Period morphisms and variation of p-adic Hodge structure (preliminary draft)

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Abstract

Given a de Rham $\mathbf{Q}_p$-local system on a connected smooth rigid analytic variety X, we use p-adic Hodge theory to define canonical p-adic period morphisms out of a suitable covering space of X. This construction is exactly analogous with the period morphisms of classical Hodge theory, and it gives a general conceptual framework for Scholze's Hodge-Tate period map. In particular, we construct the Hodge-Tate period map for any Shimura variety.

Our main tools are relative p-adic Hodge theory and its formalization via the pro-étale site, together with the theory of diamonds.

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1 Introduction

Let X be a connected complex manifold, and let $H = (H_{\mathbf{Z}}, \mathrm{Fil}^{\bullet} \subset H_{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O}_X)$ be a rank n variation of $\mathbf{Z}$-Hodge structure over X. Let $\tilde{X}$ be the $\mathrm{GL}_n(\mathbf{Z})$-covering of X parametrizing trivializations $\alpha : \underline{\mathbf{Z}}^n \overset{\sim}{\to} H_{\mathbf{Z}}$, and let $\mathrm{Fl}_{\mathbf{h}} = P_{\mathbf{h}}(\mathbf{C}) \backslash \mathrm{GL}_n(\mathbf{C})$ be the variety parametrizing flags in $\mathbf{C}^n$ with successive graded pieces having ranks $\mathbf{h} = \{h^i = \mathrm{rank}\, \mathrm{Fil}^i / \mathrm{Fil}^{i+1}\}$. Then we have a natural *period map* $\pi : \tilde{X} \to \mathrm{Fl}_{\mathbf{h}}$, sitting in a $\mathrm{GL}_n(\mathbf{Z})$-equivariant diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] \arrow[d] & \mathrm{Fl}_{\mathbf{h}} \\ X & \end{tikzcd}$$

of complex manifolds. These period maps have been the subject of intense and fruitful study: among many other results, we mention Griffiths's famous transversality theorem constraining the image of π [Gri70], and Schmid's profound study of the degeneration of π near the boundary of $\tilde{X}$ [Sch73].

Until recently, p-adic Hodge theory has featured a wide variety of p-adic periods, but not so many period morphisms.[1] The situation changed dramatically with Scholze's discovery [Sch15b] of the Hodge-Tate period map for Hodge type Shimura varieties (which was then refined in [CS15]). More precisely, let $\mathcal{S}_{K^pK_p}$ be a Hodge-type Shimura variety at some level K^pK_p considered as a rigid analytic space,[2] and let $\mathcal{S}_{K^p}$ be the associated perfectoid Shimura variety with infinite level at p [Sch15b]. The Hodge-Tate period map is a $\mathbf{G}(\mathbf{Q}_p)$-equivariant morphism $\pi_{\mathrm{HT}} : \mathcal{S}_{K^p} \to \mathscr{F}\!\ell_{\mathbf{G},\mu}$, which again sits in an equivariant diagram

$$\begin{tikzcd} \mathcal{S}_{K^p} \arrow[r, "\pi_{\mathrm{HT}}"] \arrow[d] & \mathscr{F}\!\ell_{\mathbf{G},\mu} \\ \mathcal{S}_{K^pK_p} & \end{tikzcd}$$

[1] The notable exception here being the Grothendieck-Messing crystalline period maps defined and studied in [GH94] and then vastly generalized in [RZ96]. However, these period maps differ rather significantly from those studied in classical Hodge theory and in this paper: they only exist in very specific situations, they depend on noncanonical choices (namely, a horizontal trivialization of some F-isocrystal), and they don't utilize Fontaine's p-adic period rings in their constructions.

[2] Cf. §1.2 for an explanation of the notation here.

of adic spaces over $\mathbf{Q}_p$. This map is a p-adic analogue of the Borel embedding $X \hookrightarrow X^c$ of a Hermitian symmetric space into its compact dual. Note that $X^c = \mathrm{Fl}_{\mathbf{G},\mu}(\mathbf{C})$ is a flag variety and the Borel embedding is really a special case of the archimedean period map π considered above.[3] This suggests the natural question of whether $\mathcal{S}_{K^p}$ and π_{HT} are also special cases of some general p-adic construction.

In this article, we answer this question affirmatively. As a particular corollary of our results, we obtain the existence of the Hodge-Tate period map for any Shimura variety. Along the way, we prove a number of results about the geometry of $\mathbf{Q}_p$-local systems (and their analogues with $\mathbf{G}$-structure) on p-adic analytic spaces.

First of all, if X is any analytic adic space and $\mathbf{V}$ is a $\mathbf{Q}_p$-local system of rank n on X, we construct a space $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ over X parametrizing trivializations of $\mathbf{V}$. We'd like to say that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is a $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor over X, but unfortunately it's not clear whether $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is even a reasonable adic space in general. The correct setting here seems to be that of *diamonds* [Sch14, SW16]: we prove that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is naturally a diamond, and that its natural map to the diamond $X^\diamondsuit$ associated with X is a pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor. The necessity of changing to a "looser" category of geometric objects here seems roughly analogous to the fact that, in the complex analytic situation above, $\tilde{X} \to X$ is defined only in the complex analytic category even when X begins its life as a projective variety. We also take $\mathbf{G}$-structures into account from the beginning: we define and study $\mathbf{G}(\mathbf{Q}_p)$-local systems uniformly for any reductive group $\mathbf{G}/\mathbf{Q}_p$, and we prove that the associated space $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is a diamond and a well-behaved pro-étale $\mathbf{G}(\mathbf{Q}_p)$-torsor over $X^\diamondsuit$. These spaces recover both perfectoid Shimura varieties (cf. §1.2) and the moduli spaces of local shtukas introduced in [Sch14] (cf. Remark 4.15) as special cases.

When X is a smooth rigid space and $\mathbf{V}$ is a *Hodge-Tate* local system, we construct a canonical *Hodge-Tate period morphism* π_{HT} from $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ to a flag variety $\mathscr{F}\ell^\diamondsuit_{\mathrm{GL}_n,\mathbf{h}}$ (where $\mathbf{h}$ again records the Hodge numbers of $\mathbf{V}$). In fact, when $\mathbf{V}$ is de Rham, we go beyond Hodge-Tate periods and construct a *de Rham period morphism* π_{dR} from $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ to an open Schubert cell $\mathrm{Gr}_{\mathrm{GL}_n,\mathbf{h}}$ in a de Rham affine Grassmannian, from which π_{HT} can be recovered. We remark that the target $\mathrm{Gr}_{\mathrm{GL}_n,\mathbf{h}}$ of the de Rham period morphism *only* exists as a diamond in general; analogously, the open subspace of $\mathrm{Fl}_{\mathbf{h}}$ parametrizing Hodge structures (and through which $\pi : \tilde{X} \to \mathrm{Fl}_{\mathbf{h}}$ factors) is a non-algebraic complex manifold.

We now turn to a more detailed description of our results.

1.1 Torsors for $\mathbf{Q}_p$-local systems, and period maps

We start with a very brief recollection on $\mathbf{Q}_p$-local systems.

Proposition 1.1. *Let X be a locally Noetherian adic space, or a perfectoid space, or a diamond. Then we have a natural category $\mathbf{Q}_p\mathrm{Loc}(X)$ of $\mathbf{Q}_p$-*local systems *on X; these are (certain) sheaves of $\mathbf{Q}_p$-vector spaces on the pro-étale site X_{proet} which are locally free of finite rank. When X is locally Noetherian or perfectoid, with associated diamond $X^\diamondsuit$, there is a natural equivalence $\mathbf{Q}_p\mathrm{Loc}(X) \cong \mathbf{Q}_p\mathrm{Loc}(X^\diamondsuit)$.*

These arise quite naturally in geometry: if $f : Y \to X$ is a proper smooth morphism of locally Noetherian adic spaces over $\mathbf{Q}_p$, then any $R^i f_{\mathrm{proet}*}\mathbf{Q}_p$ is naturally a $\mathbf{Q}_p$-local system on X (as announced by Gabber, Kedlaya-Liu, and Scholze-Weinstein). We also remark that if X is a variety over $\mathbf{Q}_p$, then any lisse $\mathbf{Q}_p$-sheaf on X_{et} induces a $\mathbf{Q}_p$-local system on X^{ad}.

Our first result is the following theorem.

[3] At least after generalizing the construction of π to account for "VHSs with $\mathbf{G}$-structure".

Theorem 1.2 (cf. Theorems 4.11, 4.13)**.** *Let $\mathcal{D}$ be any diamond, and let $\mathbf{V}$ be a $\mathbf{Q}_p$-local system on $\mathcal{D}$ of constant rank n. Then the functor*

$$\begin{array}{rcl} \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} : \mathrm{Perf}_{/\mathcal{D}} & \to & \mathrm{Sets} \\ \{f : T \to \mathcal{D}\} & \mapsto & \mathrm{Isom}_{\mathbf{Q}_p\mathrm{Loc}(T)}(\underline{\mathbf{Q}_p}^n, f^*\mathbf{V}) \end{array}$$

is representable by a diamond pro-étale over $\mathcal{D}$, and the natural map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ is a pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor. The functor from rank n $\mathbf{Q}_p$-local systems on $\mathcal{D}$ to pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsors over $\mathcal{D}$ given by $\mathbf{V} \mapsto \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$ is an equivalence of categories, with essential inverse given by $\tilde{\mathcal{D}} \mapsto \tilde{\mathcal{D}} \times_{\underline{\mathrm{GL}_n(\mathbf{Q}_p)}} \underline{\mathbf{Q}_p}^n$.

With applications in mind, we also consider the following more general situation. Let $\mathbf{G}$ be a reductive group over $\mathbf{Q}_p$, so $G = \mathbf{G}(\mathbf{Q}_p)$ is a locally profinite group. Then for X as in Proposition 1.1, we define *G-local systems on X* as additive exact tensor functors

$$\begin{array}{rcl} \mathbf{V} : \mathrm{Rep}(\mathbf{G}) & \to & \mathbf{Q}_p\mathrm{Loc}(X) \\ (W, \rho) & \mapsto & \mathbf{V}_W \end{array}$$

in the obvious way.[4] This generality is not artificial; for example, polarized $\mathbf{Q}_p$-local systems correspond to GSp_{2n}- or GO_n-local systems, and we get even more general examples from Shimura varieties (as we'll see below). When X is a diamond, we prove that the natural functor $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is a pro-étale G-torsor over X, and we again prove (cf. Theorem 4.13) that the association $\mathbf{V} \mapsto \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is an equivalence of categories, with essential inverse given by sending a pro-étale G-torsor $\tilde{X}$ over X to the G-local system

$$\begin{array}{rcl} \mathbf{V}(\tilde{X}) : \mathrm{Rep}(\mathbf{G}) & \to & \mathbf{Q}_p\mathrm{Loc}(X) \\ (W, \rho) & \mapsto & \tilde{X} \times_{\underline{G},\rho} \underline{W}. \end{array}$$

Next we recall the definition of a *de Rham $\mathbf{Q}_p$-local system* on a smooth rigid analytic space X, following the notation and terminology in [Sch13]. This seems to be the correct p-adic analogue of a variation of ($\mathbf{Q}-$ or $\mathbf{Z}-$)Hodge structures on a complex manifold.

Definition 1.3. Let $E/\mathbf{Q}_p$ be a discretely valued nonarchimedean field with perfect residue field of characteristic p, and let X be a smooth rigid analytic space over $\mathrm{Spa}\, E$. Let $\lambda : X_{\mathrm{proet}} \to X_{\mathrm{an}}$ be the natural projection of sites, and let $\mathbb{B}^+_{\mathrm{dR}}, \mathbb{B}_{\mathrm{dR}}, \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}$ be the usual period sheaves on X_{proet} as defined in [Sch13]. Given a $\mathbf{Q}_p$-local system $\mathbf{V}$ on X, define

$$\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) = \lambda_*(\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}).$$

This is a locally free $\mathcal{O}_X$-module of finite rank, equipped with a decreasing exhaustive separated filtration by $\mathcal{O}_X$-local-direct summands and with an integrable connection satisfying Griffiths transversality. There is a natural injective map

$$\alpha_{\mathrm{dR}} : \lambda^*\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) \otimes_{\lambda^*\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}$$

of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$-modules compatible with all structures. We say $\mathbf{V}$ is *de Rham* if α_{dR} is an isomorphism.[5]

[4] Here of course $\mathrm{Rep}(\mathbf{G})$ denotes the tensor category of pairs (W, ρ) with W a finite-dimensional $\mathbf{Q}_p$-vector space and $\rho : \mathbf{G} \to \mathbf{GL}(W)$ a morphism of algebraic groups over $\mathbf{Q}_p$.

[5] Our definition of de Rham is not the same as the definition given in [Sch13], but one easily checks that the two formulations are equivalent.

Maintain the setup of the previous definition, and assume $\mathbf{V}$ is de Rham. Then we get two $\mathbb{B}_{\mathrm{dR}}^{+}$-lattices in the $\mathbb{B}_{\mathrm{dR}}$-local system $\mathbf{V}\otimes_{\underline{\mathbf{Q}_p}}\mathbb{B}_{\mathrm{dR}}$: one given by $\mathbf{M}:=\mathbf{V}\otimes_{\underline{\mathbf{Q}_p}}\mathbb{B}_{\mathrm{dR}}^{+}$ and the other given by the image of

$$\left(\lambda^{*}\mathbf{D}_{\mathrm{dR}}(\mathbf{V})\otimes_{\lambda^{*}\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+}\right)^{\nabla=0}$$

under the isomorphism

$$\alpha_{\mathrm{dR}}^{\nabla=0}:\left(\lambda^{*}\mathbf{D}_{\mathrm{dR}}(\mathbf{V})\otimes_{\lambda^{*}\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{dR}}\right)^{\nabla=0}\xrightarrow{\sim}\mathbf{V}\otimes_{\underline{\mathbf{Q}_p}}\mathbb{B}_{\mathrm{dR}}$$

induced by α_{dR}. Call this second one $\mathbf{M}_0$. When $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(\mathbf{V})=\mathbf{D}_{\mathrm{dR}}(\mathbf{V})$, we have $\mathbf{M}_0\subseteq\mathbf{M}$. More generally, let

$$h^i=\mathrm{rank}_{\mathcal{O}_X}\mathrm{gr}^i\mathbf{D}_{\mathrm{dR}}(\mathbf{V})=\mathrm{rank}_{\mathcal{O}_X}(\mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}/\mathrm{Fil}^{i+1}\mathbf{D}_{\mathrm{dR}})$$

be the Hodge numbers of $\mathbf{V}$, and let $\mu_{\mathbf{V}}:\mathbf{G}_m\to\mathrm{GL}_n$ be the cocharacter in which the weight i appears with multiplicity h^i. We remark that the h^i's are constant on connected components of X, and thus $\mu_{\mathbf{V}}$ is as well.

Proposition 1.4. *The relative positions of $\mathbf{M}_0$ and $\mathbf{M}$ inside $\mathbf{V}\otimes_{\underline{\mathbf{Q}_p}}\mathbb{B}_{\mathrm{dR}}$ are given by $\mu_{\mathbf{V}}$.*

More generally, we say a G-local system $\mathbf{V}$ is de Rham if the associated $\mathbf{Q}_p$-local systems $\mathbf{V}_W$ are de Rham for all $(W,\rho)\in\mathrm{Rep}(\mathbf{G})$. Using the fact that the comparison isomorphism α_{dR} is compatible with direct sums, tensor products, and subquotients, the Tannakian formalism (plus a little more) gives a conjugacy class of Hodge cocharacters $\mu_{\mathbf{V}}:\mathbf{G}_{m,\overline{\mathbf{Q}_p}}\to\mathbf{G}_{\overline{\mathbf{Q}_p}}$ such that $\rho\circ\mu_{\mathbf{V}}$ measures the Hodge filtration on $\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_W)$ for any (W,ρ).

The following theorem is our main result on period morphisms.

Theorem 1.5. *Let $\mathbf{V}$ be a de Rham $G=\mathbf{G}(\mathbf{Q}_p)$-local system on a smooth rigid analytic space X, with constant Hodge cocharacter μ. Then we have natural G-equivariant period morphisms*

$$\pi_{\mathrm{dR}}:\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^{\diamond}}\to\mathrm{Gr}_{\mathbf{G},\mu}$$

and

$$\pi_{\mathrm{HT}}:\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^{\diamond}}\to\mathscr{F}\ell_{\mathbf{G},\mu}^{\diamond}$$

of diamonds over $\mathrm{Spd}\,E$*, fitting into a G-equivariant commutative diagram*

$$\begin{tikzcd}
\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^{\diamond}} \arrow[r, "\pi_{\mathrm{dR}}"] \arrow[dr, "\pi_{\mathrm{HT}}"'] & \mathrm{Gr}_{\mathbf{G},\mu} \arrow[d, "\sigma_{\mathrm{BB}}"] \\
& \mathscr{F}\ell_{\mathbf{G},\mu}^{\diamond}
\end{tikzcd}$$

where σ_{BB} is the Bialynicki-Birula morphism.

The map σ_{BB} was defined and studied in [CS15], where the authors also prove that σ_{BB} is an isomorphism exactly when μ is minuscule; we reprove this by a direct group-theoretic calculation (cf. Corollary 3.19). We also define π_{HT} directly, in the more general situation where $\mathbf{V}$ is only assumed to be Hodge-Tate.

1.2 Some applications to Shimura varieties

Let $(\mathbf{G}, X)$ be a Shimura datum, with associated Hodge cocharacter μ and reflex field E. For any sufficiently small open subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$, the associated Shimura variety Sh_K is a smooth quasiprojective variety over E. Choose a prime $\mathfrak{p}$ of E lying over p, and let $\mathcal{S}_K = (Sh_K \times_E E_{\mathfrak{p}})^{\mathrm{ad}}$ be the associated rigid analytic space over $\operatorname{Spa} E_{\mathfrak{p}}$. Let $\mathscr{S}_K = \mathcal{S}_K^{\diamondsuit}$ be the associated diamond over $\operatorname{Spd} E_{\mathfrak{p}}$. Finally, let $G = \mathbf{G}(\mathbf{Q}_p)$ be as before.

Proposition 1.6. *For any sufficiently small $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$, there is a diamond $\mathscr{S}_{K^p}$ over $\operatorname{Spd} E_{\mathfrak{p}}$ with an action of G such that*

$$\mathscr{S}_{K^p} \cong \varprojlim_{K_p} \mathscr{S}_{K^p K_p}$$

G-equivariantly and compatibly with changing K^p (or even with changing the Shimura data).

When $(\mathbf{G}, X)$ is of Hodge type, Scholze [Sch15b] constructed a *perfectoid Shimura variety*, i.e. a perfectoid space $\mathcal{S}_{K^p}$ with continuous G-action such that

$$\mathcal{S}_{K^p} \sim \varprojlim_{K_p} \mathcal{S}_{K^p K_p}$$

as adic spaces. Whenever such an $\mathcal{S}_{K^p}$ exists, we necessarily have $\mathscr{S}_{K^p} \cong \mathcal{S}_{K^p}^{\diamondsuit}$ for formal reasons. However, unlike the difficult construction of $\mathcal{S}_{K^p}$, the proof of Proposition 1.6 is *essentially trivial* once the theory of diamonds is set up: the inverse limit of any projective system of diamonds with finite étale transition maps is a diamond. We also point out that knowledge of $\mathscr{S}_{K^p}$ is strictly weaker than knowledge of $\mathcal{S}_{K^p}$: the construction of the latter object also gives very rich information about the existence of certain affinoid coverings, formal models, compactifications, etc., and $\mathscr{S}_{K^p}$ doesn't a priori have the same applications to p-adic automorphic forms as $\mathcal{S}_{K^p}$. In any case, it seems likely that $\mathcal{S}_{K^p}$ always exists (and X. Shen has recently constructed such perfectoid Shimura varieties when $(\mathbf{G}, X)$ is of abelian type), but this is probably out of reach in general. Therefore, it might be surprising that the following result is within reach.

Theorem 1.7. *There is a natural G-equivariant Hodge-Tate period morphism*

$$\pi_{\mathrm{HT}} : \mathscr{S}_{K^p} \to \mathscr{F}\!\ell_{\mathbf{G},\mu}^{\diamondsuit}$$

of diamonds over $\operatorname{Spd} E_{\mathfrak{p}}$, compatible with changing K^p and functorial in morphisms of arbitrary *Shimura data. When $(\mathbf{G}, X)$ is of Hodge type (or more generally, of abelian type), this is the morphism of diamonds associated with the "refined Hodge-Tate period morphism"*

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p} \to \mathscr{F}\!\ell_{\mathbf{G},\mu}$$

of Caraiani-Scholze.

