverse coconnective and $\tilde{i}^* R\tilde{j}_*\tilde{j}^*\mathcal{F}$ is relatively perverse coconnective. By the induction hypothesis, $\tilde{i}^* R\tilde{j}_*\tilde{j}^*\mathcal{F}$ is also fiberwise perverse coconnective. Now look at the triangle

$$R\tilde{i}^!\mathcal{F} \to \tilde{i}^*\mathcal{F} \to \tilde{i}^* R\tilde{j}_*\tilde{j}^*\mathcal{F} \to .$$

Then $R\tilde{i}^!\mathcal{F}$ is relatively perverse coconnective, so again by the induction hypothesis $R\tilde{i}^!\mathcal{F}$ is also fiberwise perverse coconnective. Therefore $\tilde{i}^*\mathcal{F}$ is fiberwise perverse coconnective. Since $\tilde{j}^*\mathcal{F}$ is also fiberwise perverse coconnective, we deduce that $\mathcal{F}$ is fiberwise perverse connective.

"If." Suppose that $\mathcal{F} \in D_c^b$ is fiberwise perverse coconnective. By Theorem 0.11, we can choose a dense open $j : U \to S$ with closed complement $i : Z \to S$ such that $\tilde{j}^*\mathcal{F}$ is relatively perverse coconnective and $\tilde{i}^* R\tilde{j}_*\tilde{j}^*\mathcal{F}$ is relatively perverse coconnective. Again, look at the triangle

$$R\tilde{i}^!\mathcal{F} \to \tilde{i}^*\mathcal{F} \to \tilde{i}^* R\tilde{j}_*\tilde{j}^*\mathcal{F} \to .$$

By assumption, the middle term is fiberwise perverse coconnective, hence also relatively perverse coconnective by the induction hypothesis. Therefore the middle and right terms are relatively perverse coconnective, so $R\tilde{i}^!\mathcal{F}$ is relatively perverse coconnective. Since $\tilde{i}_*$ is t-exact and $R\tilde{j}_*$ is left t-exact for the relative perverse t-structures, we now see that the outer terms in the triangle

$$\tilde{i}_* R\tilde{i}^!\mathcal{F} \to \mathcal{F} \to R\tilde{j}_*\tilde{j}^*\mathcal{F} \to$$

are relatively perverse coconnective, so $\mathcal{F}$ is relatively perverse coconnective as desired.

Finally, we extend the theorem to arbitrary objects of $D(X)$. Suppose given some $\mathcal{F} \in {}^{p/S}D^{\geq 0}(X)$. Since ${}^{p/S}D^{\geq 0}(X)$ is stable under filtered colimits, any $\mathcal{F} \in {}^{p/S}D^{\geq 0}(X)$ can be written as a filtered colimit $\mathrm{colim}\mathcal{F}_i$ for some $\mathcal{F}_i \in {}^{p/S}D^{\geq 0}(X) \cap D_c^b$. Then each $\mathcal{F}_i$ is fiberwise perverse coconnective by the bounded constructible case of the theorem, so $\mathcal{F}$ is fiberwise perverse coconnective.

Conversely, suppose $\mathcal{F} \in D(X)$ is fiberwise perverse coconnective. Pick an isomorphism $\mathcal{F} = \mathrm{colim}_i\mathcal{F}_i$ with $\mathcal{F}_i \in D_c^b(X)$, so we get a map

$$\alpha : \mathcal{F} = \mathrm{colim}_i\mathcal{F}_i \to \mathrm{colim}_i {}^{p/S}\tau^{\geq 0}\mathcal{F}_i.$$

Note that $\mathrm{colim}_i {}^{p/S}\tau^{\geq 0}\mathcal{F}_i \in {}^{p/S}D^{\geq 0}(X)$ since ${}^{p/S}D^{\geq 0}(X)$ is stable under colimits. It thus suffices to show that α is an isomorphism. This can be checked on geometric fibers of f. But for any geometric point $\overline{s} \to S$,

$$(\mathrm{colim}_i {}^{p/S}\tau^{\geq 0}\mathcal{F}_i)|X_{\overline{s}} \cong \mathrm{colim}_i {}^{p}\tau^{\geq 0}(\mathcal{F}_i|X_{\overline{s}}) \cong {}^{p}\tau^{\geq 0}(\mathrm{colim}_i\mathcal{F}_i|X_{\overline{s}}) = {}^{p}\tau^{\geq 0}(\mathcal{F}|X_{\overline{s}}) = \mathcal{F}|X_{\overline{s}},$$

using the fact that the relative perverse truncations of $\mathcal{F}_i$ commute with base change on S. This gives the result. □

This implies some much stronger t-exactness results.

Proposition 0.14. *Let $f : X \to S$ be a finite type map of nice schemes. Then we have the following t-exactness properties for the relative perverse t-structures.*

1) *For any morphism $g : T \to S$ with base change $\tilde{g} : X_T \to X$, $\tilde{g}^*$ is t-exact. In particular, the relative perverse truncation functors commute with any base change on S.*

2) *If $h : Y \to X$ is affine and quasifinite, $h_!$ is t-exact. If $h : Y \to X$ is quasifinite, $h_!$ is right t-exact.*

3) *If $h : Y \to X$ is smooth of pure relative dimension d, $h^*[d]$ is t-exact.*

Proof. Immediate from the previous result. □

Proposition 0.15. *Let $f : X \to S$ be a finite type map of nice schemes. Then $\mathcal{F} \in D^b_c(X)$ lies in ${}^{p/S}D^{\geq 0}(X)$ if and only if for all étale maps $j : U \to X$ such that $f \circ j$ is affine, we have $R(f \circ j)_! j^* \mathcal{F} \in D^{\geq 0}(S)$.*

Proof. By an easy spreading out argument, this reduces (via Theorem 0.13) to the following statement, which follows by duality from [BBDG, Reciproque 4.1.6]: for X of finite type over a separably closed field k, $\mathcal{F} \in D^b_c(X)$ lies in ${}^pD^{\geq 0}(X)$ iff $R\Gamma_c(U, \mathcal{F}) \in D^{\geq 0}(\Lambda)$ for all open affines $U \subseteq X$. □

Proposition 0.16. *Let $f : X \to S$ be a finite type map of nice schemes. Then the relative duality functor $\mathbf{D}_{X/S} = R\mathscr{H}\mathrm{om}(-, Rf^! \Lambda)$ induces a contravariant equivalence of categories between ${}^{p/S}D^{\geq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}$ and ${}^{p/S}D^{\leq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}$, and satisfies biduality on each of these categories.*

Proof. By [LZ20, Corollary 2.27], $\mathbf{D}_{X/S}$ preserves f-ULA objects, and on f-ULA objects it commutes with any pullback on S. If $\mathcal{F} \in {}^{p/S}D^{\geq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}$ is given, we then compute that

$$i^*_{\overline{s}} \mathbf{D}_{X/S} \mathcal{F} = \mathbf{D}_{X_{\overline{s}}}(\mathcal{F}|X_{\overline{s}}) \in {}^pD^{\leq 0}(X_{\overline{s}})$$

for all geometric points $\overline{s} \to S$, using Theorem 0.13 to see that $\mathcal{F}|X_{\overline{s}}$ is perverse coconnective. This shows that $\mathbf{D}_{X/S}\mathcal{F}$ is relatively perverse connective. An identical argument handles the opposite case. Finally, one checks that $\mathbf{D}_{X/S}$ satisfies biduality on $D^b_{c,f-\mathrm{ULA}}$: this can be checked after pullback to any geometric point by [LZ20, Corollary 2.27], so it reduces to usual biduality for varieties. □

Proposition 0.17. *Let $f : X \to S$ be a finite type map of nice schemes. Then ${}^{p/S}D^{\geq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}(X)$ and ${}^{p/S}D^{\geq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}(X)$ define a t-structure on $D^b_{c,f-\mathrm{ULA}}(X)$. In particular, if $\mathcal{F} \in D^b_c(X)$ is f-ULA, then any relative perverse truncation of $\mathcal{F}$ is also f-ULA.*

Proof. Fix an f-locally acyclic object $\mathcal{F} \in D^b_c(X)$. By [LZ19, Theorem 6.5 and Remark 6.7(1)], it suffices to show that relative perverse truncations of $\mathcal{F}$ are f-locally acyclic after base change along all maps $T \to S$ where T is the spectrum of a strictly Henselian DVR. Since relative perverse truncations commute with arbitrary base change, this reduces us to the case where S is a strictly Henselian DVR. In this situation, universal local acyclicity of $\mathcal{G} \in D^b_c(X)$ is equivalent to the vanishing of the vanishing cycles $R\Phi(\mathcal{G})$.