We remark that when $\mathscr{S}_{K^p}$ is the diamond of a perfectoid Shimura variety $\mathcal{S}_{K^p}$, any morphism of diamonds

$$f^{\diamondsuit} : \mathscr{S}_{K^p} = \mathcal{S}_{K^p}^{\diamondsuit} \to \mathscr{F}\!\ell_{\mathbf{G},\mu}^{\diamondsuit}$$

arises uniquely from a morphism

$$f : \mathcal{S}_{K^p} \to \mathscr{F}\!\ell_{\mathbf{G},\mu}$$

of adic spaces. In particular, the proof of Theorem 1.7 actually gives a new construction of the refined Hodge-Tate period map for perfectoid Shimura varieties of Hodge type (or abelian type).

This proof is of course related to Caraiani-Scholze's proof, but the details are rather different: in particular, we never need to make any of the noncanonical choices which arise from choosing an embedding of a Hodge-type Shimura datum into a Siegel Shimura datum.

We briefly sketch the construction of π_{HT}. For any fixed level K_p, the map $\mathscr{S}_{K^p} \to \mathscr{S}_{K^pK_p}$ is naturally a pro-étale K_p-torsor,[6] so the pushout

$$\widetilde{\mathscr{S}_{K^pK_p}} := \mathscr{S}_{K^p} \times_{\underline{K_p}} \underline{G}$$

is naturally a pro-étale G-torsor over $\mathscr{S}_{K^pK_p}$, compatibly with varying K^pK_p. Note that via the G-action on $\mathscr{S}_{K^p}$, we get a canonical splitting

$$\widetilde{\mathscr{S}_{K^pK_p}} \cong \mathscr{S}_{K^p} \times K_p \backslash G$$

which is G-equivariant for the diagonal G-action on the right-hand side. By our previous equivalence of categories, $\widetilde{\mathscr{S}_{K^pK_p}}$ gives rise to a G-local system $\mathbf{V}$ over $\mathscr{S}_{K^pK_p}$, or equivalently over $\mathcal{S}_{K^pK_p}$, which we call the *tautological G-local system*. The crucial ingredient is then:

Theorem 1.8 (R. Liu-X. Zhu [LZ16])**.** *The tautological G-local system $\mathbf{V}$ over $\mathcal{S}_{K^pK_p}$ is de Rham (and in particular Hodge-Tate) with Hodge cocharacter μ, for any Shimura variety.*

Liu-Zhu prove, surprisingly, that if the stalk of a $\mathbf{Q}_p$-local system (or G-local system) $\mathbf{V}$ on a connected rigid analytic space is de Rham at one classical point (in the classical sense of Fontaine), then $\mathbf{V}$ is de Rham. For a Shimura variety, this allows one to check the de Rham property at "special points", where everything is explicit.

Thanks to this result, we can apply Theorem 1.5 to get a G-equivariant morphism of diamonds

$$\pi_{\mathrm{HT},K_p} : \widetilde{\mathscr{S}_{K^pK_p}} \to \mathscr{F}\ell^{\diamondsuit}_{\mathbf{G},\mu}$$

over $\operatorname{Spd} E_{\mathfrak{p}}$ (since μ is minuscule, there is no difference between Hodge-Tate periods and de Rham periods in this situation). Finally, we check by hand, using the aforementioned splitting, that π_{HT,K_p} descends along the G-equivariant projection $\widetilde{\mathscr{S}_{K^pK_p}} \to \mathscr{S}_{K^p}$ to a G-equivariant morphism

$$\pi_{\mathrm{HT}} : \mathscr{S}_{K^p} \to \mathscr{F}\ell^{\diamondsuit}_{\mathbf{G},\mu}$$

independent of the auxiliary choice of K_p.

1.3 Newton strata

As a curious consequence of Theorem 1.5, any de Rham $\mathbf{Q}_p$-local system $\mathbf{V}$ on a rigid space induces a "generic fiber Newton stratification" of the space:

Corollary 1.9. *Let X be a connected rigid analytic space over* $\operatorname{Spa} E$ *as above, and let $\mathbf{V}$ be a de Rham $\mathbf{Q}_p$-local system on X of rank n with constant Hodge cocharacter $\mu_{\mathbf{V}}$. Then $\mathbf{V}$ induces a natural* Newton stratification *of X into locally closed subsets*

$$|X| = \coprod_{b \in B(\mathrm{GL}_n/\mathbf{Q}_p, \mu_{\mathbf{V}})} |X|^b$$

indexed by Newton polygons lying above the Hodge polygon of $\mathbf{V}$ and with matching endpoints.

[6] This is not quite true unless the Shimura datum satisfies Milne's axiom SV5.

Proof. Pulling back the Newton strata of $\mathrm{Gr}_{\mathrm{GL}_n,\mu_{\mathbf{V}}}$ defined by Caraiani-Scholze[7] under π_{dR}, we get a stratification of $|\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond}|$ by $\mathrm{GL}_n(\mathbf{Q}_p)$-stable locally closed subsets, which descend under the identifications

$$|\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond}|/\mathrm{GL}_n(\mathbf{Q}_p) \cong |X^\diamond| \cong |X|$$

to a stratification of $|X|$. $\square$

Remark. The subsets $|X|^b$ are determined by their rank one points: $x \in |X|$ lies in a given $|X|^b$ if and only if its unique rank one generization $\tilde{x}$ lies in $|X|^b$. Also, it's not clear whether the closure of any given $|X|^b$ is a union of $|X|^{b'}$'s, so the term "stratification" is being used in a loose sense here.

Next we compare this "generic fiber" Newton stratification with a suitable "special fiber" Newton stratification, in a situation where the latter exists in a reasonably canonical way. More precisely, let $\mathbf{f} : \mathfrak{Y} \to \mathfrak{X}$ be a smooth proper morphism of connected flat p-adic formal schemes over $\mathrm{Spf}\,\mathcal{O}_E$, with associated morphism of adic generic fibers $f : Y \to X$ over $\mathrm{Spa}\,E$. Assume X is smooth. Let $\mathcal{M} = R^i\mathbf{f}_{\mathrm{crys}*}(\mathcal{O}/W)[\frac{1}{p}]$ be the ith relative crystalline cohomology, and let $\mathbf{V} = R^i f_{\mathrm{proet}*}\mathbf{Q}_p$ be the ith relative p-adic étale cohomology. Then, on the one hand, the Frobenius action on the specializations of $\mathcal{M}$ at geometric points of $\mathfrak{X}$ gives a natural stratification

$$|\mathfrak{X}| = \coprod_{b \in B(\mathrm{GL}_n/\mathbf{Q}_p,\mu_{\mathbf{V}})} |\mathfrak{X}|^b.$$

In this context, the fact that the Newton polygon lies over the Hodge polygon is a famous theorem of Mazur [Maz73]. On the other hand, the previous corollary applied to $\mathbf{V}$ gives a decomposition $|X| = \coprod_{b\in B(\mathrm{GL}_n/\mathbf{Q}_p,\mu_{\mathbf{V}})} |X|^b$ with the same indexing set.

Theorem 1.10. *The stratification of $|X|$ associated with $\mathbf{V}$ agrees on rank one points with the pullback (under the specialization map $\mathbf{s} : |X| \to |\mathfrak{X}|$) of the stratification of $|\mathfrak{X}|$ associated with $\mathcal{M}$.*

The proof of this result makes use of Fargues's results on local shtukas and Breuil-Kisin modules over the ring $\mathbb{A} = W(\mathcal{O}_{C^\flat})$, together with Bhatt-Morrow-Scholze's construction of a new cohomology theory valued in Breuil-Kisin modules over $\mathbb{A}$.

Finally, we compare the generic and special fiber Newton strata in the setting of Shimura varieties. Again $\mathrm{Gr}_{\mathbf{G},\mu} \cong \mathscr{F}\ell^\diamond_{\mathbf{G},\mu}$ has a Newton stratification by G-invariant locally closed subsets indexed by the Kottwitz set $B(\mathbf{G},\mu)$, so pulling back under π_{HT} and descending gives a "generic fiber" Newton stratification of $|\mathcal{S}_{K^pK_p}|$ as before. Suppose now that $(\mathbf{G}, X)$ is a Shimura datum of Hodge type, with $p > 2$ and $\mathbf{G}/\mathbf{Q}_p$ unramified. Let $K_p \subset G$ be hyperspecial, so by work of Kisin [Kis10], $Sh_{K^pK_p} \times_E E_{\mathfrak{p}}$ has (among other things) a good integral model $S^\circ_{K^pK_p}$ over $\mathcal{O}_{E_{\mathfrak{p}}}$ and the special fiber of $S^\circ_{K^pK_p}$ has a natural Newton stratification. Let $\mathfrak{S}_{K^pK_p}$ be the formal scheme given as the p-adic completion of $S^\circ_{K^pK_p}$, so the associated rigid analytic space identifies with the locus of good reduction $\mathcal{S}^{gd}_{K^pK_p} \subset \mathcal{S}_{K^pK_p}$. Again we have a specialization map

$$\mathbf{s} : |\mathcal{S}^{gd}_{K^pK_p}| \to |\mathfrak{S}_{K^pK_p}| = |S^\circ_{K^pK_p} \times_{\mathcal{O}_{E_{\mathfrak{p}}}} \mathbf{F}_q|.$$

Theorem 1.11. *The generic fiber Newton strata of $|\mathcal{S}^{gd}_{K^pK_p}|$ coincide on rank one points with the pullback under $\mathbf{s}$ of the special fiber Newton strata.*

[7] Warning: Here and in what follows, Caraiani-Scholze use $B(\mathbf{G},\mu^{-1})$ everywhere that we use $B(\mathbf{G},\mu)$, but everything matches after noting that the two are in bijection via $b \mapsto b^{-1}$.

The proof naturally combines (the ideas in the proof of) Theorem 1.10 with the Tannakian formalism and some unwinding of Kisin's construction. When the Sh_K's are compact PEL Shimura varieties of type A or C, this is a result of Caraiani-Scholze (cf. Section 4.3 of their paper, especially Lemma 4.3.20 and the diagram immediately following).

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2 Preliminaries

We assume basic familiarity with adic spaces and perfectoid spaces. Unless explicitly stated otherwise, all adic spaces are (honest) analytic adic spaces over $\mathrm{Spa}\,\mathbf{Z}_p$; we denote this category by Adic. We also follow the convention that perfectoid spaces do *not* live over a fixed perfectoid base field.

If A is a topological ring, we denote by A° the subring of powerbounded elements and by $A^{\circ\circ}$ the subset of topologically nilpotent elements.

We reserve the notation $\cong$ for canonical isomorphisms.

2.1 Sheaves on sites

Let $\mathcal{C}$ be a site. We write $\mathrm{PSh}(\mathcal{C})$ and $\mathrm{Sh}(\mathcal{C})$ for the categories of presheaves and sheaves on $\mathcal{C}$. Given $X \in \mathcal{C}$, we write $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$ for the Yoneda embedding of X into $\mathrm{PSh}(\mathcal{C})$. The following definition is standard.

Definition 2.1. A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves is *surjective* if for any $U \in \mathcal{C}$ and any section $s \in \mathcal{G}(U)$, there exists a covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ such that $s|_{U_i} \in \mathcal{G}(U_i)$ lies in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$ for each $i \in I$.

We also adopt the following conventions.

Definition 2.2. Let $\mathcal{C}$ be a site. A morphism $\mathcal{F} \to \mathcal{G}$ of (pre)sheaves on $\mathcal{C}$ is *representable* if for every $X \in \mathcal{C}$ and every morphism of (pre)sheaves $h_X \to \mathcal{G}$, there is an isomorphism $s : h_Y \overset{\sim}{\to} \mathcal{F} \times_{\mathcal{G}} h_X$ for some $Y \in \mathcal{C}$.

As usual, the pair (Y, s) in the previous definition is unique up to unique isomorphism, so we'll typically suppress s and just write "...an isomorphism $h_Y \simeq \mathcal{F} \times_{\mathcal{G}} h_X$...". Note that we follow the Stacks Project in saying "representable" instead of "relatively representable".

Definition 2.3. Let $\mathcal{C}$ be a site, and let "blah" be a property of morphisms in the category underlying $\mathcal{C}$ which is preserved under arbitrary pullback. A morphism $\mathcal{F} \to \mathcal{G}$ of (pre)sheaves on $\mathcal{C}$ is "blah" if it is representable and, for every $X \in \mathcal{C}$ and every morphism of (pre)sheaves $h_X \to \mathcal{G}$, the morphism $Y \to X$ corresponding to the upper horizontal arrow in the pullback diagram

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{G}} h_X \simeq h_Y \arrow[r] \arrow[d] & h_X \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{G} \end{tikzcd}$$

is “blah”.

Proposition 2.4. *If $\mathcal{F} \to \mathcal{G}$ is “blah”, then for any $\mathcal{H} \to \mathcal{G}$ the pullback $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{H}$ is “blah”.*

Proof. Given $h_X \to \mathcal{H}$, we have $(\mathcal{F} \times_{\mathcal{G}} \mathcal{H}) \times_{\mathcal{H}} h_X = \mathcal{F} \times_{\mathcal{G}} h_X$, so this is immediate. □

We also use the following terminology once or twice:

Definition 2.5. Let $\mathcal{C}' \to \mathcal{C}$ be a morphism of sites, with $u : \mathcal{C} \to \mathcal{C}'$ the associated (continuous) functor on underlying categories. We say $\mathcal{C}'$ is *finer than* $\mathcal{C}$ if u is fully faithful and all $\mathcal{C}$-covers are $\mathcal{C}'$-covers.

2.2 Perfectoid spaces, tilting and untilting

Here we recall some material from [Sch12, Sch14] (cf. also [KL15, KL16]).

Recall that a *Tate ring* is a topological ring A containing an open subring A_0 whose induced topology is the ϖ-adic topology for some $\varpi \in A_0^{\circ\circ} \cap A^\times$. It's easy to check that $A = A_0[\frac{1}{\varpi}]$. Conversely, if A is Tate, then for any $\varpi \in A^{\circ\circ} \cap A^\times$ we can find an open bounded subring A_0 containing ϖ whose induced topology is the ϖ-adic topology. Any $\varpi \in A^{\circ\circ} \cap A^\times$ is a *pseudouniformizer* of A. A Tate ring A is *perfectoid* if A is complete and the subring of powerbounded elements A° is bounded, and if there exists a pseudouniformizer ϖ such that $p \in \varpi^p \cdot A^\circ$ and such that the Frobenius on A°/ϖ is surjective. An *affinoid perfectoid space* is an adic space of the form $\mathrm{Spa}(A, A^+)$ where A is a perfectoid Tate ring. By definition, an adic space is perfectoid if it admits an open covering by affinoid perfectoid spaces.

Let $\widetilde{\mathrm{Perf}}$ denote the category of all perfectoid spaces. Let Perf denote the category of perfectoid spaces in characteristic p, so tilting defines a functor

$$\begin{array}{rcl} \widetilde{\mathrm{Perf}} & \to & \mathrm{Perf} \\ X & \mapsto & X^\flat. \end{array}$$

We remind the reader that on affinoid perfectoid spaces, tilting sends $\mathrm{Spa}(A, A^+)$ to $\mathrm{Spa}(A^\flat, A^{+\flat})$, where $(-)^\flat$ is the endofunctor on multiplicative monoids defined by

$$A^\flat = \left\{ \mathbf{a} = (a_i)_{i \geq 0} \in A^{\mathbf{N}} \mid a_{i+1}^p = a_i \, \forall i \geq 0 \right\};$$

it is true but not obvious that for any perfectoid Tate ring A, the monoid $A^\flat$ is a perfectoid $\mathbf{F}_p$-algebra, with addition given by the formula

$$(\mathbf{a} + \mathbf{a}')_i = \lim_{j \to \infty} (a_{i+j} + a'_{i+j})^{p^j}.$$

We also remind the reader that tilting has essentially every compatibility one could dream of: there is a natural homeomorphism $|X| \cong |X^\flat|$ compatible with affinoid perfectoid subspaces and with rational subsets thereof, there are natural equivalences $X_{\mathrm{an}} \cong X^\flat_{\mathrm{an}}$ and $X_{\text{ét}} \cong X^\flat_{\text{ét}}$ with associated equivalences of topoi, etc. However, tilting is forgetful: there are often many non-isomorphic spaces with tilts isomorphic to a given $S \in \mathrm{Perf}$, so one needs to specify extra data to recover X from $X^\flat$. Before explaining what extra data is needed, we define precisely the notion of an "untilt".

Definition 2.6. Given any $X \in \mathrm{Perf}$, an *untilt of* X is a pair $(X^\sharp, \iota)$ where $X^\sharp \in \widetilde{\mathrm{Perf}}$ and $\iota : X^{\sharp\flat} \xrightarrow{\sim} X$ is an isomorphism. A *morphism* $m : (X^\sharp, \iota) \to (X^{\sharp\prime}, \iota')$ of untilts of X is a morphism of perfectoid spaces $m : X^\sharp \to X^{\sharp\prime}$ such that $\iota' \circ m^\flat = \iota$.

As motivation for the name, observe that any $Y \in \widetilde{\mathrm{Perf}}$ is canonically an untilt of its tilt $Y^\flat$, by taking $Y^{\flat\sharp} = Y$ and $\iota = \mathrm{id}$. Note that for any morphism $m : (X^\sharp, \iota) \to (X^{\sharp\prime}, \iota')$ of untilts of X, $m^\flat = \iota'^{-1} \circ \iota$ is an isomorphism, so m is necessarily an isomorphism as well. Thus the untilts of X naturally form a groupoid. Note also that a given $(X^\sharp, \iota)$ has no automorphisms, since if $m : X^\sharp \to X^\sharp$ tilts to the identity map, then m must be the identity; in particular, the untilts of X form a *setoid* in the terminology of the Stacks Project.

Proposition 2.7. *If X is a perfectoid space, then the assignment $U \mapsto W(\mathcal{O}^+_X(U)^\flat)$ on open affinoid perfectoid subsets $U \subset X$ defines a sheaf of rings $\mathbb{A}_X$ on X which depends only on the tilt of X. This sheaf of rings comes with a natural surjective ring map $\theta_X : \mathbb{A}_X \to \mathcal{O}^+_X$.*

Definition 2.8 (after Fargues-Fontaine, Scholze, Kedlaya-Liu (cf. [KL16, §3.2]))**.** Let X be a perfectoid space. An ideal sheaf $\mathcal{J} \subset \mathbb{A}_X$ is *primitive of degree one* if locally on a covering $X = \cup_i U_i$ by affinoid perfectoids, $\mathcal{J}|_{U_i} = (\xi_i)$ is principal and generated by an element $\xi_i \in W(\mathcal{O}^+_X(U_i)^\flat)$ of the form $\xi_i = p + [\varpi_i]\alpha_i$ with $\varpi_i \in \mathcal{O}^+_X(U_i)^\flat$ some pseudouniformizer and with $\alpha_i \in W(\mathcal{O}^+_X(U_i)^\flat)$ arbitrary.

Let Perf^+ denote the category of pairs $(X, \mathcal{J})$ where $X \in \mathrm{Perf}$ and $\mathcal{J} \subset \mathbb{A}_X$ is an ideal sheaf which is primitive of degree one, with morphisms $(X, \mathcal{J}) \to (Y, \mathcal{I})$ given by morphisms $f : X \to Y$ such that $f^{-1}\mathcal{I} \cdot \mathbb{A}_X = \mathcal{J}$. Note that Perf^+ is a category fibered in sets over Perf. For any $X \in \widetilde{\mathrm{Perf}}$, the ideal sheaf $\ker \theta_X \subset \mathbb{A}_X = \mathbb{A}_{X^\flat}$ is primitive of degree one, so we get a functor

$$\begin{array}{rcl} \widetilde{\mathrm{Perf}} & \to & \mathrm{Perf}^+ \\ X & \mapsto & \left(X^\flat, \ker \theta_X \subset \mathbb{A}_{X^\flat}\right). \end{array}$$

Proposition 2.9 (Scholze, Kedlaya-Liu)**.** *This functor defines a natural equivalence*[8] $\widetilde{\mathrm{Perf}} \cong \mathrm{Perf}^+$*, and untilts of a given $X \in \mathrm{Perf}$ correspond to fibers of the map $\mathrm{Perf}^+ \to \mathrm{Perf}$. Explicitly, the untilt $X^\sharp \in \widetilde{\mathrm{Perf}}$ associated with a pair $(X, \mathcal{J}) \in \mathrm{Perf}^+$ has integral structure sheaf $\mathcal{O}^+_{X^\sharp} = \mathbb{A}_X/\mathcal{J}$.*

For any perfectoid space S, tilting defines a natural equivalence $\widetilde{\mathrm{Perf}}_{/S} \cong \mathrm{Perf}_{/S^\flat}$.

We will freely use the equivalence $\widetilde{\mathrm{Perf}}_{/S} \cong \mathrm{Perf}_{/S^\flat}$. The reader may wish to think of this remarkable result as a kind of "crystalline" property of tilting: an untilt of $S^\flat$ induces unique and compatible untilts of all perfectoid spaces over $S^\flat$.