Fix such an S, with special and generic points $s, \eta \in S$, and equipped with the canonical dimension function sending s to 0. By a theorem of Gabber [I, Corollaire 4.6], the functor $R\Phi[-1] : D^b_c(X) \to D^b_c(X_s)$ is perverse t-exact. In particular, since the perverse truncation functors commute with $R\Phi$ (up to shift), the perverse truncation functors preserve the property of being f-ULA. In other words, ${}^{\mathfrak{p}}D^{\leq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}(X)$ and ${}^{\mathfrak{p}}D^{\geq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}(X)$ define a t-structure on $D^b_{c,f-\mathrm{ULA}}(X)$. Now, observe that

$${}^{\mathfrak{p}}D^{\leq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}(X) = {}^{p/S}D^{\leq -1}(X) \cap D^b_{c,f-\mathrm{ULA}}(X)$$

and ${}^{\mathfrak{p}}D^{\geq 0}(X) \subset {}^{p/S}D^{\geq -1}(X)$ by Proposition 0.7. Applying Lemma 0.18 below with $\mathcal{D} = D(X)$, $\mathcal{D}' = D^b_{c,f-\mathrm{ULA}}(X)$, $\mathcal{C}^{\leq 0}_1 = {}^{\mathfrak{p}}D^{\leq -1}(X)$, and $\mathcal{C}^{\leq 0}_2 = {}^{p/S}D^{\leq 0}(X)$, we deduce that ${}^{p/S}D^{\leq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}(X)$ and ${}^{\mathfrak{p}}D^{\leq -1}(X) \cap D^b_{c,f-\mathrm{ULA}}(X)$ define (the connective parts of) the same t-structure on $D^b_{c,f-\mathrm{ULA}}(X)$. In particular, the truncation functors ${}^{\mathfrak{p}}\tau^{\leq n-1}$ and ${}^{p/S}\tau^{\leq n}$ coincide on $D^b_{c,f-\mathrm{ULA}}(X)$, so the relative perverse truncation functors preserve $D^b_{c,f-\mathrm{ULA}}(X)$ as required. This gives the claim. □

Lemma 0.18. *Let $\mathcal{D}$ be a triangulated category, let $\mathcal{D}' \subset \mathcal{D}$ be a thick triangulated subcategory, and let $\mathcal{C}_1 = (\mathcal{C}_1^{\leq 0}, \mathcal{C}_1^{\geq 0})$ and $\mathcal{C}_2 = (\mathcal{C}_2^{\leq 0}, \mathcal{C}_2^{\geq 0})$ be two t-structures on $\mathcal{D}$. Suppose that $\mathcal{C}_1 \cap \mathcal{D}'$ defines a t-structure on $\mathcal{D}'$, and that $\mathcal{C}_1^{\leq 0} \cap \mathcal{D}' = \mathcal{C}_2^{\leq 0} \cap \mathcal{D}'$ and $\mathcal{C}_1^{\geq 0} \cap \mathcal{D}' \subset \mathcal{C}_2^{\geq 0} \cap \mathcal{D}'$.*

Then $\mathcal{C}_2 \cap \mathcal{D}'$ defines a t-structure on $\mathcal{D}'$ which coincides with the t-structure $\mathcal{C}_1 \cap \mathcal{D}'$.

The final condition holds e.g. if $\mathcal{C}_1^{\leq 0} \subset \mathcal{C}_2^{\leq 0}$.

Proof. For any $X \in \mathcal{D}'$, there are natural maps $\tau_2^{\leq n} X \leftarrow \tau_2^{\leq n} \tau_1^{\leq n} X \rightarrow \tau_1^{\leq n} X$. Since $\tau_1^{\leq n} X \in \mathcal{C}_1^{\leq n} \cap \mathcal{D}' = \mathcal{C}_2^{\leq n} \cap \mathcal{D}'$ by assumption, the second arrow is an isomorphism. The cone of the first arrow is $\tau_2^{\leq n} \tau_1^{> n} X$, which vanishes since $\tau_1^{> n} X \in \mathcal{C}_1^{> n} \cap \mathcal{D}' \subset \mathcal{C}_2^{> n} \cap \mathcal{D}' \subset \mathcal{C}_2^{> n}$ by assumption. Therefore both arrows are isomorphisms, so the truncation functors $\tau_2^{\leq n}$ preserve $\mathcal{D}'$ and coincide with $\tau_1^{\leq n}$ on objects of $\mathcal{D}'$, as required. □

Theorem 0.19. *Let $f : X \to S$ be a finite type map of nice schemes, with S regular. Then ${}^{\mathfrak{p}}D^{\leq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}$ and ${}^{\mathfrak{p}}D^{\geq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}$ define a t-structure on $D^b_{c,f-\mathrm{ULA}}(X)$ which agrees up to shift with the relative perverse t-structure on $D^b_{c,f-\mathrm{ULA}}(X)$ (which makes sense by Proposition 0.17). In particular, if $\mathcal{F} \in D^b_c(X)$ is f-ULA, then all perverse cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^n(\mathcal{F})$ are also f-ULA.*

The final claim here reproves and generalizes a theorem of Gaitsgory, who proved this when S is a smooth variety [Gai, Theorem F].

Proof. We want to apply Lemma 0.18, with $\mathcal{D} = D(X)$, $\mathcal{D}' = D^b_{c,f-\mathrm{ULA}}(X)$, and the t-structures with connective parts $\mathcal{C}_1^{\leq 0} = {}^{p/S}D^{\leq 0}(X)$ and $\mathcal{C}_2^{\leq 0} = {}^{\mathfrak{p}}D^{\leq 0}(X)[-\mathrm{dim}_S]$. By Proposition 0.17, $\mathcal{C}_1 \cap \mathcal{D}'$ defines a t-structure on $\mathcal{D}'$, and $\mathcal{C}_1^{\leq 0} \cap \mathcal{D}' = \mathcal{C}_2^{\leq 0} \cap \mathcal{D}'$ by Proposition 0.7. It remains to see that $\mathcal{C}_1^{\geq 0} \cap \mathcal{D}' \subset \mathcal{C}_2^{\geq 0} \cap \mathcal{D}'$, i.e. that

$$ {}^{p/S}D^{\geq 0}(X) \cap D^b_{c,f-\mathrm{ULA}}(X) \subset {}^{\mathfrak{p}}D^{\geq 0}(X)[-\mathrm{dim}_S] \cap D^b_{c,f-\mathrm{ULA}}(X). $$

However, we know that $\mathbf{D}_{X/S}$ defines a contravariant equivalence from $\mathcal{C}_1^{\leq 0} \cap \mathcal{D}'$ to $\mathcal{C}_1^{\geq 0} \cap \mathcal{D}'$ by Proposition 0.16. We also know that $\mathbf{D}_X = \mathbf{D}_{X/S}[2\mathrm{dim}_S]$ defines a contravariant equivalence from ${}^{\mathfrak{p}}D_c^{\leq 0}(X)$ to ${}^{\mathfrak{p}}D_c^{\geq 0}(X)$; after accounting for the various shifts, this means that $\mathbf{D}_{X/S}$ gives a contravariant equivalence from $\mathcal{C}_2^{\leq 0} \cap \mathcal{D}'$ to $\mathcal{C}_2^{\geq 0} \cap \mathcal{D}'$. Therefore, applying $\mathbf{D}_{X/S}$ to $\mathcal{C}_1^{\leq 0} \cap \mathcal{D}' = \mathcal{C}_2^{\leq 0} \cap \mathcal{D}'$ gives $\mathcal{C}_2^{\geq 0} \cap \mathcal{D}' = \mathcal{C}_1^{\geq 0} \cap \mathcal{D}'$, which is more than we require.