We'll also repeatedly make use of "almost purity" in the following form (cf. [KL16, Proposition 3.3.18]):

Proposition 2.10. *If A is any perfectoid Tate ring and B is a finite étale A-algebra, then B is perfectoid. In particular, if X is any perfectoid space and $f : Y \to X$ is any finite étale morphism, then Y is automatically perfectoid. Furthermore, tilting induces an equivalence $X_{\mathrm{fet}} \cong X^\flat_{\mathrm{fet}}$.*

[8] Here we are implicitly replacing $\widetilde{\mathrm{Perf}}$, which is a category fibered in setoids over Perf, with the equivalent category fibered in sets; cf. Tag 04S9 of the Stacks Project.

2.3 The big pro-étale site

In this section we recall some material from [Sch14], and prove some basic properties of the big pro-étale site.

Definition 2.11.

i. A morphism $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces is *affinoid pro-étale* if (B, B^+) admits a presentation as the completion of a filtered direct limit

$$(B, B^+) = \widehat{\varinjlim_{i \in I}(A_i, A_i^+)}$$

of perfectoid (A, A^+)-algebras with each $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ étale.

ii. A morphism $f : Y \to X$ of perfectoid spaces is *pro-étale* if for every point $y \in Y$, there is an open affinoid perfectoid subset $V \subset Y$ containing y and an open affinoid perfectoid subset $U \subset X$ containing $f(V)$ such that the induced morphism $V \to U$ is affinoid pro-étale.

iii. A pro-étale morphism of perfectoid spaces $f : Y \to X$ is a *pro-étale cover(ing)* if for every quasicompact open subset $U \subset X$, there is some quasicompact open subset $V \subset Y$ with $f(V) = U$.

We record the following basic properties of pro-étale morphisms.

Proposition 2.12. *Suppose $f : Y \to X$ is a pro-étale morphism of perfectoid spaces.*

i. *If $Z \to X$ is any morphism of perfectoid spaces, then $Z \times_X Y \to Z$ is a pro-étale morphism.*

ii. *If $Z \to X$ is pro-étale, then $Z \times_X Y \to X$ is pro-étale.*

iii. *If $g : Z \to Y$ is any morphism of perfectoid spaces, then $f \circ g$ is pro-étale if and only if g is pro-étale.*

In order to see that perfectoid spaces with pro-étale covers form a site, we need to verify the following proposition.

Proposition 2.13. *Suppose $f : Y \to X$ is a pro-étale covering of perfectoid spaces. If $g : Z \to X$ is any morphism of perfectoid spaces, then $Z \times_X Y \to Z$ is a pro-étale covering.*

Proof. Let $W \subset Z$ be a quasicompact open subset, so $W \subseteq g^{-1}(g(W))$. Choose a quasicompact open $U \subset X$ with $g(W) \subseteq U$, and then choose a quasicompact open $V \subset Y$ with $f(V) = U$. Then $W \times_U V \subset Z \times_X Y$ is a quasicompact open with image W in Z. $\square$

Proposition 2.14. *Suppose*

$$\begin{tikzcd} Z \arrow[rr, "f\circ g"] \arrow[dr, "g"'] & & X \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

is a diagram of pro-étale morphisms of perfectoid spaces, and suppose $f \circ g$ is a pro-étale cover. Then f is a pro-étale cover.

Proof. Let $U \subset X$ be any qc open; we need to find a qc open $V \subset Y$ with $f(V) = U$. By assumption, we may choose some qc open $W \subset Z$ with $(f \circ g)(W) = U$. Choose a covering $\cup_{i \in I} V_i$ of $f^{-1}(U)$ by qc opens in Y, so $\cup_{i \in I} g^{-1}(V_i) \cap W$ is an open covering of W. Since W is qc, we can find a finite subset $I' \subset I$ such that $\cup_{i \in I'} g^{-1}(V_i) \cap W$ is an open covering of W. Then

$$g(W) = \cup_{i \in I'} g\left(g^{-1}(V_i) \cap W\right) \subseteq \cup_{i \in I'} V_i,$$

so $V := \cup_{i \in I'} V_i$ is a qc open subset of Y such that $g(W) \subseteq V \subseteq f^{-1}(U)$, and therefore $f(V) = U$ as desired. $\square$

Definition 2.15. The *big pro-étale site* is the site $\mathrm{Perf}^{\mathrm{proet}}$ whose underlying category is the category Perf of perfectoid spaces in characteristic p, with coverings given by pro-étale coverings.

There is also an analogous site $\widetilde{\mathrm{Perf}}^{\text{proét}}$ with arbitrary perfectoid spaces as objects. We also have a small pro-étale site $X_{\text{proét}}$ for any perfectoid space X, with objects given by perfectoid spaces pro-étale over X and covers given by pro-étale covers. These sites are compatible with tilting in the obvious sense.

We now turn to sheaves on $\mathrm{Perf}^{\mathrm{proet}}$. Every structural result we'll prove for $\mathrm{Sh}(\mathrm{Perf}^{\mathrm{proet}})$ has an obvious analogue for $\mathrm{Sh}(\widetilde{\mathrm{Perf}}^{\mathrm{proet}})$ compatible with tilting; we do not spell this out.

Proposition 2.16. *For any $X \in \mathrm{Perf}$, the presheaf $h_X = \mathrm{Hom}(-, X)$ is a sheaf on the big pro-étale site.*

Proof sketch. One reduces to the case of X affinoid. Recall that for any affinoid adic space $X = \mathrm{Spa}(R, R^+)$ and any adic space Y, there's a natural identification

$$\mathrm{Hom}(Y, X) = \mathrm{Hom}\left((R, R^+), (\mathcal{O}(Y), \mathcal{O}(Y)^+)\right).$$

Let $Y \in \mathrm{Perf}$ be some perfectoid space with a given pro-étale cover $\tilde{Y} \to Y$, so we need to show exactness of the sequence

$$0 \to \mathrm{Hom}(Y, X) \to \mathrm{Hom}(\tilde{Y}, X) \rightrightarrows \mathrm{Hom}(\tilde{Y} \times_Y \tilde{Y}, X).$$

But $\mathcal{O}_Y$ and $\mathcal{O}_Y^+$ are sheaves on Y_{proet}, so we get an exact sequence

$$0 \to (\mathcal{O}(Y), \mathcal{O}(Y)^+) \to (\mathcal{O}(\tilde{Y}), \mathcal{O}(\tilde{Y})^+) \rightrightarrows (\mathcal{O}(\tilde{Y} \times_Y \tilde{Y}), \mathcal{O}(\tilde{Y} \times_Y \tilde{Y})^+),$$

and we're done upon applying the left-exact functor $\mathrm{Hom}\left((R, R^+), -\right)$. $\square$

By our conventions, given any property of morphisms of perfectoid spaces preserved under arbitrary pullback, there is a corresponding notion for morphisms of sheaves on $\mathrm{Perf}^{\mathrm{proet}}$. In particular, we may speak of a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ on $\mathrm{Perf}^{\mathrm{proet}}$ being an open immersion, Zariski closed immersion, finite étale, étale, pro-finite étale, pro-étale, a pro-étale cover, etc. (Note, however, that "surjective" will always mean "surjective as a morphism of sheaves.")

Proposition 2.17. *A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves on $\mathrm{Perf}^{\mathrm{proet}}$ is a pro-étale cover if and only if it is surjective and pro-étale.*

Proof. Let $\mathcal{F} \to \mathcal{G}$ be a pro-étale morphism. By definition, the sheaf morphism $\mathcal{F} \to \mathcal{G}$ is surjective if, for any $h_X \to \mathcal{G}$ with associated $h_Y \simeq \mathcal{F} \times_{\mathcal{G}} h_X$, we can find some pro-étale cover $\tilde{X} \to X$ and a section $h_{\tilde{X}} \to \mathcal{F}$ such that the diagram

$$\begin{tikzcd}
& \mathcal{F} \arrow[r] & \mathcal{G} \\
& h_Y \arrow[u] \arrow[r] & h_X \arrow[u] \\
h_{\tilde{X}} \arrow[uur] \arrow[urr] & &
\end{tikzcd}$$

commutes. If $\mathcal{F} \to \mathcal{G}$ is a pro-étale cover, then $\tilde{X} = Y$ does the job, so $\mathcal{F} \to \mathcal{G}$ is surjective.

Suppose conversely that $\mathcal{F} \to \mathcal{G}$ is a surjective pro-étale morphism, and consider any map $h_X \to \mathcal{G}$, so we get the same diagram as above. We need to show that the associated morphism $Y \to X$ is a pro-étale cover. Since the square is cartesian, we can fill in another arrow from $h_{\tilde{X}}$ to h_X, vis.

$$\begin{tikzcd}
& \mathcal{F} \arrow[r] & \mathcal{G} \\
& h_Y \arrow[u] \arrow[r] & h_X \arrow[u] \\
h_{\tilde{X}} \arrow[ur] \arrow[uur] \arrow[urr] & &
\end{tikzcd}$$

Since $Y \to X$ and $\tilde{X} \to X$ are both pro-étale, the morphism $\tilde{X} \to Y$ is pro-étale by Proposition 2.12.iii, so the diagram

$$\begin{tikzcd}
\tilde{X} \arrow[rr] \arrow[dr] & & X \\
& Y \arrow[ur] &
\end{tikzcd}$$

satisfies the hypotheses of Proposition 2.14 and thus $Y \to X$ is a pro-étale cover. $\square$

Finally, we record some facts about pro-finite étale morphisms, which we define as follows:

Definition 2.18. A morphism $f : Y \to X$ of perfectoid spaces is *pro-finite étale* if for some covering of X by open affinoid perfectoid subsets $U_i = \mathrm{Spa}(A_i, A_i^+) \subset X$ we have $Y \times_X U_i = \mathrm{Spa}(B_i, B_i^+)$ where (B_i, B_i^+) admits a presentation as the completion of a filtered direct limit

$$(B_i, B_i^+) = \widehat{\varinjlim_{j \in J_i} (A_{ij}, A_{ij}^+)}$$

of perfectoid (A_i, A_i^+)-algebras with each A_{ij} finite étale over A_i and with each A_{ij}^+ given as the integral closure of A_i^+ in A_{ij}. Let $X_{\text{profét}}$ be the category with objects given by perfectoid spaces pro-finite étale over X, and with the obvious morphisms.

Note in particular that pro-finite étale morphisms are pro-étale, so they enjoy the same stability properties: the natural analogues of the results in Proposition 2.12 are true, and in particular, any pullback of a pro-finite étale morphism is pro-finite étale.

Proposition 2.19. *For any perfectoid space X and any inverse system $Y = (Y_i)_{i \in I} \in \mathrm{pro} - X_{\text{fét}}$, there is a unique perfectoid space $\hat{Y} \in X_{\text{profét}}$ with compatible morphisms $\hat{Y} \to Y_i$ such that*

$$\hat{Y} \sim \varprojlim_{i \in I} Y_i$$

in the sense of [SW]. The functor $Y \mapsto \hat{Y}$ induces a fully faithful embedding $\mathrm{pro} - X_{\text{fét}} \to X_{\text{profét}}$ which identifies $X_{\text{profét}}$ with the stackification of the prestack $U \in X_{\text{an}} \mapsto \mathrm{pro} - U_{\text{fét}}$.

3 Diamonds

In this section we introduce the notion of a diamond, following [Sch14]. Due to the relative novelty of this theory and the lack (as yet) of a permanent written reference, we develop strictly more material than we'll need later in terms of examples and basic constructions. We hope this increases the readability of the paper enough to justify some occasional long-windedness in this section.

On the other hand, our goal here is *not* to establish in detail the foundational properties of diamonds. Indeed, the properties we need are summarized in Fact 3.4 below, and in our experience knowing the *proofs* of these results does not help one *use* them in practice. Instead, we advise the reader to simply take the properties enumerated in Fact 3.4 on faith and then to work with diamonds as they would any other category of reasonably behaved algebro-geometric objects. The reader wishing to understand these foundational matters should consult [Sch14] or [Han16]. Full foundations will appear in [SW16].

3.1 Definition and key properties

Definition 3.1 ([Sch14, Definition 8.2.8/2])**.** A *diamond* is a sheaf $\mathcal{D}$ on $\mathrm{Perf}^{\mathrm{proet}}$ which admits a surjective and pro-étale morphism $h_X \to \mathcal{D}$ from a representable sheaf. A *morphism* of diamonds is a morphism of sheaves on $\mathrm{Perf}^{\mathrm{proet}}$. We write Dia for the category of diamonds.

If $\mathcal{D}$ is a diamond, we refer to any choice of a surjective pro-étale morphism $h_X \to \mathcal{D}$ as a *presentation* of $\mathcal{D}$. Here is a first sanity check.

Proposition 3.2. *The functor*

$$\begin{array}{ccc} \mathrm{Perf} & \to & \mathrm{Dia} \\ X & \mapsto & h_X \end{array}$$

is fully faithful.

Proof. This is an immediate consequence of Proposition 2.16 and the Yoneda lemma. □

Given $X \in \mathrm{Perf}$, we denote h_X interchangeably by $X^\diamondsuit$ (cf. Fact 3.4.1 below).

We note the following easy bootstrap.

Proposition 3.3. *Let $\mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on* $\mathrm{Perf}^{\mathrm{proet}}$*. If $\mathcal{F} \to \mathcal{G}$ is surjective and pro-étale, then $\mathcal{F}$ is a diamond if and only if $\mathcal{G}$ is a diamond. If $\mathcal{G}$ is a diamond and $\mathcal{F} \to \mathcal{G}$ is representable, then $\mathcal{F}$ is a diamond.*

Fact 3.4. We record here the salient facts concerning diamonds (cf. [Sch14, SW16, Han16]).

1. There is a natural functor $X \mapsto X^\diamondsuit$ from the category Adic to the category of diamonds, defined as follows. Given any perfectoid space $Y \in \mathrm{Perf}$, an *untilt of Y over X* is a triple $(Y^\sharp, \iota, f)$ where $(Y^\sharp, \iota)$ is an untilt of Y and $f : Y^\sharp \to X$ is a morphism of adic spaces. A *morphism* $m : (Y^\sharp, \iota, f) \to (Y^{\sharp\prime}, \iota', f')$ of untilts of Y over X is a morphism of perfectoid spaces $m : Y^\sharp \to Y^{\sharp\prime}$ such that $\iota' \circ m^\flat = \iota$ and $f' \circ m = f$. Any morphism of untilts of Y over X is again an isomorphism, and a given $(Y^\sharp, \iota, f)$ has no automorphisms. Then

$$X^\diamondsuit : \mathrm{Perf}^{\mathrm{proet}} \to \mathrm{Sets}$$

is the presheaf sending $Y \in \mathrm{Perf}$ to the set of isomorphism classes of untilts of Y over X. The association $X \mapsto X^\diamondsuit$ is clearly a functor.

When X is a perfectoid space, the equivalence $\widetilde{\mathrm{Perf}}_{/X} \cong \mathrm{Perf}_{/X^\flat}$ easily implies that $X^\diamondsuit \cong h_{X^\flat}$ (explaining our previous notation), so $X^\diamondsuit$ is a diamond in this case. It is true, but not obvious, that $X^\diamondsuit$ is a diamond in general. We sketch the proof for affinoid X in Lemma 3.10 below.

Notation: If R (resp. (R, R^+)) is a Tate ring (resp. a Tate-Huber pair) over $\mathbf{Z}_p$, we set $\mathrm{Spd}\, R := \mathrm{Spa}(R, R^\circ)^\diamondsuit$ (resp. $\mathrm{Spd}(R, R^+) := \mathrm{Spa}(R, R^+)^\diamondsuit$).

2. If $E/\mathbf{Q}_p$ is a discretely valued nonarchimedean field with perfect residue field of characteristic p, the functor from normal rigid analytic spaces over $\mathrm{Spa}\, E$ to diamonds over $\mathrm{Spd}\, E$ is fully faithful.

3. There is a natural functor $\mathcal{D} \mapsto |\mathcal{D}|$ from diamonds to topological spaces such that for any $X \in \mathrm{Adic}$, there is a natural homeomorphism $|X^\diamondsuit| \cong |X|$.

4. For any diamond $\mathcal{D}$, open immersions $\mathcal{E} \hookrightarrow \mathcal{D}$ are in natural bijection with open subsets $|\mathcal{E}| \subseteq |\mathcal{D}|$.

5. The category of diamonds admits fiber products and products of pairs, and well-behaved notions of quasicompact and quasiseparated objects and morphisms.

6. For any $S \in \mathrm{Adic}$, there is a natural equivalence $\mathrm{Perf}_{/S^\diamondsuit} \cong \widetilde{\mathrm{Perf}}_{/S}$, so a diamond over $S^\diamondsuit$ can be regarded as a functor on perfectoid spaces over S. For any morphisms $Y \to S$ and $X \to S$ with Y perfectoid, we have
$$\mathrm{Hom}_S(Y, X) \cong \mathrm{Hom}_{S^\diamondsuit}\left(Y^\diamondsuit, X^\diamondsuit\right).$$

7. Let $\mathcal{D}$ be a diamond, and let $\bullet$ be one of the decorations $\bullet \in \{\mathrm{an}, \text{fét}, \text{ét}, \text{profét}, \text{proét}\}$, respectively. Then there is a well-behaved site "$\mathcal{D}_\bullet$" with objects given by diamonds $\mathcal{E}$ over $\mathcal{D}$ such that the map $\mathcal{E} \to \mathcal{D}$ is (respectively) an open immersion, a finite étale map, an étale map, a pro-finite étale map, or a pro-étale map, and with covers given by collections $\{\mathcal{E}_i \to \mathcal{E}\}$ such that $\coprod \mathcal{E}_i \to \mathcal{E}$ is surjective as a map of sheaves.

8. If $X \in \mathrm{Adic}$ is locally Noetherian or perfectoid, the functor $(-)^\diamondsuit$ induces a natural equivalence $X_\bullet \cong X^\diamondsuit_\bullet$ for $\bullet \in \{\mathrm{an}, \text{fét}, \text{ét}\}$.

3.2 Pro-étale torsors

This section is partly based on Sections 4.2-4.3 of [Wei15]. Throughout, G denotes a locally profinite group.

Definition 3.5. Let G be a locally profinite group. A morphism of perfectoid spaces $\tilde{X} \to X$ is a *pro-étale G-torsor* if there is a G-action on $\tilde{X}$ lying over the trivial G-action on X such that $\tilde{X} \times_X \tilde{X} \cong \underline{G} \times \tilde{X}$ and such that there exists a G-equivariant isomorphism $\tilde{X} \times_X X' \simeq X' \times \underline{G}$ after pullback to some pro-étale cover $X' \to X$.[9]

A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves on $\mathrm{Perf}^{\mathrm{proet}}$ is a *pro-étale G-torsor* if there is a G-action on $\mathcal{F}$ lying over the trivial G-action on $\mathcal{G}$ such that, pro-étale locally on $\mathcal{G}$, we have $\mathcal{F} \simeq \mathcal{G} \times G$.

[9]These conditions actually guarantee the continuity of the G-action on $\tilde{X}$, in the sense of [Sch15a].

Remark. If $\mathcal{F} \to \mathcal{G}$ is a pro-étale G-torsor, the induced map $\mathcal{F}/\underline{G} \to \mathcal{G}$ is an isomorphism (since it becomes an isomorphism pro-étale-locally on $\mathcal{G}$, and everything is a pro-étale sheaf), and the action map $\underline{G} \times \mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is an isomorphism as well.

Proposition 3.6. *If $\mathcal{F} \to \mathcal{G}$ is a pro-étale G-torsor with G profinite, then $\mathcal{F} \to \mathcal{G}$ is surjective and pro-étale. In particular, $\mathcal{G}$ is a diamond if and only if $\mathcal{F}$ is a diamond.*

This is slightly tricky, since we don't know *a priori* that $\mathcal{F} \to \mathcal{G}$ is even representable.

Proof (after Weinstein). Surjectivity is clear. Since G is profinite, we may write $G = \lim_{\leftarrow i} G/H_i$ with H_i open normal in G. Then $\mathcal{F} = \lim_{\leftarrow i} \mathcal{F}/\underline{H_i}$ as sheaves, so it suffices to show that each $\mathcal{F}/\underline{H_i} \to \mathcal{G}$ is finite étale. This is true on a pro-étale cover of $\mathcal{G}$, by definition, so we're reduced to showing that whether or not a morphism $\mathcal{F} \to \mathcal{G}$ is *finite étale* can be checked pro-étale-locally on $\mathcal{G}$. We may clearly assume $\mathcal{G}$ is representable.

So let $\mathcal{F} \to h_X$ be any sheaf map, and let $h_{\tilde{X}} \to h_X$ be a surjective pro-étale map such that $\mathcal{F} \times_{h_X} h_{\tilde{X}}$ is representable, say by $h_{\tilde{Y}}$, and such that $h_{\tilde{Y}} \to h_{\tilde{X}}$ is finite étale. Since

$$\begin{aligned} \tilde{Y} \times_{\tilde{X}, \mathrm{pr}_i} (\tilde{X} \times_X \tilde{X}) &\cong \mathcal{F} \times_X \tilde{X} \times_{\tilde{X}, \mathrm{pr}_i} (\tilde{X} \times_X \tilde{X}) \\ &\cong \mathcal{F} \times_X (\tilde{X} \times_X \tilde{X}), \end{aligned}$$

where $\mathrm{pr_i} : \tilde{X} \times_X \tilde{X} \to \tilde{X}$ denotes either of the two projections, the two pullbacks of $\tilde{Y}$ under pr_i are canonically isomorphic, and one easily checks that these isomorphisms satisfy the usual cocycle condition. In particular, the finite étale map $\tilde{Y} \to \tilde{X}$ inherits a descent datum relative to the pro-étale cover $\tilde{X} \to X$. Now the key technical point is that finite étale maps form a stack for the pro-étale topology on any perfectoid space X: precisely, the fibered category $p : \mathrm{F\acute{E}t} \to \mathrm{Perf}$ defined by

$$p^{-1}(U) = \{V \to U \,\text{finite étale}\}$$

for any $U \in \mathrm{Perf}$ is a stack on $\mathrm{Perf}^{\mathrm{proet}}$. (This is Lemma 4.2.4 of [Wei15].) Thus $\tilde{Y}$ descends uniquely to a finite étale map $Y \to X$, and one easily checks that $\mathcal{F} \cong h_Y$.

The final claim follows immediately from Proposition 3.3. □

Remark 3.7. Let us a say a pro-étale G-torsor $\mathcal{F} \to \mathcal{G}$ (or $\tilde{X} \to X$) is *good* if the map $\mathcal{F} \to \mathcal{G}$ is pro-étale (and in particular, representable). By the previous proposition, any pro-étale G-torsor for a profinite group G is good. It's unclear, however, if goodness holds for arbitrary locally profinite G. We'll prove in Corollary 4.14 below that when $\mathcal{D}$ is a diamond and G is (open in) the $\mathbf{Q}_p$-points of a reductive group $\mathbf{G}/\mathbf{Q}_p$, any pro-étale G-torsor $\tilde{\mathcal{D}} \to \mathcal{D}$ is good, by taking a somewhat circuitous route through $\mathbf{Q}_p$-local systems and relative p-adic Hodge theory.

We also have a nice "punctual" criterion for when the formation of a quotient by a profinite group action gives a pro-étale torsor:

Theorem 3.8. *Let G be a profinite group, and let $\mathcal{F}$ be a sheaf on $\mathrm{Perf}^{\mathrm{proet}}$ equipped with an action of G. Suppose G acts freely on the sections $\mathcal{F}(C, \mathcal{O}_C)$ for any algebraically closed nonarchimedean field C in characteristic p. Then $\mathcal{F} \to \mathcal{F}/\underline{G}$ is a good pro-étale G-torsor.*

Here $\mathcal{F}(C, \mathcal{O}_C)$ is of course shorthand for the set of sheaf maps $h_{\mathrm{Spa}(C, \mathcal{O}_C)} \to \mathcal{F}$.

Proof. This is exactly Proposition 4.3.2 of [Wei15]. Note however that Proposition 4.3.2 of *loc. cit.* is only stated in the case where $\mathcal{F}$ is a representable sheaf living over $h_{\mathrm{Spa}(C_0, \mathcal{O}_{C_0})}$ for some fixed algebraically closed nonarchimedean field $C_0/\mathbf{F}_p$, but a careful reading shows that Weinstein's proof doesn't use these assumptions in any way. □

The special case of Theorem 3.8 where $\mathcal{F}$ is representable gives the first real mechanism for constructing diamonds. Due to its importance, we state it separately:

Theorem 3.9. *Let X be a perfectoid space in characteristic p, and let G be a profinite group acting on X. Suppose G acts freely on the set $X(C, \mathcal{O}_C)$ for any algebraically closed nonarchimedean field C in characteristic p. Then $h_X \to h_X/\underline{G}$ is a good pro-étale G-torsor, so $h_X/\underline{G}$ is a diamond and $h_X \to h_X/\underline{G}$ is a presentation. Furthermore, there is a natural homeomorphism $|h_X/\underline{G}| \cong |X|/G$.*

Proof. By Theorem 3.8, $h_X \to h_X/\underline{G}$ is a pro-étale G-torsor, so the sheaf map $h_X \to h_X/\underline{G}$ is surjective and pro-étale by Proposition 3.6. The claim on topological spaces follows from the definition of $|h_X/\underline{G}|$, cf. the proof of [Han16, Proposition 3.15]. ◻

Lemma 3.10. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid adic space. Choose a directed system $(R_i, R_i^+), i \in I$ of finite étale Galois (R, R^+)-algebras such that*

$$(\tilde{R}, \tilde{R}^+) = \widehat{\lim_{i\to}(R_i, R_i^+)}$$

is perfectoid, where the completion is for the topology making $\lim_{i\to} R_i^+$ open and bounded. (By an argument of Colmez and Faltings, such a direct system always exists.) Let G_i be the Galois group of R_i over R, so G_i acts on $X_i = \mathrm{Spa}(R_i, R_i^+)$ and by continuity $G = \lim_{\leftarrow i} G$ acts on $\tilde{X} = \mathrm{Spa}(\tilde{R}, \tilde{R}^+)$. Then:

1. *The space $\tilde{X}^\flat$ with its induced action of G satisfies the hypotheses of Theorem 3.9. Consequently, $h_{\tilde{X}^\flat} \to h_{\tilde{X}^\flat}/\underline{G}$ is a pro-étale G-torsor and $h_{\tilde{X}^\flat}/\underline{G}$ is a diamond.*
2. *There is a natural isomorphism*
$$X^\diamond \cong h_{\tilde{X}^\flat}/\underline{G}.$$

In particular, $X^\diamond$ is a diamond.

3.3 The diamond of $\mathrm{Spa}\,\mathbf{Q}_p$

By Fact 3.4.1, the adic space $\mathrm{Spa}\,\mathbf{Q}_p$ has an associated diamond $\mathrm{Spd}\,\mathbf{Q}_p$. We explain here its canonical presentation, as a special case of Theorem 3.10. Note that as a functor, $\mathrm{Spd}\,\mathbf{Q}_p$ is quote simple: the set of maps $h_S \to \mathrm{Spd}\,\mathbf{Q}_p$ is canonically identified with the set of untilts $(S^\sharp, \iota)$ where p is invertible in $\mathcal{O}_{S^\sharp}$, i.e. the set of untilts where $S^\sharp$ is a perfectoid space over $\mathrm{Spa}\,\mathbf{Q}_p$.

Let $\zeta_{p^n}, n \geq 1$ be a compatible sequence of primitive p^nth roots of unity, and let $\mathbf{Q}_p^{\mathrm{cyc}} = \widehat{\mathbf{Q}_p(\zeta_{p^\infty})}$. Let us describe the tilt of $\mathbf{Q}_p^{\mathrm{cyc}}$. Setting $\mathcal{O}_n = \mathbf{Z}_p[\zeta_{p^n}]$, we have an isomorphism of $\mathbf{F}_p$-algebras

$$\begin{array}{rcl} \mathcal{O}_n/(\zeta_p - 1) & = & \mathbf{F}_p[t]/(t^{p^{n-1}(p-1)}) \\ \zeta_{p^n} & \mapsto & 1 + t, \end{array}$$

or equivalently

$$\begin{array}{rcl} \mathcal{O}_n/(\zeta_p - 1) & = & \mathbf{F}_p[t^{p^{1-n}}]/(t^{p-1}) \\ \zeta_{p^n} & \mapsto & 1 + t^{p^{1-n}}. \end{array}$$

Taking the inductive limit over n, we get

$$\mathbf{Z}_p^{\mathrm{cyc}}/(\zeta_p - 1) = \mathbf{F}_p[t^{1/p^\infty}]/(t^{p-1}),$$

and then applying $(-)^\flat = \lim_{\substack{\leftarrow \\ x\mapsto x^p}}(-)$ gives $\mathbf{Z}_p^{\mathrm{cyc},\flat} \cong \mathbf{F}_p[[t^{1/p^\infty}]]$ and $\mathbf{Q}_p^{\mathrm{cyc},\flat} \cong \mathbf{F}_p((t^{1/p^\infty}))$. Note that the sharp map $\mathbf{Q}_p^{\mathrm{cyc},\flat} \to \mathbf{Q}_p^{\mathrm{cyc}}$ sends t to

$$\varpi := \lim_{n\to\infty}(\zeta_{p^n}-1)^{p^{n-1}}.$$

Since $\mathbf{Q}_p^{\mathrm{cyc}} = \widehat{\mathbf{Q}_p(\zeta_{p^\infty})}$ is a perfectoid pro-étale $\mathbf{Z}_p^\times$-torsor over $\mathbf{Q}_p$ with tilt $\mathbf{Q}_p^{\mathrm{cyc},\flat} \cong \mathbf{F}_p((t^{1/p^\infty}))$, Lemma 3.10 implies the following result.

Proposition 3.11. *There is a natural isomorphism* $\operatorname{Spd}\mathbf{Q}_p \cong h_{\operatorname{Spa}\mathbf{F}_p((t^{1/p^\infty}))}/\underline{\mathbf{Z}_p^\times}$*, where* $a \in \mathbf{Z}_p^\times$ *acts by sending* t^{1/p^n} *to* $(1+t^{1/p^n})^a - 1$.

A similar result holds for any finite extension $K/\mathbf{Q}_p$. Precisely, let $\mathbf{F}_q$ be the residue field of K, and let $\mathcal{G} \in \mathcal{O}_K[[X,Y]]$ be a Lubin-Tate formal $\mathcal{O}_K$-module law. For $a \in \mathcal{O}_K$, let $[a](T) = \exp_{\mathcal{G}}(a\log_{\mathcal{G}}(T)) \in \mathcal{O}_K[[T]]$ be the series representing multiplication by a on $\mathcal{G}$. Then $\operatorname{Spd} K \cong h_{\operatorname{Spa}\mathbf{F}_q((t^{1/q^\infty}))}/\underline{\mathcal{O}_K^\times}$ where $a \in \mathcal{O}_K^\times$ acts by sending t^{1/q^n} to $[a](t^{1/q^n})$.

Self-products of the diamond of $\operatorname{Spa}\mathbf{Q}_p$

Since diamonds admit products of pairs, the diamond $(\operatorname{Spd}\mathbf{Q}_p)^n$ is well-defined. This is an intricate object for $n>1$, no longer in the essential image of the functor $(-)^\diamond$. We note, among other things, that $|(\operatorname{Spd}\mathbf{Q}_p)^n|$ has Krull dimension $n-1$. This seems to suggest that the (nonexistent) structure map "$\operatorname{Spa}\mathbf{Q}_p \to \operatorname{Spa}\mathbf{F}_1$" has relative dimension one.

3.4 The diamond $B_{\mathrm{dR}}^+/\mathrm{Fil}^n$

For any perfectoid Tate ring $R/\mathbf{Q}_p$, we have the de Rham period ring $\mathbb{B}_{\mathrm{dR}}^+(R)$, defined as the completion of $W(R^{\flat\circ})[\frac{1}{p}]$ along the kernel of the natural surjection $\theta : W(R^{\flat\circ})[\frac{1}{p}] \twoheadrightarrow R$. When $R = \mathbf{C}_p$, this is the usual Fontaine ring B_{dR}^+. In general $\ker\theta$ is principal and generated by some non-zerodivisor ξ, so $\mathbb{B}_{\mathrm{dR}}^+(R)$ is filtered by the ideals $\mathrm{Fil}^i = (\ker\theta)^i$ with associated gradeds $\mathrm{gr}^i\mathbb{B}_{\mathrm{dR}}^+(R) \simeq \xi^i R$.

Let $B_{\mathrm{dR}}^+/\mathrm{Fil}^n \to \operatorname{Spd}\mathbf{Q}_p$ be the functor whose sections over a given map $\operatorname{Spa}(R,R^+)^\diamond \to \operatorname{Spd}\mathbf{Q}_p$ for any $\operatorname{Spa}(R,R^+) \in \mathrm{Perf}$ are given by the set $\mathbb{B}_{\mathrm{dR}}^+(R^\sharp)/\mathrm{Fil}^n$, where $R^\sharp$ is the untilt of R determined by the given map to $\operatorname{Spd}\mathbf{Q}_p$.

Theorem 3.12. *The functor* $B_{\mathrm{dR}}^+/\mathrm{Fil}^n$ *is a diamond.*

This is Proposition 18.2.3 in the Berkeley notes, where a "qpf" proof is given. We complement this with a "pro-étale" proof here. Note also that $B_{\mathrm{dR}}^+/\mathrm{Fil}^n$ is a ring diamond (we recommend avoiding jokes about $B_{\mathrm{dR}}^+/\mathrm{Fil}^n$ being a "diamond ring").

Proof. Induction on n. The case $n=1$ is clear, since $B_{\mathrm{dR}}^+/\mathrm{Fil}^1 \cong \mathbf{A}^{1,\diamond}$ where $\mathbf{A}^1 = \cup_{n\geq 1}\operatorname{Spa}\mathbf{Q}_p\langle p^n x\rangle$ is the affine line over $\operatorname{Spa}\mathbf{Q}_p$.

Let $\mathbf{B} = (\operatorname{Spa}\mathbf{Z}_p^{\mathrm{cyc}}[[T]])_\eta$, and let $\tilde{\mathbf{B}} = \lim_{\leftarrow\varphi}\mathbf{B}$ where φ is the endomorphism of $\mathbf{B}$ given by the map $T \mapsto (1+T)^p - 1$. Explicitly, $\tilde{\mathbf{B}} = (\operatorname{Spa} A)_\eta$ where A is the (p,T)-adic completion of

$$\mathbf{Z}_p^{\mathrm{cyc}}[[T]][T_1,T_2,\dots]/(\varphi(T_1)-T,\varphi(T_2)-T_1,\dots,\varphi(T_{i+1})-T_i,\dots).$$

Then $\tilde{\mathbf{B}}$ is perfectoid, and the map $\pi : \tilde{\mathbf{B}} \to \mathbf{A}^1$ characterized by $\pi^* x = \log(1+T)$ is a perfectoid pro-étale covering of $\mathbf{A}^1$. (More precisely, $\tilde{\mathbf{B}}^\lozenge \to \mathbf{A}^{1,\lozenge}$ is a good pro-étale G-torsor, with $G = \mathbf{Q}_p(1) \rtimes \mathbf{Z}_p^\times$.)

Claim: *There is an isomorphism of functors*

$$B_{\mathrm{dR}}^+/\mathrm{Fil}^n \times_{\mathbf{A}^{1,\lozenge}} \tilde{\mathbf{B}}^\lozenge \quad \simeq \quad B_{\mathrm{dR}}^+/\mathrm{Fil}^{n-1} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \tilde{\mathbf{B}}^\lozenge$$

for any $n \geq 2$.

Granted this claim, we deduce the theorem as follows: Since $\tilde{\mathbf{B}}^\lozenge \to \mathbf{A}^{1,\lozenge}$ is surjective and pro-étale, the map

$$B_{\mathrm{dR}}^+/\mathrm{Fil}^n \times_{\mathbf{A}^{1,\lozenge}} \tilde{\mathbf{B}}^\lozenge \to B_{\mathrm{dR}}^+/\mathrm{Fil}^n$$

is surjective and pro-étale, so by Proposition 3.3 it's enough to show that $B_{\mathrm{dR}}^+/\mathrm{Fil}^n \times_{\mathbf{A}^{1,\lozenge}} \tilde{\mathbf{B}}^\lozenge$ is a diamond. But the claim together with the existence of fiber products reduces this to the fact that $B_{\mathrm{dR}}^+/\mathrm{Fil}^{n-1}$ is a diamond, which is exactly our induction hypothesis.

It remains to prove the claim. In general, the map θ induces a natural transformation $B_{\mathrm{dR}}^+/\mathrm{Fil}^n \to \mathbf{A}^{1,\lozenge}$ of functors over $\mathrm{Spd}\,\mathbf{Q}_p$. Let $s : \tilde{\mathbf{B}} \to B_{\mathrm{dR}}^+/\mathrm{Fil}^n$ be the natural transformation sending $r = (r_0, r_1, r_2, \dots) \in \tilde{\mathbf{B}}(R^\sharp)$ to $\log[1 + r^\flat]$ where $r^\flat = (\overline{r_0}, \overline{r_1}, \dots) \in (R^{\sharp\circ}/p)^\flat$. Then $(\theta \circ s)(r) = \log(1 + r_0) = \pi(r)$. In particular, the diagram

$$\begin{tikzcd} & \tilde{\mathbf{B}}^\lozenge \arrow[dl, "s"'] \arrow[d, "\pi"] \\ B_{\mathrm{dR}}^+/\mathrm{Fil}^n \arrow[r, "\theta"] & \mathbf{A}^{1,\lozenge} \end{tikzcd}$$

commutes, and π is surjective and pro-étale. The claim is then deduced as follows: given $(a, b) \in B_{\mathrm{dR}}^+/\mathrm{Fil}^n \times \tilde{\mathbf{B}}^\lozenge$ with $\theta(a) = \pi(b)$, the element $a - s(b)$ lives in $\ker\theta \subset B_{\mathrm{dR}}^+/\mathrm{Fil}^n$, so we get a natural isomorphism

$$\begin{array}{rcl} B_{\mathrm{dR}}^+/\mathrm{Fil}^n \times_{\mathbf{A}^{1,\lozenge}} \tilde{\mathbf{B}}^\lozenge & \cong & \ker\theta \times_{\mathrm{Spd}\,\mathbf{Q}_p} \tilde{\mathbf{B}}^\lozenge \\ (a, b) & \to & (a - s(b), b) \\ (x + s(y), y) & \leftarrow & (x, y). \end{array}$$

Furthermore, after base change to $\mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}$ we have an exact sequence

$$0 \to B_{\mathrm{dR}}^+/\mathrm{Fil}^{n-1} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}} \xrightarrow{t} B_{\mathrm{dR}}^+/\mathrm{Fil}^n \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}} \to \mathbf{A}^{1,\lozenge}_{\mathbf{Q}_p^{\mathrm{cyc}}} \to 0$$

where t is the usual $\log[\varepsilon]$ of p-adic Hodge theory, so we get an isomorphism

$$\ker\theta \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}} \simeq B_{\mathrm{dR}}^+/\mathrm{Fil}^{n-1} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}$$

after making a choice of t. But $\tilde{\mathbf{B}}$ naturally lives over $\mathbf{Q}_p^{\mathrm{cyc}}$, so then

$$\begin{array}{rcl} \ker\theta \times_{\mathrm{Spd}\,\mathbf{Q}_p} \tilde{\mathbf{B}}^\lozenge & \cong & (\ker\theta \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}) \times_{\mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}} \tilde{\mathbf{B}}^\lozenge \\ & \simeq & \left(B_{\mathrm{dR}}^+/\mathrm{Fil}^{n-1} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}\right) \times_{\mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}} \tilde{\mathbf{B}}^\lozenge \\ & \simeq & B_{\mathrm{dR}}^+/\mathrm{Fil}^{n-1} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \tilde{\mathbf{B}}^\lozenge, \end{array}$$

and the claim follows. $\square$

3.5 The diamond $B^{+,\varphi_q=\pi^n}_{\mathrm{crys},E}$

If R is any perfectoid Tate ring over $\mathbf{Q}_p$, we have the Fontaine rings $\mathbb{A}_{\mathrm{crys}}(R)$ and $\mathbb{B}^+_{\mathrm{crys}}(R) = \mathbb{A}_{\mathrm{crys}}(R)[\frac{1}{p}]$, where $\mathbb{A}_{\mathrm{crys}}(R)$ is defined as the p-adic completion of the PD envelope of $W(R^{\flat\circ})$ with respect to the ideal $\ker(\theta : W(R^{\flat\circ}) \to R^\circ/p)$. Note that $\mathbb{A}_{\mathrm{crys}}(R)$ is flat over $\mathbf{Z}_p$, and $\mathbb{B}^+_{\mathrm{crys}}(R)$ is naturally a Banach ring over $\mathbf{Q}_p$ with unit ball $\mathbb{A}_{\mathrm{crys}}(R)$. We also recall that $W(R^{\flat\circ})$ is a subring of $\mathbb{A}_{\mathrm{crys}}(R)$, the Witt vector Frobenius φ on $W(R^{\flat\circ})$ extends to these rings, and there is a natural injective ring map $\mathbb{B}^+_{\mathrm{crys}}(R) \to \mathbb{B}^+_{\mathrm{dR}}(R)$ extending the map $W(R^{\flat\circ}) \to \mathbb{B}^+_{\mathrm{dR}}(R)$.