Applying Lemma 0.18 and Proposition 0.17, we deduce that the relative perverse t-structure on $D^b_c(X)$ restricts to a t-structure on $D^b_{c,f-\mathrm{ULA}}(X)$ which coincides (up to shift) with the restriction of the absolute perverse t-structure. In particular, the truncation functors ${}^{\mathfrak{p}}\tau^{\leq n}$ preserve $D^b_{c,f-\mathrm{ULA}}(X)$, which is what we wanted to show. □

Proposition 0.20. *Let S be a nice scheme equipped with a dimension function, and let $f : X \to S$ be a finite type map. For any $\mathcal{F} \in D^b_c(X)$, the following are equivalent.*

1) *$\mathcal{F}$ is relatively perverse.*

2) *There is a finite stratification by locally closed regular subschemes $S = \coprod S_i$ such that $(\mathcal{F}|X_i)[\mathrm{dim}_{S_i}]$ is perverse and ULA for the morphism $X_i = X \times_S S_i \to S_i$.*

Proof. Exercise. □

Proposition 0.21. *Let $f : X \to S$ be a finite type map of nice schemes, with S regular and equipped with a dimension function. Let $\mathcal{F} \in \mathrm{Perv}_{f-\mathrm{ULA}}(X)$ be any object. Let $j : U \to S$ be the complement of a nowhere-dense closed subscheme $Z \subset S$, with base change $\tilde{j} : X_U \to X$. Then $\mathcal{F} \cong \tilde{j}_{!*}\tilde{j}^*\mathcal{F}$. If moreover Z has codimension ≥ 2, then $\mathcal{F} \cong {}^{\mathfrak{p}}\mathcal{H}^0(\tilde{j}_!\tilde{j}^*\mathcal{F}) \cong {}^{\mathfrak{p}}\mathcal{H}^0(R\tilde{j}_*\tilde{j}^*\mathcal{F})$.*

When S and D are smooth varieties, this was proved by Reich in his thesis.

The argument only depends on the ULA condition in a mild way. In fact, the argument shows that any $\mathcal{F} \in D^b_c(X)$ such that $\mathcal{F}[-\dim_S]$ and $\mathbf{D}_{X/S}(\mathcal{F}[-\dim_S])$ are relatively perverse connective is automatically perverse and satisfies the conclusion of the Proposition.

Proof. Let $\tilde{i} : X_Z \to X$ be the base change of the evident closed immersion $i : Z \to S$. It's enough to show that the nonzero perverse cohomology sheaves of $\tilde{i}^*\mathcal{F}$ resp. $R\tilde{i}^!\mathcal{F}$ are concentrated in degrees ≤ -1 (resp. ≥ 1), with strict inequalities if Z has codimension ≥ 2. To begin, note that $\mathcal{F}[-\dim_S]$ is relatively perverse, so $\tilde{i}^*\mathcal{F}[-\dim_S]$ is relatively perverse, and thus $\tilde{i}^*\mathcal{F}[-\dim_S] \in {}^{\mathfrak{p}}D^{\leq 0}(X_Z)[-\dim_Z]$. Therefore $\tilde{i}^*\mathcal{F}[\dim_Z - \dim_S]$ has vanishing perverse cohomology in degrees ≥ 1, so $\tilde{i}^*\mathcal{F}$ has vanishing perverse cohomology in degrees $\geq 1 + \dim_Z - \dim_S$. Since Z is nowhere-dense, $1 + \dim_Z - \dim_S \leq 0$ with strict inequality if Z has codimension ≥ 2. Therefore $\tilde{i}^*\mathcal{F}$ has vanishing perverse cohomology in degrees ≥ 0, and in degrees ≥ -1 when Z has codimension ≥ 2, as required.

To analyze $R\tilde{i}^!\mathcal{F}$, use biduality to write $R\tilde{i}^!\mathcal{F} = \mathbf{D}_{X_Z}\tilde{i}^*\mathbf{D}_X\mathcal{F}$. Then we understand $\tilde{i}^*\mathbf{D}_X\mathcal{F}$ by the argument in the previous paragraph, which implies the necessary concentration of $R\tilde{i}^!\mathcal{F}$ by duality. □

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