Now let E be a finite extension of $\mathbf{Q}_p$, with uniformizer π, maximal unramified subfield E_0, and residue field

$$\mathcal{O}_{E_0}/p = \mathcal{O}_E/\pi = \mathbf{F}_q = \mathbf{F}_{p^f}.$$

If R is any perfectoid Tate ring over E, then $\mathbb{B}^+_{\mathrm{crys}}(R)$ is naturally an E_0-algebra and the action of $\varphi_q = \varphi^f$ is E_0-linear. In particular, φ_q acts E-linearly on the base extension $\mathbb{B}^+_{\mathrm{crys},E}(R) = \mathbb{B}^+_{\mathrm{crys}}(R) \otimes_{E_0} E$, so we get an E-vector space $\mathbb{B}^{+,\varphi_q=\pi^n}_{\mathrm{crys},E}(R)$ inside $\mathbb{B}^+_{\mathrm{crys},E}(R)$. The association $R \mapsto \mathbb{B}^{+,\varphi_q=\pi^n}_{\mathrm{crys},E}(R)$ defines a functor from perfectoid Tate rings over E to E-Banach spaces; we write $B^{+,\varphi_q=\pi^n}_{\mathrm{crys},E}$ for the associated sheaf on $\mathrm{Perf}^{\mathrm{proet}}_{/\mathrm{Spd}\,E}$. We also note that the map $\mathbb{B}^+_{\mathrm{crys}}(R) \to \mathbb{B}^+_{\mathrm{dR}}(R)$ extends (via the canonical E-algebra structure on $\mathbb{B}^+_{\mathrm{dR}}(R)$) to an injective E-algebra map $\mathbb{B}^+_{\mathrm{crys},E}(R) \to \mathbb{B}^+_{\mathrm{dR}}(R)$. In particular, we get a natural transformation $B^{+,\varphi_q=\pi^n}_{\mathrm{crys},E} \to B^+_{\mathrm{dR}}/\mathrm{Fil}^n \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,E$.

Theorem 3.13. i. *The natural transformation $B^{+,\varphi_q=\pi^n}_{\mathrm{crys},E} \to B^+_{\mathrm{dR}}/\mathrm{Fil}^n \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,E$ induces a short exact sequence*

$$0 \to V_{\pi,n} \to B^{+,\varphi_q=\pi^n}_{\mathrm{crys},E} \to B^+_{\mathrm{dR}}/\mathrm{Fil}^n \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,E \to 0$$

of sheaves of E-vector spaces on $\mathrm{Perf}^{\mathrm{proet}}_{/\mathrm{Spd}\,E}$, where $V_{\pi,n} \simeq \underline{E}$ pro-étale-locally on $\mathrm{Spd}\,E$.

ii. *The sheaf $B^{+,\varphi_q=\pi^n}_{\mathrm{crys},E}$ is a diamond.*

Remark. Explicitly, $V_{\pi,n}$ is the n-fold tensor product $\otimes^n_E V_p\mathcal{G}_\pi$, where $V_p\mathcal{G}_\pi$ is (the sheaf associated with) the functor sending a perfectoid $(E, \mathcal{O}_E)$-algebra (R, R^+) to the E-vector space $V_p\mathcal{G}_\pi(R, R^+) = (\lim_{\leftarrow n} \mathcal{G}_\pi[\pi^n](R^+)) \otimes_{\mathcal{O}_E} E$. Here $\mathcal{G}_\pi$ is the Lubin-Tate formal $\mathcal{O}_E$-module law associated with the uniformizer π. In particular, if $E = \mathbf{Q}_p$ and $\pi = p$, then $V_{\pi,n} \cong \underline{\mathbf{Q}_p(n)}$.

Remark. The idea that the functors $B^+_{\mathrm{dR}}/\mathrm{Fil}^n$ and $B^{+,\varphi_q=\pi^n}_{\mathrm{crys},E}$ might have some reasonable geometric structure is due to Colmez, who defined [Col02] a beautiful category of "finite-dimensional Banach Spaces" (with a capital "S") containing these objects. (We note that Colmez notates these spaces as $\mathbb{B}_n$ and $\mathbb{U}_{E,n}$, respectively.)

3.6 The de Rham affine Grassmannian

Fix a reductive group $\mathbf{G}$ over $\mathbf{Q}_p$, and let $\mu : \mathbf{G}_{m,E} \to \mathbf{G}_E$ be a cocharacter defined over a fixed finite extension $E/\mathbf{Q}_p$. Let $\breve{E}$ denote the completion of the maximal unramified extension of E. The functors of interest in this section are the following:

Definition 3.14. The *de Rham affine Grassmannian* $\mathrm{Gr}_{\mathbf{G}}$ is the functor on perfectoid $(\mathbf{Q}_p, \mathbf{Z}_p)$-algebras sending (R, R^+) to the set of (isomorphism classes of) $\mathbf{G}$-torsors over $\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}(R)$ equipped with a trivialization over $\operatorname{Spec}\mathbb{B}_{\mathrm{dR}}(R)$. We regard $\mathrm{Gr}_{\mathbf{G}}$ as a functor fibered over $\operatorname{Spd}\mathbf{Q}_p$ in the obvious way.

For any cocharacter as above, let $\mathrm{Gr}_{\mathbf{G},\leq\mu}\subset\mathrm{Gr}_{\mathbf{G}}\times_{\operatorname{Spd}\mathbf{Q}_p}\operatorname{Spd}E$ be the functor on perfectoid $(E,\mathcal{O}_E)$-algebras sending (R,R^+) to the set of $\mathbf{G}$-torsors over $\operatorname{Spec}\mathbb{B}^+_{\mathrm{dR}}(R)$ equipped with a μ-*bounded* trivialization over $\operatorname{Spec}\mathbb{B}_{\mathrm{dR}}(R)$, and let $\mathrm{Gr}_{\mathbf{G},\mu}\subset\mathrm{Gr}_{\mathbf{G},\leq\mu}$ be the subfunctor where the relative position of the trivialization is given exactly by μ.

One checks directly that $\mathrm{Gr}_{\mathbf{G},\mu}$ and $\mathrm{Gr}_{\mathbf{G},\leq\mu}$ are sheaves on $\mathrm{Perf}^{\mathrm{proet}}_{/\operatorname{Spd}E}$. We remark that $\mathrm{Gr}_{\mathbf{G},\leq\mu}$ coincides with $\mathrm{Gr}_{\mathbf{G},\mu}$ when μ is minuscule, but in general there is a nontrivial stratification $\mathrm{Gr}_{\mathbf{G},\leq\mu}=\coprod_{\nu\leq\mu}\mathrm{Gr}_{\mathbf{G},\nu}$. For later use, we also give the Tannakian interpretation of $\mathrm{Gr}_{\mathbf{G},\leq\mu}$ and $\mathrm{Gr}_{\mathbf{G},\mu}$:

Proposition 3.15. $\mathrm{Gr}_{\mathbf{G},\leq\mu}$ *(resp.* $\mathrm{Gr}_{\mathbf{G},\mu}$*) is the functor on perfectoid* $(E,\mathcal{O}_E)$*-algebras sending* (R,R^+) *to the set of associations*

$$\Lambda:(\rho,W)\in\operatorname{Rep}(\mathbf{G})\to\{\Lambda_W\subset W\otimes_{\mathbf{Q}_p}\mathbb{B}_{\mathrm{dR}}(R)\text{ a }\mathbb{B}^+_{\mathrm{dR}}(R)\text{−lattice}\}$$

compatible with tensor products and short exact sequences, such that for all $(\rho,W)\in\operatorname{Rep}(\mathbf{G})$ *and all geometric points* $s=\operatorname{Spa}(C,C^+)\to\operatorname{Spa}(R,R^+)$*, there is a* $\mathbb{B}^+_{\mathrm{dR}}(C)$*-basis* $v_1,\dots,v_{\dim W}$ *(resp.* $e_1,\dots,e_{\dim W}$*) of* $W\otimes_{\mathbf{Q}_p}\mathbb{B}^+_{\mathrm{dR}}(C)$ *(resp. of* $s^*\Lambda_W$*) such that*

$$\begin{pmatrix} e_1\\ e_2\\ \vdots\\ e_{\dim W}\end{pmatrix}=(\rho\circ\nu)(\xi)\cdot\begin{pmatrix} v_1\\ v_2\\ \vdots\\ v_{\dim W}\end{pmatrix}$$

for some generator ξ *of* $\ker\left(\theta:\mathbb{B}^+_{\mathrm{dR}}(C)\to C\right)$ *and some* $\nu\in X_*(\mathbf{T})$ *with* $\nu\leq\mu$ *(resp. with* $\nu=\mu$*).*

We'll only need the space $\mathrm{Gr}_{\mathbf{G},\mu}$, but we take the trouble to define $\mathrm{Gr}_{\mathbf{G},\leq\mu}$ since in some ways it's better behaved:

Theorem 3.16 (Scholze)**.** *The functors* $\mathrm{Gr}_{\mathbf{G},\leq\mu}$ *and* $\mathrm{Gr}_{\mathbf{G},\mu}$ *are "spatial" diamonds over* $\operatorname{Spd}E$*, and* $\mathrm{Gr}_{\mathbf{G},\leq\mu}$ *is quasicompact.*

This is one of the main theorems from [Sch14], and the proof is difficult and indirect. However, if one restricts attention to $\mathrm{Gr}_{\mathbf{G},\mu}$, things are more straightforward: using Theorem 3.12, we give below a direct proof[10] that $\mathrm{Gr}_{\mathbf{G},\mu}$ is a diamond.

We also have the flag variety $\mathscr{F}\ell_{\mathbf{G},\mu}$, defined as the adic space over E associated with the flag variety $\mathbf{P}^{\mathrm{std}}_{\mu}\backslash\mathbf{G}$. Here $\mathbf{P}^{\mathrm{std}}_{\mu}\subset\mathbf{G}$ is the parabolic with Levi factor $\mathbf{M}_\mu=\operatorname{Cent}(\mu)$ and with Lie algebra

$$\operatorname{Lie}(\mathbf{P}^{\mathrm{std}}_{\mu})=\oplus_{i\leq0}\mathfrak{g}_\mu(i).$$

We note that in the remainder of this section, we will sometimes pass casually from schemes over E to adic spaces over E, to functors on perfectoid $(E,\mathcal{O}_E)$-algebras, to diamonds over $\operatorname{Spd}E$; in a parallel abuse, we sometimes write $\mathscr{F}\ell_{\mathbf{G},\mu}$ instead of $\mathscr{F}\ell^{\lozenge}_{\mathbf{G},\mu}$ to lighten notation. This should cause no confusion.

In what follows we set $\mathbb{G}(R)=\mathbf{G}(\mathbb{B}^+_{\mathrm{dR}}(R))$, regarded as a group-valued functor on perfectoid $(E,\mathcal{O}_E)$-algebras (R,R^+). Note that θ induces a natural group homomorphism $\theta:\mathbb{G}\to\mathbf{G}$; set $\mathbb{G}^1=\ker\theta$. More generally, if $\mathbf{H}$ is some algebraic subgroup of $\mathbf{G}$ defined over E, we set $\mathbb{H}=$

[10] To be added.

$\mathbf{H}(\mathbb{B}_{\mathrm{dR}}^+) < \mathbb{G}$; change of font in this manner always indicates this convention. We always have a theta map $\theta : \mathbb{H} \to \mathbf{H}$.

If $\xi \in \mathbb{B}_{\mathrm{dR}}^+(R)$ is any element generating $\ker\theta$, we consider the subgroup

$$\mathbb{G}_\mu(R) = \mathbf{G}(\mathbb{B}_{\mathrm{dR}}^+(R)) \cap \mu(\xi)\mathbf{G}(\mathbb{B}_{\mathrm{dR}}^+(R))\mu(\xi)^{-1} < \mathbb{G}(R).$$

This is well-defined independently of choosing ξ, and is functorial in R.

Proposition 3.17. *The sheaf $\mathrm{Gr}_{\mathbf{G},\mu}$ is the pro-étale sheafification of the presheaf on perfectoid $(E, \mathcal{O}_E)$-algebras sending (R, R^+) to $\mathbb{G}_\mu(R)\backslash\mathbb{G}(R)$. Under this identification, the Tannakian interpretation of $\mathrm{Gr}_{\mathbf{G},\mu}$ sends $\mathbb{G}_\mu(R)g$ to the tensor functor*

$$\Lambda : (\rho, W) \in \mathrm{Rep}(\mathbf{G}) \to \{\Lambda_W = \rho(g^{-1}\mu(\xi)) \cdot (W \otimes_{\mathbf{Q}_p} \mathbb{B}_{\mathrm{dR}}^+(R)) \subset W \otimes_{\mathbf{Q}_p} \mathbb{B}_{\mathrm{dR}}(R)\}.$$

Proof. Straightforward. □

Proposition 3.18. *We have*

$$\mathbb{G}_\mu(R) \subseteq \theta^{-1}(\mathbf{P}_\mu^{\mathrm{std}}(R))$$

as subgroups of $\mathbb{G}(R)$, with equality if and only if μ is minuscule.

Proof. Let $\mathbf{P}_\mu$ denote the opposite parabolic to $\mathbf{P}_\mu^{\mathrm{std}}$. After enlarging E if necessary, we may assume $\mathbf{G}$ is split. Let $\mathbf{T}$ be a split maximal torus containing $\mathrm{im}(\mu)$, and choose a Borel $\mathbf{B}$ contained in $\mathbf{P}_\mu$ such that $\mu \in X_*(\mathbf{T})$ is $\mathbf{B}$-dominant. The groups in question are generated as subgroups of $\mathbb{G}(R)$ by their intersections with $\mathbb{T} = \mathbf{T}(\mathbb{B}_{\mathrm{dR}}^+(R))$ (which is plainly all of $\mathbb{T}$ for either group) together with their intersections with the root subgroups of $\mathbf{G}$. Therefore it suffices to check that the claim holds after intersecting both sides with any root group.

Let $\mathbf{U}_\alpha$ be a root group of $\mathbf{G}$, with isomorphism $u_\alpha : \mathbf{G}_a \xrightarrow{\sim} \mathbf{U}_\alpha$, so $\mathbb{U}_\alpha(R) \simeq \mathbb{B}_{\mathrm{dR}}^+(R)$. Note that $\xi^n\mathbb{U}_\alpha(R) = u_\alpha(\xi^n\mathbb{B}_{\mathrm{dR}}^+(R))$ is a well-defined subgroup of $\mathbb{U}_\alpha(R)$. If α is negative, then

$$\mathbb{G}_\mu(R) \cap \mathbb{U}_\alpha(R) = \theta^{-1}(\mathbf{P}_\mu^{\mathrm{std}}(R)) \cap \mathbb{U}_\alpha(R) = \mathbb{U}_\alpha(R).$$

If α is positive, then $\mathbb{G}_\mu(R)\cap\mathbb{U}_\alpha(R) = \xi^{\langle\mu,\alpha\rangle}\mathbb{U}_\alpha(R)$ while $\theta^{-1}(\mathbf{P}_\mu^{\mathrm{std}}(R))\cap\mathbb{U}_\alpha(R) = \xi^{\min(\langle\mu,\alpha\rangle,1)}\mathbb{U}_\alpha(R)$. This gives the claimed inclusion. Finally we note that a $\mathbf{B}$-dominant cocharacter μ is minuscule if and only if $\langle\mu,\alpha\rangle = \min(\langle\mu,\alpha\rangle, 1)$ for all positive roots α. □

Corollary 3.19. *There is a natural G-equivariant Bialynicki-Birula morphism*

$$\sigma_{\mathrm{BB}} : \mathrm{Gr}_{\mathbf{G},\mu} \to \mathscr{F}\!\ell_{\mathbf{G},\mu},$$

which is an isomorphism if and only if μ is minuscule.

Proof. This map is given by the projection

$$\mathrm{Gr}_{\mathbf{G},\mu} \cong \mathbb{G}_\mu\backslash\mathbb{G} \to \theta^{-1}(\mathbf{P}_\mu^{\mathrm{std}})\backslash\mathbb{G} \cong \mathbf{P}_\mu^{\mathrm{std}}\backslash\mathbf{G} \cong \mathscr{F}\!\ell_{\mathbf{G},\mu},$$

so the result follows from the previous proposition. □

Connection with the Fargues-Fontaine curve

The functor $\mathrm{Gr}_{\mathbf{G},\mu}$ also has an important interpretation in terms of bundles on the Fargues-Fontaine curve. We briefly recall this circle of ideas, and use it to define the Newton strata of $\mathrm{Gr}_{\mathbf{G},\mu}$, following [FF15, KL15, CS15].

Proposition 3.20. *Fix a finite extension $E/\mathbf{Q}_p$ with residue field $\mathbf{F}_q$. Then for any perfectoid space $S \in \mathrm{Perf}_{\mathbf{F}_q}$ we have a functorially associated adic space $\mathcal{X}_S = \mathcal{X}_{S,E}$ over* $\mathrm{Spa}\, E$*, the* adic Fargues-Fontaine curve*, with diamond given by $\mathcal{X}_S^\lozenge \cong S^\lozenge/\mathrm{Frob}_q^{\mathbf{Z}} \times_{\mathbf{F}_q} \mathrm{Spd}\, E$. Any untilt $(S^\sharp, \iota) \in (\mathrm{Spd}\, E)(S)$ determines a canonical closed immersion $i : S^\sharp \hookrightarrow \mathcal{X}_S$.*

In particular, if S is a perfectoid space over E and $\mathcal{E}$ is a vector bundle or $\mathbf{G}$-bundle on $\mathcal{X}_{S^\flat}$, we can speak of bundles obtained by modifying $\mathcal{E}$ along the closed immersion $i : S \hookrightarrow \mathcal{X}_{S^\flat}$. The formal completion of $\mathcal{X}_{S^\flat}$ along S is $\mathbb{B}_{\mathrm{dR}}^+(S)$, roughly, in the sense that if $S = \mathrm{Spa}(A, A^+)$ and $\mathcal{I}$ is the ideal sheaf cutting out S in $\mathcal{X}_{S^\flat}$, then $H^0(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n) \cong \mathbb{B}_{\mathrm{dR}}^+(A)/\mathrm{Fil}^n$ for any n. The formal completion of a vector bundle on $\mathcal{X}_{S^\flat}$ along S is then a finite projective $\mathbb{B}_{\mathrm{dR}}^+$-module, and modifying the bundle amounts to changing this lattice. Making these ideas precise, one obtains the following (cf. [CS15, §3.5]).

Proposition 3.21. *The functor $\mathrm{Gr}_{\mathbf{G},\mu}$ coincides with the functor on $\widetilde{\mathrm{Perf}}_{/\mathrm{Spa}\, E}$ sending S to the set of isomorphism classes of pairs $(\mathcal{F}, u)$ where $\mathcal{F}$ is a $\mathbf{G}$-bundle on $\mathcal{X}_{S^\flat}$ and $u : \mathcal{F}|_{\mathcal{X}_{S^\flat} \smallsetminus S} \xrightarrow{\sim} \mathcal{E}_{\mathrm{triv}}|_{\mathcal{X}_{S^\flat} \smallsetminus S}$ is an isomorphism extending to a μ-positioned modification of the trivial $\mathbf{G}$-bundle along the immersion $i : S \hookrightarrow \mathcal{X}_{S^\flat}$.*

When $\mathbf{G} = \mathrm{GL}_n$ and $\mu = (k_1 \geq \cdots \geq k_n)$ with $k_n \geq 0$, the functor $\mathrm{Gr}_{\mathbf{G},\mu}$ parametrizes certain sub-$\mathbb{B}_{\mathrm{dR}}^+$-modules $\Xi \subseteq (\mathbb{B}_{\mathrm{dR}}^+)^n$; the associated bundle $\mathcal{F}$ is then defined by the exactness of the sequence

$$0 \to \mathcal{F} \xrightarrow{u} \mathcal{O}_{\mathcal{X}}^n \to i_* \left((\mathbb{B}_{\mathrm{dR}}^+)^n/\Xi\right) \to 0$$

of coherent sheaves on $\mathcal{X}_S$.

When $S = \mathrm{Spa}(C, C^+)$ is a geometric point, we have a classification result due to Fargues-Fontaine (for GL_n) and Fargues (for general $\mathbf{G}$):

Proposition 3.22. *For any perfectoid space $S/\overline{\mathbf{F}_q}$, any element $b \in \mathbf{G}(\breve{E})$ gives rise to a functorially associated $\mathbf{G}$-bundle $\mathcal{E}_{b,S}$ on $\mathcal{X}_S$ such that the isomorphism class of $\mathcal{E}_{b,S}$ depends only on the σ-conjugacy class of b. When S is a geometric point, the induced map*

$$\begin{array}{rcl} B(\mathbf{G}) & \to & \mathrm{Bun}_{\mathbf{G}}(\mathcal{X}_S) \\ b & \mapsto & \mathcal{E}_{b,S}. \end{array}$$

is surjective on objects.

Proof. See [FF15] and [Far15]. □

In particular, for any $\mathbf{G}$-bundle $\mathcal{E}$ on any $\mathcal{X}_S$, we get an associated function $b(-) : |S| \to B(\mathbf{G})$ such that for any point $x = \mathrm{Spa}(K, K^+) \to S$ with associated geometric point $\overline{x}$, we have $x^*\mathcal{E} \simeq \mathcal{E}_{b(x),\overline{x}}$ as $\mathbf{G}$-bundles on $\mathcal{X}_{\overline{x}}$. (Note that for $\mathbf{G} = \mathrm{GL}_n$, the slopes of the F-isocrystal defined by b are inverse to the slopes of the vector bundle $\mathcal{E}_b$: the bundle $\mathcal{O}(1)$ corresponds to taking $b = \pi^{-1} \in \mathbf{G}_m(\breve{E})$.)

Definition 3.23. For any $b \in B(\mathbf{G})$, the Newton stratum $\mathrm{Gr}^b_{\mathbf{G},\mu} \subset \mathrm{Gr}_{\mathbf{G},\mu}$ is the subfunctor defined as follows: a given S-point

$$(\mathcal{F}, u) \in \mathrm{Gr}_{\mathbf{G},\mu}(S)$$

as above factors through $\mathrm{Gr}^b_{\mathbf{G},\mu}$ if and only if $\overline{x}^*\mathcal{F} \simeq \mathcal{E}_{b,\overline{x}}$ for all geometric points $\overline{x} = \mathrm{Spa}(C, C^+) \to S^\flat$.

By [CS15, Proposition 3.5.3], the stratum $\mathrm{Gr}^b_{\mathbf{G},\mu}$ is empty unless $b \in B(\mathbf{G}, \mu)$ (and in fact the converse holds as well).

4 Geometry of local systems

4.1 $\mathbf{Q}_p$-local systems

In this section we define $\mathbf{Q}_p$-local systems on arbitrary adic spaces and on diamonds. There is a little bit of bookkeeping to be done here, since there are multiple competing definitions of "the pro-étale site" in the literature.

Let X be any adic space.[11] Then X has a pro-étale site X_{proet} defined as in [KL15, Definition 9.1.4] (generalizing the definition given in [Sch13] for locally Noetherian X). Any $Y \in X_{\mathrm{proet}}$ has an associated topological space $|Y|$.

Definition 4.1. *Let X be any adic space. For any topological ring A, let $\underline{A}$ be the sheaf of rings on X_{proet} with sections given by $\mathrm{Cont}(|U|, A)$ over any $U \in X_{\mathrm{proet}}$. Then we have a natural category $A\mathrm{Loc}(X)$ of* A-local systems *on X: an object of $A\mathrm{Loc}(X)$ is a sheaf $\mathcal{F}$ of flat $\underline{A}$-modules on X_{proet} such that $\mathcal{F}|_{U_{i,\mathrm{proet}}} \simeq \underline{A}^{n_i}$ locally on some pro-étale covering $\{U_i \to X\}$.*

We mostly care about the particular cases $A = \mathbf{Q}_p, \mathbf{Z}_p$, or $\mathbf{Z}/p^j\mathbf{Z}$ (in descending order of both interest and difficulty).

Remark 4.2. When X is a perfectoid space, we also have the "new" pro-étale site as defined in §2.3, which we momentarily denote by $X_{\mathrm{proet,new}}$. There is a natural map of sites $\eta : X_{\mathrm{proet,new}} \to X_{\mathrm{proet}}$ corresponding to the inclusion of categories $X_{\mathrm{proet}} \subset X_{\mathrm{proet,new}}$, and the topology $X_{\mathrm{proet,new}}$ is finer than X_{proet}. However, [KL16, Theorem 4.5.11] implies that for any perfectoid space X and any $A \in \{\mathbf{Q}_p, \mathbf{Z}_p, \mathbf{Z}/p^j\mathbf{Z}\}$, the pullback map $\eta^* : A\mathrm{Loc}(X_{\mathrm{proet}}) \to A\mathrm{Loc}(X_{\mathrm{proet,new}})$ is an equivalence of categories, with essential inverse given by η_*. In particular, when X is perfectoid, we can and do regard $A\mathrm{Loc}(X)$ as being interchangeably defined in terms of sheaves on $X_{\mathrm{proet,new}}$ or X_{proet}. In light of this, we will essentially always omit the the subscript "new": the double meaning of X_{proet} for perfectoid X causes no ambiguity in the discussion of $\mathbf{Q}_p$-local systems and allied objects. Similar remarks apply to diamonds of the form $X^\diamondsuit$, where again $(X^\diamondsuit)_{\mathrm{proet}}$ is *a priori* finer than $(X_{\mathrm{proet}})^\diamondsuit$.

Now suppose $\mathcal{D}$ is a diamond; again, any $\mathcal{E} \in \mathcal{D}_{\mathrm{proet}}$ has an associated topological space $|\mathcal{E}|$, so the definitions of $\underline{A}$ and $A\mathrm{Loc}(-)$ as above go through verbatim, and we get a category $A\mathrm{Loc}(\mathcal{D})$.

Proposition 4.3. *Let X be any adic space, with associated diamond $X^\diamondsuit$. Then there is a natural equivalence $A\mathrm{Loc}(X) \cong A\mathrm{Loc}(X^\diamondsuit)$ for any $A \in \{\mathbf{Q}_p, \mathbf{Z}_p, \mathbf{Z}/p^j\mathbf{Z}\}$.*

[11] The constructions in this section work equally well on the larger category of "generalized adic spaces" in the sense of [SW13]. Note that spaces of this type play an implicit role in what follows: if X is an analytic adic space, then we can make sense of generalized adic spaces étale over X, and these spaces may not be true adic spaces.

Proof. If X is perfectoid, the functor $Y \mapsto Y^\diamondsuit$ induces an equivalence $X_{\mathrm{proet}} \cong X^\diamondsuit_{\mathrm{proet}}$ compatible with $|\cdot|$, so the claim is immediate in this case. (Here again X_{proet} denotes the "new" pro-étale site on perfectoid spaces.)

Now let X be arbitrary, and choose a pro-étale cover $f : U \to X$ with U perfectoid such that $f^\diamondsuit : U^\diamondsuit = h_{U^\flat} \to X^\diamondsuit$ is a presentation of $X^\diamondsuit$. Note that $U \times_X U$ and $U \times_X \times U \times_X U$ exist as perfectoid spaces[12]: $f^\diamondsuit$ is representable and pro-étale by assumption, so $U^\diamondsuit \times_{f,X^\diamond,f} U^\diamondsuit \simeq h_Y$ for some $Y \in \mathrm{Perf}_{/X^\diamond}$, and then the equivalence $\mathrm{Perf}_{/X^\diamond} \cong \widetilde{\mathrm{Perf}}_{/X}$ gives an untilt $Y^\sharp/X$ representing $U \times_X U$. Let $A\mathrm{Loc}(U, \mathrm{DD}(f))$ be the category of A-local systems on U equipped with a descent datum relative to the covering $f : U \to X$. Then

$$\begin{aligned} A\mathrm{Loc}(X) &\cong A\mathrm{Loc}(U, \mathrm{DD}(f)) \\ &\cong A\mathrm{Loc}(U^\diamondsuit, \mathrm{DD}(f^\diamondsuit)) \\ &\cong A\mathrm{Loc}(X^\diamondsuit). \end{aligned}$$

Here the first isomorphism follows from the (tautological) fact that the fibered category over X_{proet} given by $V \in X_{\mathrm{proet}} \mapsto A\mathrm{Loc}(V)$ is a stack (with the third isomorphism following analogously), and the second isomorphism follows from the equivalence $A\mathrm{Loc}(U) \cong A\mathrm{Loc}(U^\flat) = A\mathrm{Loc}(U^\diamondsuit)$ and the compatibility of the descent data with tilting. Note that in passing from the first to the second line, we are implicitly passing through the equivalence

$$A\mathrm{Loc}(U_{\mathrm{proet}}, \mathrm{DD}(f)) \cong A\mathrm{Loc}(U_{\mathrm{proet,new}}, \mathrm{DD}(f)),$$

which accounts for the restriction on A. $\square$

4.2 The functors of trivializations, lattices, and sections

Definition 4.4. Let X be an adic space, and choose some $\mathbf{V} \in \mathbf{Q}_p\mathrm{Loc}(X)$ of rank n. Then we consider the presheaves on $\widetilde{\mathrm{Perf}}_{/X}$ defined as follows:

$$\begin{aligned} \mathcal{T}\mathrm{riv}_{\mathbf{V}/X} : \widetilde{\mathrm{Perf}}_{/X} &\to \mathrm{Sets} \\ \{f : T \to X\} &\mapsto \mathrm{Isom}_{\mathbf{Q}_p\mathrm{Loc}(T)}(\underline{\mathbf{Q}_p}^n, f^*\mathbf{V}), \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}\mathrm{at}_{\mathbf{V}/X} : \widetilde{\mathrm{Perf}}_{/X} &\to \mathrm{Sets} \\ \{f : T \to X\} &\mapsto \{\mathbf{L} \subset f^*\mathbf{V} \mid \mathbf{L} \in \mathbf{Z}_p\mathrm{Loc}(T)\,\text{of rank}\,n\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}\mathrm{ect}_{\mathbf{V}/X} : \widetilde{\mathrm{Perf}}_{/X} &\to \mathrm{Sets} \\ \{f : T \to X\} &\mapsto H^0_{\mathrm{proet}}(T, f^*\mathbf{V}). \end{aligned}$$

For $\mathcal{D}$ a diamond and $\mathbf{V} \in \mathbf{Q}_p\mathrm{Loc}(\mathcal{D})$ of rank n, we make the analogous definitions with $\widetilde{\mathrm{Perf}}_{/X}$ replaced by $\mathrm{Perf}_{/\mathcal{D}}$.

The following proposition is an easy verification from the definitions, but it plays a very important role in all that follows.

[12] In general, fiber products $Y \times_X Z$ with Y, Z perfectoid and X arbitrary are a bit delicate.

Proposition 4.5. *If X is an adic space (resp. diamond) with a $\mathbf{Q}_p$-local system $\mathbf{V}$, and $\mathcal{F}_{\mathbf{V}/X}$ is one of the functors of Definition 4.4, then $\mathcal{F}_{\mathbf{V}/X}$ is a sheaf on $\widetilde{\mathrm{Perf}}^{\mathrm{proet}}_{/X}$ (resp. $\mathrm{Perf}^{\mathrm{proet}}_{/X}$). If $f : Y \to X$ is any morphism of adic spaces (resp. diamonds) then $\mathcal{F}_{\mathbf{V}/X} \times_{X,f} Y \cong \mathcal{F}_{f^*\mathbf{V}/Y}$. If X is any adic space, the equivalences $\mathbf{Q}_p\mathrm{Loc}(X) \cong \mathbf{Q}_p\mathrm{Loc}(X^\diamond)$ and*

$$\mathrm{Sh}(\widetilde{\mathrm{Perf}}^{\mathrm{proet}}_{/X}) \cong \mathrm{Sh}(\mathrm{Perf}^{\mathrm{proet}}_{/X^\diamond})$$

induce an isomorphism $\mathcal{F}_{\mathbf{V}/X} \cong \mathcal{F}_{\mathbf{V}/X^\diamond}$.

Remark 4.6. These functors are related in some natural and pleasant ways. Since we'll use some of these relationships explicitly in what follows, we spell them out fully:

1. We have a natural transformation

$$\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to \mathcal{L}\mathrm{at}_{\mathbf{V}/X}$$

given on T-points by sending $\beta \in \mathrm{Isom}_{\mathbf{Q}_p\mathrm{Loc}(T)}(\underline{\mathbf{Q}_p}^n, f^*\mathbf{V})$ to $\beta(\underline{\mathbf{Z}_p}^n)$. This induces an isomorphism

$$\mathcal{L}\mathrm{at}_{\mathbf{V}/X} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}/\underline{\mathrm{GL}_n(\mathbf{Z}_p)}.$$

2. Sending β to $\beta(e_1) \in H^0_{\mathrm{proet}}(T, f^*\mathbf{V})$ (where $e_1 = (1, 0, \ldots, 0)^t \in \mathbf{Q}_p^n$) induces a natural transformation

$$\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to \mathcal{S}\mathrm{ect}_{\mathbf{V}/X}$$

which gives rise to an isomorphism

$$\mathcal{S}\mathrm{ect}^\times_{\mathbf{V}/X} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}/\underline{P}.$$

Here

$$P = \begin{pmatrix} 1 & \mathbf{Q}_p^{n-1} \\ 0 & \mathrm{GL}_{n-1}(\mathbf{Q}_p) \end{pmatrix}$$

is the "mirabolic" subgroup of $\mathrm{GL}_n(\mathbf{Q}_p)$, and $\mathcal{S}\mathrm{ect}^\times_{\mathbf{V}/X} \subset \mathcal{S}\mathrm{ect}_{\mathbf{V}/X}$ denotes the subfunctor of nowhere-vanishing sections.

3. We have a natural isomorphism

$$\begin{array}{rcl} \mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \times_{\underline{\mathrm{GL}_n(\mathbf{Q}_p)}} \underline{\mathbf{Q}_p}^n & \cong & \mathcal{S}\mathrm{ect}_{\mathbf{V}/X} \\ (\beta, v) & \mapsto & \beta(v). \end{array}$$

4. We have a natural isomorphism

$$\begin{array}{rcl} \mathcal{T}\mathrm{riv}_{\mathbf{V}/X} & \cong & \left(\mathcal{S}\mathrm{ect}_{\mathbf{V}/X}\right)^n \times_{\mathcal{S}\mathrm{ect}_{\wedge^n \mathbf{V}/X}} \mathcal{S}\mathrm{ect}^\times_{\wedge^n \mathbf{V}/X} \\ \beta & \mapsto & \left((\beta(e_i))_{1 \le i \le n}, \beta(e_1) \wedge \cdots \wedge \beta(e_n)\right). \end{array}$$

Our immediate goal for now is the following result.

Theorem 4.7. *Let X be a perfectoid space, and let $\mathbf{V}$ be a rank n $\mathbf{Q}_p$-local system on X. Then each of the sheaves of Definition 4.4 is representable by a perfectoid space pro-étale over X. Furthermore:*

i. *$\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to X$ is a pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor.*

ii. *$\mathcal{L}\mathrm{at}_{\mathbf{V}/X} \to X$ is étale.*

iii. *$\mathcal{S}\mathrm{ect}_{\mathbf{V}/X} \to X$ is naturally a $\mathbf{Q}_p$-module object in X_{proet}. The subfunctor $\mathcal{S}\mathrm{ect}^{\times}_{\mathbf{V}/X}$ of nowhere-vanishing sections, defined as the complement of the zero section $s : X \to \mathcal{S}\mathrm{ect}_{\mathbf{V}/X}$, is open and pro-étale over X.*

We'll repeatedly use the following lemma.

Lemma 4.8. *If X is any adic space and $\mathbf{V}$ is any $\mathbf{Q}_p$-local system on X, then $\mathbf{V}$ admits a $\mathbf{Z}_p$-lattice locally in the analytic topology on X. Precisely, we can find a covering of X by open affinoids U_i together with $\mathbf{Z}_p$-local systems $\mathbf{L}_i \subset \mathbf{V}|_{U_i}$ such that $\mathbf{V}|_{U_i} \simeq \mathbf{L}_i \otimes_{\underline{\mathbf{Z}_p}} \underline{\mathbf{Q}_p}$.*

Proof. This follows immediately from Remark 8.4.5 and Corollary 8.4.7 in [KL15]. ☐

Lemma 4.9. *Let X be a perfectoid space, and let $\mathbf{L}$ be a $\mathbf{Z}_p$-local system on X of constant rank n. Then the functor*

$$\begin{array}{rcl} \mathcal{T}\mathrm{riv}_{\mathbf{L}/X} : \widetilde{\mathrm{Perf}}_{/X} & \to & \mathrm{Sets} \\ \{f : T \to X\} & \mapsto & \mathrm{Isom}_{\mathbf{Z}_p\mathrm{Loc}(T)}(\underline{\mathbf{Z}_p}^n, f^*\mathbf{L}) \end{array}$$

is representable by a perfectoid space profinite-étale over X, and the natural map $\mathcal{T}\mathrm{riv}_{\mathbf{L}/X} \to X$ is a pro-étale $\mathrm{GL}_n(\mathbf{Z}_p)$-torsor.

We remark that again, $\mathcal{T}\mathrm{riv}_{\mathbf{L}/X}$ is clearly a sheaf on $\widetilde{\mathrm{Perf}}^{\mathrm{proet}}_{/X}$.

Proof. Set $\mathbf{L}/p^j = \mathbf{L} \otimes_{\underline{\mathbf{Z}_p}} \underline{\mathbf{Z}/p^j}$, so $\mathbf{L}/p^j$ is a sheaf of flat $\mathbf{Z}/p^j$-modules on X_{proet} which is locally free of rank n, and $\mathbf{L} \cong \lim_{\leftarrow j} \mathbf{L}/p^j$. Let

$$\begin{array}{rcl} \mathcal{T}\mathrm{riv}_{(\mathbf{L}/p^j)/X} : \widetilde{\mathrm{Perf}}_{/X} & \to & \mathrm{Sets} \\ \{f : T \to X\} & \mapsto & \mathrm{Isom}_{\mathbf{Z}/p^j\mathrm{Loc}(T)}(\underline{(\mathbf{Z}/p^j)}^n, f^*\mathbf{L}/p^j) \end{array}$$

be the analogous functor with its natural action of $\mathrm{GL}_n(\mathbf{Z}/p^j)$. Clearly $\mathcal{T}\mathrm{riv}_{\mathbf{L}/X} \cong \lim_{\leftarrow j} \mathcal{T}\mathrm{riv}_{(\mathbf{L}/p^j)/X}$, so it suffices to show that each $\mathcal{T}\mathrm{riv}_{(\mathbf{L}/p^j)/X}$ is representable by a finite étale $\mathrm{GL}_n(\mathbf{Z}/p^j)$-torsor over X. To show this, note that we may find some pro-étale covering $g : Y \to X$ such that $g^*(\mathbf{L}/p^j)$ is constant, so $g^*\mathcal{T}\mathrm{riv}_{(\mathbf{L}/p^j)/X} \cong \mathcal{T}\mathrm{riv}_{g^*(\mathbf{L}/p^j)/Y}$ is representable by a perfectoid space Y' finite étale over Y, in fact with $Y' \simeq Y \times \mathrm{GL}_n(\mathbf{Z}/p^j)$. We then get a descent datum for the finite étale morphism $Y' \to Y$ relative to the pro-étale covering $Y \to X$; as already noted in the proof of Proposition 3.6, any such descent datum is effective, so $Y' \to Y$ descends to a finite étale morphism $X' \to X$, and the isomorphism $\mathcal{T}\mathrm{riv}_{g^*(\mathbf{L}/p^j)/Y} \simeq Y'$ then descends to an isomorphism $\mathcal{T}\mathrm{riv}_{(\mathbf{L}/p^j)/X} \simeq X'$, as required. ☐

Before proving Theorem 4.7, we need one more lemma.

Lemma 4.10. *Let G be a locally profinite group with an open subgroup H, and let Y be a perfectoid space with a continuous action of H. Then the pushout $Y \times_{\underline{H}} \underline{G}$ exists as a perfectoid space with continuous G-action. If $Y \to X$ is a pro-étale H-torsor, then $Y \times_{\underline{H}} \underline{G} \to X$ is a pro-étale G-torsor. If $Y \to X$ is pro-étale then $Y \times_{\underline{H}} \underline{G} \to X$ is pro-étale.*

Proof. For existence, one checks that $Y \times_{\underline{H}} \underline{G}$ fibers G-equivariantly over the discrete space $H\backslash G$, with fibers isomorphic to Y. The remaining statements are then an easy exercise. $\square$

Proof of Theorem 4.7. Fix X and $\mathbf{V}$ as in the statement. As in Lemma 5.6, we fix a covering of X by open affinoid perfectoid subsets U_i together with $\mathbf{Z}_p$-local systems $\mathbf{L}_i \subset \mathbf{V}|_{U_i}$ such that $\mathbf{V}|_{U_i} \simeq \mathbf{L}_i \otimes_{\underline{\mathbf{Z}_p}} \underline{\mathbf{Q}_p}$.

First we prove the claims regarding $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$. By Lemma 4.9, each $\mathcal{T}\mathrm{riv}_{\mathbf{L}_i/U_i}$ is a perfectoid space which is naturally profinite-étale over U_i and a pro-étale $\mathrm{GL}_n(\mathbf{Z}_p)$-torsor over U_i. Lemma 4.10 then shows that the pushout $\mathcal{T}\mathrm{riv}_{\mathbf{L}_i/U_i} \times_{\underline{\mathrm{GL}_n(\mathbf{Z}_p)}} \mathrm{GL}_n(\mathbf{Q}_p)$ exists as a perfectoid space and gives a good pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor over U_i. On the other hand, the isomorphism $\mathbf{V}|_{U_i} \simeq \mathbf{L}_i \otimes_{\underline{\mathbf{Z}_p}} \underline{\mathbf{Q}_p}$ induces a $\mathrm{GL}_n(\mathbf{Z}_p)$-equivariant natural transformation

$$\mathcal{T}\mathrm{riv}_{\mathbf{L}_i/U_i} \to \mathcal{T}\mathrm{riv}_{\mathbf{V}|_{U_i}/U_i}$$

which extends along the pushout to an isomorphism

$$\mathcal{T}\mathrm{riv}_{\mathbf{L}_i/U_i} \times_{\underline{\mathrm{GL}_n(\mathbf{Z}_p)}} \underline{\mathrm{GL}_n(\mathbf{Q}_p)} \xrightarrow{\sim} \mathcal{T}\mathrm{riv}_{\mathbf{V}|_{U_i}/U_i},$$

so each $\mathcal{T}\mathrm{riv}_{\mathbf{V}|_{U_i}/U_i}$ is a perfectoid space and a good pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor over U_i. Since $\mathcal{T}\mathrm{riv}_{\mathbf{V}|_{U_i}/U_i} \times_{U_i} U_{ij} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}|_{U_{ij}}/U_{ij}}$ canonically, the spaces $\mathcal{T}\mathrm{riv}_{\mathbf{V}|_{U_i}/U_i}$ glue over the covering $\{U_i\}$ in the obvious way, giving a perfectoid space $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ pro-étale over X such that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \times_X U_i \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}|_{U_i}/U_i}$.

For $\mathcal{L}\mathrm{at}_{\mathbf{V}/X}$, we argue as follows. It again suffices to show that each $\mathcal{L}\mathrm{at}_{\mathbf{V}|_{U_i}/U_i}$ is representable and étale over U_i. As in [KL15], Remark 1.4.7, let $L_m(\mathbf{L}_i)$ be the functor parametrizing $\mathbf{Z}_p$-local systems $\mathbf{L}_i' \subset \mathbf{L}_i \otimes_{\underline{\mathbf{Z}_p}} \underline{\mathbf{Q}_p}$ such that $p^m \mathbf{L}_i \subseteq \mathbf{L}_i' \subseteq p^{-m}\mathbf{L}_i$. This functor is representable by an affinoid perfectoid space finite étale over U_i, and $L_m(\mathbf{L}_i)$ is naturally a subfunctor of $\mathcal{L}\mathrm{at}_{\mathbf{V}|_{U_i}/U_i}$; note also that $L_m(\mathbf{L}_i) \to L_{m'}(\mathbf{L}_i)$ is an open and closed immersion for $m \leq m'$. Since U_i is quasicompact, every section of $\mathcal{L}\mathrm{at}_{\mathbf{V}|_{U_i}/U_i}$ factors through $L_m(\mathbf{L}_i)$ for some m, so $\mathcal{L}\mathrm{at}_{\mathbf{V}|_{U_i}/U_i} \simeq \cup_{m \geq 0} L_m(\mathbf{L}_i)$, and this latter union is clearly étale.

For $\mathcal{S}\mathrm{ect}_{\mathbf{V}/X}$, it again suffices to show that each $\mathcal{S}\mathrm{ect}_{\mathbf{V}|_{U_i}/U_i}$ has the claimed properties. We note that $\mathcal{S}\mathrm{ect}_{(\mathbf{L}_i/p^j)/U_i}$ is clearly representable and finite étale over U_i: pro-étale-locally on U_i, it becomes a disjoint union of p^n copies of U_i. Since each zero section $s : U_i \to \mathcal{S}\mathrm{ect}_{(\mathbf{L}_i/p^n)/U_i}$ is open-closed, it has an open-closed complement $\mathcal{S}\mathrm{ect}^{\times}_{(\mathbf{L}_i/p^n)/U_i}$ which is finite étale over U_i. The inverse limit

$$\mathcal{S}\mathrm{ect}_{\mathbf{L}_i/U_i} \cong \varprojlim \mathcal{S}\mathrm{ect}_{(\mathbf{L}_i/p^n)/U_i}$$

is clearly representable and pro-finite étale. On the other hand, the natural inclusion $\mathcal{S}\mathrm{ect}_{\mathbf{L}_i/U_i} \subset \mathcal{S}\mathrm{ect}_{\mathbf{V}|_{U_i}/U_i}$ extends to an isomorphism

$$\mathcal{S}\mathrm{ect}_{\mathbf{V}|_{U_i}/U_i} \cong \varinjlim_{\times p} \mathcal{S}\mathrm{ect}_{\mathbf{L}_i/U_i},$$

and the transition maps here are open-closed immersions, so $\mathcal{S}\mathrm{ect}_{\mathbf{V}|_{U_i}/U_i}$ is pro-étale over U_i. Finally, note that we can write the subfunctor of $\mathcal{S}\mathrm{ect}_{\mathbf{L}_i/U_i}$ parametrizing nonvanishing sections as

$$\mathcal{S}\mathrm{ect}^{\times}_{\mathbf{L}_i/U_i} \cong \cup_{n \geq 1} \mathcal{S}\mathrm{ect}_{\mathbf{L}_i/U_i} \times_{\mathcal{S}\mathrm{ect}_{(\mathbf{L}_i/p^n)/U_i}} \mathcal{S}\mathrm{ect}^{\times}_{(\mathbf{L}_i/p^n)/U_i},$$

where the individual fiber products here correspond to the subfunctors of $\mathcal{S}\mathrm{ect}_{\mathbf{L}_i/U_i}$ parametrizing sections which are nowhere-vanishing mod p^n. We claim each of these invididual fiber products are open. Indeed, each morphism $\mathcal{S}\mathrm{ect}^{\times}_{(\mathbf{L}_i/p^n)/U_i} \to \mathcal{S}\mathrm{ect}_{(\mathbf{L}_i/p^n)/U_i}$ is an open immersion, and the morphism $\mathcal{S}\mathrm{ect}_{\mathbf{L}_i/U_i} \to \mathcal{S}\mathrm{ect}_{(\mathbf{L}_i/p^n)/U_i}$ is an inverse limit of finite étale surjective maps, and therefore is a quotient map, so it pulls back open immersions to open immersions. Finally, $\mathcal{S}\mathrm{ect}^{\times}_{\mathbf{V}|_{U_i}/U_i}$ is again covered by p-power dilates of these open subfunctors, and hence is open. □

Theorem 4.11. *If $\mathcal{D}$ is a diamond and $\mathbf{V}$ is a rank n $\mathbf{Q}_p$-local system on $\mathcal{D}$, then any one of the functors $\mathcal{F}_{\mathbf{V}/\mathcal{D}}$ from Definition 4.4 is (representable by) a diamond pro-étale over $\mathcal{D}$ (so in particular, $\mathcal{F}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ is representable), and the exact analogues of all the statements in Theorem 4.7 are true.*

Proof. Choose any $X \in \mathrm{Perf}$ and any map $f : X^\lozenge \to \mathcal{D}$, so we get a pullback square

$$\begin{tikzcd}
\mathcal{F}_{\mathbf{V}/\mathcal{D}} \times_{\mathcal{D}} X^\lozenge \cong \mathcal{F}_{f^*\mathbf{V}/X^\lozenge} \arrow[r] \arrow[d] & X^\lozenge \arrow[d] \\
\mathcal{F}_{\mathbf{V}/\mathcal{D}} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

of sheaves on $\mathrm{Perf}^{\mathrm{proet}}$. Then $\mathcal{F}_{f^*\mathbf{V}/X^\lozenge} \cong \mathcal{F}^\lozenge_{f^*\mathbf{V}/X}$ is a representable sheaf by Proposition 4.5 and Theorem 4.7, so the map $\mathcal{F}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ is representable. In particular, if f is surjective and pro-étale, then $\mathcal{F}^\lozenge_{f^*\mathbf{V}/X} \to \mathcal{F}_{\mathbf{V}/\mathcal{D}}$ is also surjective and pro-étale, so $\mathcal{F}_{\mathbf{V}/\mathcal{D}}$ is a diamond. The analogues for the map $\mathcal{F}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ of the statements in Theorem 4.7 are clear, since the representability of $\mathcal{F}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ reduces us formally to Theorem 4.7. □

4.3 G-local systems and G-torsors

Let $\mathbf{G}$ be a reductive algebraic group over $\mathbf{Q}_p$, so $G = \mathbf{G}(\mathbf{Q}_p)$ is a locally profinite group. We denote by $\mathrm{Rep}(\mathbf{G})$ the tensor category of pairs (W, ρ) where W is a finite-dimensional $\mathbf{Q}_p$-vector space and $\rho : \mathbf{G} \to \mathbf{GL}(W)$ is a morphism of algebraic groups over $\mathbf{Q}_p$.[13]

Definition 4.12. Let X be an adic space or diamond. A *G-local system on X* is an additive exact tensor functor

$$\begin{aligned}
\mathbf{V} : \mathrm{Rep}(\mathbf{G}) &\to \mathbf{Q}_p\mathrm{Loc}(X) \\
(W, \rho) &\mapsto \mathbf{V}_W.
\end{aligned}$$

These form a category $G\mathrm{Loc}(X)$, with morphisms given by natural isomorphisms of tensor functors.

Observe that for a given $\mathbf{V} \in G\mathrm{Loc}(X)$ and $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$, the natural group acting on the space $\mathcal{T}\mathrm{riv}_{\mathbf{V}_W/X}$ is $\mathrm{GL}_{\dim W}(\mathbf{Q}_p)$ rather than $\mathrm{GL}(W)$. It will be convenient to consider the variant $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X}$ with a natural $\mathrm{GL}(W)$-action, defined as the functor sending $f : T \to X$ to $\mathrm{Isom}_{\mathbf{Q}_p\mathrm{Loc}(T)}(\underline{W}, f^*\mathbf{V}_W)$. Note that we can write this functor as the pushout

$$\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}_W/X} \times_{\underline{\mathrm{GL}_{\dim W}(\mathbf{Q}_p)}} \underline{\mathrm{Isom}_{\mathbf{Q}_p}(W, \mathbf{Q}_p^{\dim W})}.$$

[13] We write $\mathbf{GL}(W)$ for the algebraic group with functor of points $R \mapsto \mathrm{Aut}(R \otimes_{\mathbf{Q}_p} W)$, as distinguished from the locally profinite group $\mathrm{GL}(W) \simeq \mathrm{GL}_{\dim W}(\mathbf{Q}_p)$ given by its $\mathbf{Q}_p$-points.

Of course, any fixed isomorphism $W \simeq \mathbf{Q}_p^{\dim W}$ determines compatible isomorphisms $\mathrm{GL}(W) \simeq \mathrm{GL}_{\dim W}(\mathbf{Q}_p)$ and $\mathcal{T}\mathrm{riv}_{\mathbf{V}_W/X} \simeq \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X}$, so the results of Theorems 4.7 and 4.11 apply to $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X}$.

Note that we always have the *trivial G-local system* given by $\mathbf{V}^{\mathrm{triv}} : (W,\rho) \mapsto \underline{W}$, with

$$\mathrm{Aut}_{G\mathrm{Loc}(X)}(\mathbf{V}^{\mathrm{triv}}) \cong \underline{G}.$$

If X is an adic space (resp. diamond) then we again define a sheaf $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ on $\widetilde{\mathrm{Perf}}^{\mathrm{proet}}_{/X}$ (resp. $\mathrm{Perf}^{\mathrm{proet}}_{/X}$) as the functor sending $f : T \to X$ to $\mathrm{Isom}_{G\mathrm{Loc}(T)}(\mathbf{V}^{\mathrm{triv}}, f^*\mathbf{V})$. Note that G acts on $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ through its natural action on $\mathbf{V}^{\mathrm{triv}}$.

Theorem 4.13. *Let X be a perfectoid space (resp. diamond), and let $\mathbf{V}$ be a G-local system on X. Then*

i. *$\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is a perfectoid space (resp. diamond) pro-étale over X, and the map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to X$ is a pro-étale G-torsor.*

ii. *Given any pro-étale G-torsor $\tilde{X}$ over X, there is a functorially associated G-local system $\mathbf{V}(\tilde{X})$ on X characterized by the fact that for any $(W,\rho) \in \mathrm{Rep}(\mathbf{G})$, $\mathbf{V}(\tilde{X})_W$ is the sheafification of the presheaf sending $U \in X_{\mathrm{proet}}$ to the vector space $\tilde{X}(U) \times_{G,\rho} W$. Equivalently, $\mathbf{V}(\tilde{X})$ is characterized by the fact that*

$$\mathcal{S}\mathrm{ect}_{\mathbf{V}(\tilde{X})_W/X} \cong \tilde{X} \times_{\underline{G},\rho} \underline{W}$$

functorially in $(W,\rho) \in \mathrm{Rep}(\mathbf{G})$.

iii. *The functor $\tilde{X} \mapsto \mathbf{V}(\tilde{X})$ is an equivalence of categories, with essential inverse given by $\mathbf{V} \mapsto \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$.*

Proof. We may assume X is a perfectoid space: the case of diamonds follows formally from the perfectoid case exactly as in the proof of Theorem 4.11. Let $\mathbf{V}$ be a G-local system on X. Note that for any $(W,\rho) \in \mathrm{Rep}(\mathbf{G})$ we have a natural isomorphism

$$\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \times_{\underline{G},\rho} \underline{\mathrm{GL}(W)} \cong \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X},$$

so in particular $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is a subfunctor of $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X}$ if W is a faithful representation of G. We're going to prove i. by cutting out $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ explicitly inside a well-chosen $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X}$, reducing the claim to Theorems 4.7 and 4.11.

To do this, recall that by a standard result (cf. [DMOS82, Proposition I.3.1]) we may choose some faithful representation $(W,\rho) \in \mathrm{Rep}(\mathbf{G})$ together with a finite collection of vectors $\{w_\alpha \in W\}_{\alpha \in A}$ such that $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(W)$ identifies $\mathbf{G}$ with the pointwise stabilizer of the elements w_α. Then for any $f : T \to X$, the subset

$$\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}(T) \subset \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X}(T) = \mathrm{Isom}_{\mathbf{Q}_p\mathrm{Loc}(T)}(\underline{W}, f^*\mathbf{V}_W)$$

is characterized as the set of isomorphisms $\beta : \underline{W} \xrightarrow{\sim} f^*\mathbf{V}_W$ such that $\beta(\rho(g)\cdot w_\alpha) = \beta(w_\alpha)$ for all $g \in G$ and $\alpha \in A$. Choose a finite set of elements $\{g_i \in G\}_{i \in I}$ generating a Zariski-dense subgroup $\Gamma \subset G$. Then $\beta(\rho(g)\cdot w) = \beta(w)$ for all $g \in G$ if and only if $\beta(\rho(g_i)\cdot w) = \beta(w)$ for all $i \in I$: indeed, if this latter condition holds, then since β is a linear map we have the identity

$$\beta(\rho(gh)\cdot w) - \beta(w) = \beta(\rho(g)(\rho(h)-1)\cdot w) + \beta((\rho(g)-1)\cdot w)$$

which shows that $\beta(\rho(g)\cdot w) = \beta(w)$ for all $g \in \Gamma$, which then extends to all $g \in G$ by Zariski-density.

Now consider the map

$$\begin{array}{rcl} \mathrm{ev}: \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X} & \to & \left(\mathcal{S}\mathrm{ect}_{\mathbf{V}_W/X}\right)^{A\times I} \\ \beta & \mapsto & \left(\beta(\rho(g_i)\cdot w_\alpha) - \beta(w_\alpha)\right)_{\alpha\in A, i\in I}, \end{array}$$

where $\left(\mathcal{S}\mathrm{ect}_{\mathbf{V}_W/X}\right)^{A\times I}$ denotes the $A\times I$-fold fiber product over X of copies of $\mathcal{S}\mathrm{ect}_{\mathbf{V}_W/X}$. By the argument of the previous paragraph, $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is the subfunctor of $\mathcal{T}\mathrm{riv}_{\mathbf{V}_W/X}$ cut out by the fiber product

$$\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X} \times_{\mathrm{ev},\left(\mathcal{S}\mathrm{ect}_{\mathbf{V}_W/X}\right)^{A\times I},s} X,$$

where $s: X \to \left(\mathcal{S}\mathrm{ect}_{\mathbf{V}_W/X}\right)^{A\times I}$ is the (diagonal) zero section. Since $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/X}$ and $\left(\mathcal{S}\mathrm{ect}_{\mathbf{V}_W/X}\right)^{A\times I}$ are perfectoid spaces pro-étale over X, we deduce that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is perfectoid and pro-étale over X. Finally, one easily checks that the action map

$$\begin{array}{rcl} \underline{G}\times \mathcal{T}\mathrm{riv}_{\mathbf{V}/X} & \to & \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}\times_X \mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \\ (g,\beta) & \mapsto & (\beta\circ g, \beta) \end{array}$$

is an isomorphism, so $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}\times_X X' \simeq \underline{G}\times X'$ after pullback along the pro-étale cover $X'\to X$ given by taking $X' = \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ itself. This verifies that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}\to X$ is a pro-étale G-torsor.

For ii., one verifies directly that the recipe described in the statement defines a G-local system.

For iii., let $\tilde{X}\to X$ be a pro-étale G-torsor with associated G-local system $\mathbf{V}(\tilde{X})$, and choose any representation (W,ρ). Then clearly

$$\mathcal{T}\mathrm{riv}_{\mathbf{V}(\tilde{X})_W/X} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}(\tilde{X})/X} \times_{\underline{G},\rho} \underline{\mathrm{Isom}(\mathbf{Q}_p^n, W)}$$

where $n = \mathrm{dim} W$. On the other hand, for any rank n $\mathbf{Q}_p$-local system $\mathbf{S}$, we have a natural isomorphism

$$\mathcal{T}\mathrm{riv}_{\mathbf{S}/X} \cong (\mathcal{S}\mathrm{ect}_{\mathbf{S}/X})^n \times_{\mathcal{S}\mathrm{ect}_{\wedge^n\mathbf{S}/X}} \mathcal{S}\mathrm{ect}^\times_{\wedge^n\mathbf{S}/X}.$$

Applying this to $\mathbf{V}(\tilde{X})_W$ and taking into account the isomorphism $\mathcal{S}\mathrm{ect}_{\mathbf{V}(\tilde{X})_W/X}\cong \tilde{X}\times_{\underline{G},\rho}\underline{W}$, we get natural isomorphisms

$$\begin{aligned} \mathcal{T}\mathrm{riv}_{\mathbf{V}(\tilde{X})_W/X} &\cong \left(\mathcal{S}\mathrm{ect}_{\mathbf{V}(\tilde{X})_W/X}\right)^n \times_{\mathcal{S}\mathrm{ect}_{\wedge^n\mathbf{V}(\tilde{X})_W/X}} \mathcal{S}\mathrm{ect}^\times_{\wedge^n\mathbf{V}(\tilde{X})_W/X} \\ &\cong \left(\tilde{X}\times_{\underline{G},\rho}\underline{W^n}\right)\times_{(\tilde{X}\times_{\underline{G},\rho}\underline{\wedge^n W})}\left((\tilde{X}\times_{\underline{G},\rho}\underline{\wedge^n W\smallsetminus 0})\right) \\ &\cong \tilde{X}\times_{\underline{G},\rho}\left(\underline{W}^n\times_{\underline{\wedge^n W}}\underline{\wedge^n W\smallsetminus 0}\right) \\ &\cong \tilde{X}\times_{\underline{G},\rho}\underline{\mathrm{Isom}(\mathbf{Q}_p^n, W)}, \end{aligned}$$

so we have a natural $\mathrm{GL}_{\mathrm{dim}W}(\mathbf{Q}_p)$-equivariant isomorphism

$$\tilde{X}\times_{\underline{G},\rho}\underline{\mathrm{Isom}(\mathbf{Q}_p^{\mathrm{dim}W}, W)} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}(\tilde{X})/X}\times_{\underline{G},\rho}\underline{\mathrm{Isom}(\mathbf{Q}_p^n, W)}$$

for any (W,ρ). Pushing this out along $-\times_{\underline{\mathrm{GL}_{\mathrm{dim}W}(\mathbf{Q}_p)}}\underline{\mathrm{Isom}_{\mathbf{Q}_p}(W,\mathbf{Q}_p^{\mathrm{dim}W})}$ gives a natural $\mathrm{GL}(W)$-equivariant isomorphism

$$\tilde{X}\times_{\underline{G},\rho}\underline{\mathrm{GL}(W)} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}(\tilde{X})/X}\times_{\underline{G},\rho}\underline{\mathrm{GL}(W)} = \mathcal{T}\mathrm{riv}^W_{\mathbf{V}(\tilde{X})/X}.$$

Applying the recipe in the proof of part i. to this isomorphism for some faithful (W,ρ) then cuts out $\tilde{X}$ and $\mathcal{T}\mathrm{riv}_{\mathbf{V}(\tilde{X})/X}$ as naturally isomorphic subfunctors of $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}(\tilde{X})/X}$. This concludes the proof. $\square$

Corollary 4.14. *Suppose $\mathcal{D}$ is a diamond and $\tilde{\mathcal{D}} \to \mathcal{D}$ is a pro-étale G-torsor for some $G = \mathbf{G}(\mathbf{Q}_p)$ as above. Then the map $\tilde{\mathcal{D}} \to \mathcal{D}$ is pro-étale, and the map $\tilde{\mathcal{D}}/\underline{K} \to \mathcal{D}$ is étale for any open compact subgroup $K \subset G$.*

Proof. The first claim follows immediately from the identification $\tilde{\mathcal{D}} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}(\tilde{\mathcal{D}})/\mathcal{D}}$, since we proved the latter functor is pro-étale over $\mathcal{D}$.

For the second claim, we first handle the case where $G = \mathrm{GL}_n(\mathbf{Q}_p)$, so $\tilde{\mathcal{D}} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$ for some rank n $\mathbf{V} = \mathbf{V}(\tilde{\mathcal{D}}) \in \mathbf{Q}_p\mathrm{Loc}(\mathcal{D})$. After conjugation, we may assume $K \subset \mathrm{GL}_n(\mathbf{Z}_p)$, so we may factor the map in question as

$$\tilde{\mathcal{D}}/\underline{K} = \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}/\underline{K} \to \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}/\underline{\mathrm{GL}_n(\mathbf{Z}_p)} \cong \mathcal{L}\mathrm{at}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}.$$

By Theorem 4.11, $\mathcal{L}\mathrm{at}_{\mathbf{V}/\mathcal{D}}$ is étale over $\mathcal{D}$, so it remains to prove that the map

$$f : \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}/\underline{K} \to \mathcal{L}\mathrm{at}_{\mathbf{V}/\mathcal{D}}$$

is finite étale. Pulling this map back along the pro-étale cover $\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{L}\mathrm{at}_{\mathbf{V}/\mathcal{D}}$, we get a pullback diagram

$$\begin{tikzcd}
\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \times \underline{\mathrm{GL}_n(\mathbf{Z}_p)/K} \arrow[r] \arrow[d] & \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \arrow[d] \\
\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}/\underline{K} \arrow[r] & \mathcal{L}\mathrm{at}_{\mathbf{V}/\mathcal{D}}
\end{tikzcd}$$

where the upper horizontal arrow is finite étale. Thus the map f becomes finite étale after pullback along a pro-étale cover of the target, and as we've already seen in the proof of Proposition 3.6, this implies that f is finite étale.

Now we deal with the general case. Let $\mathbf{V} \in G\mathrm{Loc}(\mathcal{D})$ be the G-local system associated with $\tilde{\mathcal{D}}$, so we may identify $\tilde{\mathcal{D}} \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$. Fix some faithful $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$, and choose an open compact subgroup $K' \subset \mathrm{GL}(W)$ such that $\rho^{-1}(\rho(G) \cap K') = K$.[14] Then the monomorphism $\tilde{\mathcal{D}} = \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/\mathcal{D}} = \tilde{\mathcal{D}} \times_{\underline{G}} \underline{\mathrm{GL}(W)}$ induces a monomorphism

$$\tilde{\mathcal{D}}/\underline{K} \to \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/\mathcal{D}}/\underline{K'} = (\tilde{\mathcal{D}} \times_{\underline{G}} \underline{\mathrm{GL}(W)})/\underline{K'}$$

We claim this map is an open and closed immersion. To see this, note that

$$(\tilde{\mathcal{D}} \times_{\underline{G}} \underline{\mathrm{GL}(W)})/K' \cong \tilde{\mathcal{D}} \times_{\underline{G}} (\underline{\mathrm{GL}(W)/K'})$$

fibers over the discrete space $G\backslash\mathrm{GL}(W)/K'$, and the fiber over the identity double coset e is exactly $\tilde{\mathcal{D}}/\underline{K}$, so $\tilde{\mathcal{D}}/\underline{K} \to \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/\mathcal{D}}/\underline{K'}$ is the pullback along the map $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/\mathcal{D}}/\underline{K'} \to G\backslash\mathrm{GL}(W)/K'$ of the open and closed immersion $e \hookrightarrow G\backslash\mathrm{GL}(W)/K'$. In particular, the map $\tilde{\mathcal{D}}/\underline{K} \to \mathcal{T}\mathrm{riv}^W_{\mathbf{V}/\mathcal{D}}/\underline{K'}$ is representable and étale. Since we already proved in the previous paragraph that $\mathcal{T}\mathrm{riv}^W_{\mathbf{V}/\mathcal{D}}/\underline{K'} \to \mathcal{D}$ is étale, we're done. □

Remark 4.15. Corollary 4.14 has direct applications to the construction of moduli spaces of local shtukas and local Shimura varieties. More precisely, let $E/\mathbf{Q}_p$ be a finite extension and let $\mathbf{D} =$

[14] Fix a maximal compact subgroup $K_0 \subset \mathrm{GL}(W)$ containing $\rho(K)$, and let $N \triangleleft K_0$ be an open normal subgroup small enough that $\rho(G) \cap N \subset \rho(K)$; then we can take $K' = N\rho(K)$.

$(\mathbf{G},\mu,b)$ be a triple[15] where $\mathbf{G}/E$ is a quasisplit reductive group, $\mu:\mathbf{G}_{m,E}\to\mathbf{G}$ is a cocharacter, and $b\in\mathbf{G}(\breve{E})$; set $G=\mathbf{G}(E)$. We then have a diamond

$$\mathrm{Gr}_{\mathbf{G},\mu}^{\mathcal{E}_b}\cong\mathrm{Gr}_{\mathbf{G},\mu}\times_{\mathrm{Spd}\,E}\mathrm{Spd}\,\breve{E}$$

whose functor of points sends $S\in\widetilde{\mathrm{Perf}}_{/\mathrm{Spa}\,\breve{E}}$ to the set of $\mathbf{G}$-bundle isomorphisms $\{u:\mathcal{F}|_{\mathcal{X}_{S^\flat}\smallsetminus S}\xrightarrow{\sim}\mathcal{E}_{b,S^\flat}|_{\mathcal{X}_{S^\flat}\smallsetminus S}\}$ such that u extends to a μ-positioned modification of $\mathcal{E}_{b,S^\flat}$ along the graph of S. The subfunctor $\mathrm{Gr}_{\mathbf{D}}=\mathrm{Gr}_{\mathbf{G},\mu}^{\mathcal{E}_b,\mathrm{adm}}$ is defined by the condition that a given point $f:S\to\mathrm{Gr}_{\mathbf{G},\mu}^{\mathcal{E}_b}$ factors through $\mathrm{Gr}_{\mathbf{D}}$ if and only if the corresponding $\mathbf{G}$-bundle $\mathcal{F}/\mathcal{X}_{S^\flat}$ is pointwise-trivial. Applying results of Kedlaya-Liu, one shows that the functor $\mathrm{Gr}_{\mathbf{D}}$ is an open and partially proper subdiamond of $\mathrm{Gr}_{\mathbf{G},\mu}^{\mathcal{E}_b}$ and that there is a universal G-local system $\mathbf{V}^{\mathrm{univ}}/\mathrm{Gr}_{\mathbf{D}}$ such that $f^*\mathbf{V}^{\mathrm{univ}}$ corresponds to the pointwise-trivial $\mathbf{G}$-bundle $\mathcal{F}$ for any $f:S\to\mathrm{Gr}_{\mathbf{D}}$.[16] Applying Theorem 4.13 and Corollary 4.14 to this local system (for the group $\mathbf{G}_0=\mathrm{Res}_{E/\mathbf{Q}_p}\mathbf{G}$), we get a diamond $\mathrm{Sht}_{\mathbf{D},\infty}=\mathcal{T}\mathrm{riv}_{\mathbf{V}^{\mathrm{univ}}/\mathrm{Gr}_{\mathbf{D}}}$ which is a good pro-étale G-torsor over $\mathrm{Gr}_{\mathbf{D}}$: this is exactly the *moduli space of local shtukas with infinite level structure* associated with the datum $\mathbf{D}=(\mathbf{G},\mu,b)$. When $K\subset G$ is open compact, Corollary 4.14 shows that the quotient $\mathrm{Sht}_{\mathbf{D},K}=\mathrm{Sht}_{\mathbf{D},\infty}/\underline{K}$ parametrizing local shtukas with K-level structure is étale over $\mathrm{Gr}_{\mathbf{D}}$.

If μ is minuscule, we've seen in Corollay 3.19 that the diamond

$$\mathrm{Gr}_{\mathbf{G},\mu}^{\mathcal{E}_b}\cong\mathscr{F}\!\ell_{\mathbf{G},\mu}^{\lozenge}\times_{\mathrm{Spd}\,E}\mathrm{Spd}\,\breve{E}$$

comes from a smooth rigid space over $\mathrm{Spa}\,\breve{E}$ via the functor $(-)^\lozenge$. Since $\mathrm{Sht}_{\mathbf{D},K}\to\mathrm{Gr}_{\mathbf{G},\mu}^{\mathcal{E}_b}$ is étale, the equivalence of étale sites associated with $(-)^\lozenge$ then produces a rigid space $\mathcal{M}_{\mathbf{D},K}$ in the étale site of $\mathscr{F}\!\ell_{\mathbf{G},\mu}\times_{\mathrm{Spa}\,E}\mathrm{Spa}\,\breve{E}$ such that $\mathcal{M}_{\mathbf{D},K}^{\lozenge}\cong\mathrm{Sht}_{\mathbf{D},K}$. The space $\mathcal{M}_{\mathbf{D},K}$ is the *local Shimura variety* associated with the datum $\mathbf{D}$ (and with K-level structure) whose existence was conjectured by Rapoport and Viehmann [RV14].

4.4 Introducing pro-étale analytic stacks

In this section we consider the following structures.

Definition 4.16. Fix an adic space S. A *pro-étale analytic stack over S* is a fibered category

$$p:\mathcal{X}\to\mathrm{Perf}_{/S^\lozenge}$$

such that

1. $\mathcal{X}$ is a stack in groupoids over $\mathrm{Perf}_{/S^\lozenge}^{\mathrm{proet}}$.[17]
2. The diagonal $\mathcal{X}\xrightarrow{\Delta}\mathcal{X}\times\mathcal{X}$ is representable in diamonds.
3. There exists a diamond $\mathcal{D}$ and a 1-morphism $\mathrm{Perf}_{/\mathcal{D}}^{\mathrm{proet}}\to\mathcal{X}$[18] which is surjective and pro-étale.

By design, a pro-étale analytic stack is a perfectoid analogue of a Deligne-Mumford stack, and reasonable Deligne-Mumford stacks over $\mathrm{Spec}\,\mathbf{Q}_p$ can be "analytified" to pro-étale analytic stacks over $\mathrm{Spa}\,\mathbf{Q}_p$. However, the following theorem gives a slew of more exotic examples.

[15] A "local shtuka datum" in the terminology of [Sch14], although we certainly don't take the most general setup possible.

[16] One also shows that $\mathrm{Gr}_{\mathbf{D}}$ is nonempty iff $[b]\in B(\mathbf{G},\mu^{-1})$.

[17] Recall the natural equivalence $\mathrm{Perf}_{/S^\lozenge}^{\mathrm{proet}}\cong\widetilde{\mathrm{Perf}}_{/S}^{\mathrm{proet}}$; we'll use this without comment in what follows.

[18] Of course we'll usually just write "a morphism $\mathcal{D}\to\mathcal{X}$" like a human being.

Theorem 4.17. *Let S be a fixed adic space, and let $\mathcal{D} \to S^\diamond$ be a diamond with an action of a locally profinite group G lying over the trivial action on $S^\diamond$. Suppose that G is either profinite or isomorphic to (an open subgroup of) $\mathbf{G}(\mathbf{Q}_p)$ for some reductive group $\mathbf{G}/\mathbf{Q}_p$. Then $[\mathcal{D}/G]$ is a pro-étale analytic stack over S.*

Proof. Let us spell out the fibered category $[\mathcal{D}/G]$ for arbitrary locally profinite G. An object of $[\mathcal{D}/G]$ is a quadruple (U, P, π, φ) where $U \in \mathrm{Perf}_{/S^\diamond}$, $\pi : P \to U$ is a pro-étale G-torsor, and $\varphi : P^\diamond \to \mathcal{D}$ is a G-equivariant map compatible with the structure maps to $S^\diamond$. A morphism

$$(U', P', \pi', \varphi') \to (U, P, \pi, \varphi)$$

in $[\mathcal{D}/G]$ is a morphism $f : U' \to U$ of perfectoid spaces over $S^\diamond$ together with a G-equivariant morphism $h : P' \to P$ over f which induces a G-equivariant isomorphism $P' \cong P \times_{\pi,U,f} U'$ and such that $\varphi = \varphi' \circ h$. The functor p is given by

$$(U, P, \pi, \varphi) \mapsto U \in \mathrm{Perf}_{/S^\diamond}.$$

In other words, the fiber category of $[\mathcal{D}/G]$ over a given $U \in \mathrm{Perf}_{/S^\diamond}$ is the groupoid of pro-étale G-torsors over U equipped with a G-equivariant map to $\mathcal{D}$.

Suppose now that G is restricted as in the theorem. Then we claim $[\mathcal{D}/G]$ is a stack in groupoids over $\mathrm{Perf}^{\mathrm{proet}}_{/S^\diamond}$: i.e., pro-étale G-torsors over U satisfy effective descent for pro-étale covers. For G profinite, one easily reduces to the straightforward case of finite G and pro-étale descent for finite étale G-torsors as in the proof of Proposition 3.6. For $G = \mathbf{G}(\mathbf{Q}_p)$, the equivalence proved in Theorem 4.13.iii reduces this claim to the fact that $\mathbf{Q}_p$-local systems satisfy pro-étale descent.

Now we consider the diagonal. For any morphism $U \xrightarrow{(x_1,x_2)} \mathcal{X} \times \mathcal{X}$ with $U \in \mathrm{Perf}_{/S^\diamond}$, we need to show that

$$\mathrm{Isom}_U(x_1, x_2) = U \times_{\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X}$$

is representable by a diamond. Specifying x_1 and x_2 amounts to specifying quadruples $(U, P_i, \pi_i, \varphi_i)$ as above, and by definition $\mathrm{Isom}_U(x_1, x_2)$ is the sheaf on $\mathrm{Perf}_{/U}$ sending $f : T \to U$ to the set of isomorphisms $i : f^*P_1^\diamond \xrightarrow{\sim} f^*P_2^\diamond$ fitting into a G-equivariant commutative diagram

$$\begin{tikzcd}
& \mathcal{D} \\
f^*P_1^\diamond \arrow[ur, "\varphi_1"] \arrow[r, "\sim", "i"'] \arrow[d, "f^*\pi_1"'] & f^*P_2^\diamond \arrow[u, "\varphi_2"'] \arrow[dl, "f^*\pi_1"] \\
T^\diamond &
\end{tikzcd}$$

of diamonds over $S^\diamond$. In particular, $\mathrm{Isom}_U(x_1, x_2)$ is naturally a subfunctor of the sheaf $\mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2)$ on $\mathrm{Perf}_{/U}$ which sends $f : T \to U$ to the set of G-equivariant isomorphisms $i : f^*P_1 \xrightarrow{\sim} f^*P_2$ over

T. We claim that these sheaves naturally sit in a G-equivariant commutative diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\Delta"] & & \mathcal{D} \times_{S^\diamond} \mathcal{D} \\
\mathrm{Isom}_U(x_1, x_2) \times_{U^\diamond} P_1^\diamondsuit \arrow[u, "a"] \arrow[r] \arrow[d, "\omega"] & \mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2) \times_{U^\diamond} P_1^\diamondsuit \arrow[ur, "b"'] \arrow[d] & \\
\mathrm{Isom}_U(x_1, x_2) \arrow[r] & \mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2) &
\end{tikzcd}$$

of sheaves on $\mathrm{Perf}^{\mathrm{proet}}_{/S^\diamond}$ where both squares are pullback squares. For the lower square this is true by construction. For the upper square, we need to define the morphism b. Giving a T-point of $\mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2) \times_{U^\diamond} P_1^\diamondsuit$ is the same as specifying a triple (f, i, s) consisting morphism $f : T \to U$ together with a G-equivariant isomorphism $i : f^*P_1 \xrightarrow{\sim} f^*P_2$ and a section $s : T \to f^*P_1$ of the map $f^*P_1 \to T$. Then b sends such a triple (f, i, s) to the T point of $\mathcal{D} \times_{S^\diamond} \mathcal{D}$ given by $T \overset{(\varphi_1 \circ s, \varphi_2 \circ i \circ s)}{\longrightarrow} \mathcal{D} \times \mathcal{D}$, which factors through $\mathcal{D} \times_{S^\diamond} \mathcal{D}$ by the commutativity of the diagrams

$$\begin{tikzcd}
P_i^\diamondsuit \arrow[r, "\varphi_i"] \arrow[d, "\pi_i"] & \mathcal{D} \arrow[d] \\
U^\diamondsuit \arrow[r] & S^\diamondsuit
\end{tikzcd}$$

for $i = 1, 2$. Since i and the φ's are G-equivariant, b is G-equivariant. Furthermore, one sees by direct inspection that $b(f, i, s)$ factors through $\Delta(\mathcal{D})$ if and only if $\varphi_1 \circ s = \varphi_2 \circ i \circ s$ as morphisms $T \to \mathcal{D}$; varying s using the G-action, the G-equivariance of b shows that $\varphi_1 \circ s = \varphi_2 \circ i \circ s$ for one s if and only if the same equality holds for all $s : T \to f^*P_1$, if and only if (f, i) comes from a T-point of $\mathrm{Isom}_U(x_1, x_2)$. This shows that the upper square is a pullback square.

Next, we claim that $\mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2)$ is a diamond over $U^\diamondsuit$.[19] For $G = \lim_{\leftarrow i} G/H_i$ profinite, this easily reduces to showing that

$$\mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2) \cong \lim_{\leftarrow i} \mathrm{Isom}_{(G/H_i)\mathrm{Tor}(U)}(P_1/\underline{H_i}, P_2/\underline{H_i})$$

is an inverse limit of perfectoid spaces finite étale over U, which again reduces us to pro-étale descent for finite étale morphisms. For $G = \mathbf{G}(\mathbf{Q}_p)$, it's enough to show that $\mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2) \times_{U^\diamond} P_1^\diamondsuit$ is a diamond, since the natural projection onto $\mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2)$ is surjective and pro-étale. Let $\mathbf{V}_i$ be the G-local system associated with P_i; then

$$\begin{aligned}
\mathrm{Isom}_{G\mathrm{Tor}(U)}(P_1, P_2) \times_{U^\diamond} P_1^\diamondsuit &\cong \mathrm{Isom}_{G\mathrm{Loc}(U)}(\mathbf{V}_1, \mathbf{V}_2) \times_{U^\diamond} \mathcal{T}\mathrm{riv}_{\mathbf{V}_1/U^\diamond} \\
&\cong \mathcal{T}\mathrm{riv}_{\mathbf{V}_2/U^\diamond} \times_{U^\diamond} \mathcal{T}\mathrm{riv}_{\mathbf{V}_1/U^\diamond}
\end{aligned}$$

where the first isomorphism follows by the results of Theorem 4.13 and the second is given by sending (i, β) to $(i \circ \beta, \beta)$. Since each $\mathcal{T}\mathrm{riv}_{\mathbf{V}_i/U^\diamond}$ is a diamond, this proves the claim.

Going back to the diagram, we now know that everything in the upper square is a diamond except $\mathrm{Isom}_U(x_1, x_2) \times_{U^\diamond} P_1^\diamondsuit$. Since the upper square is a pullback square, the existence of fiber products now shows that $\mathrm{Isom}_U(x_1, x_2) \times_{U^\diamond} P_1^\diamondsuit$ is a diamond. But then

$$\omega : \mathrm{Isom}_U(x_1, x_2) \times_{U^\diamond} P_1^\diamondsuit \to \mathrm{Isom}_U(x_1, x_2)$$

[19] In fact, it is a perfectoid space, although we won't spell this out.

is surjective and pro-étale, since it's the pullback of the surjective pro-étale map $P_1^\Diamond \to U^\Diamond$ along $\mathrm{Isom}_U(x_1, x_2) \to U^\Diamond$, and the source of ω is a diamond, so the target is a diamond as well. $\square$

This construction gives a natural geometric home to some of the equivariant sites considered in [Sch15a]: the equivariant étale site $(\mathbf{P}^{n-1}/D^\times)_{\mathrm{et}}$ introduced there coincides with the literal étale site of the pro-étale analytic stack $[\mathbf{P}^{n-1,\Diamond}/D^\times]$.

We also note that these stacks can be quite strange. For example, let $\mathcal{X}_\infty$ be the infinite level perfectoid modular curve (with some tame level K^p), and consider the stack $\mathcal{X} = [\mathcal{X}_\infty/\mathrm{GL}_2(\mathbf{Q}_p)]$. This is the stack of rigid analytic families of elliptic curves with K^p-level structure taken up to p-power isogeny, roughly. One can then show, using Rapoport-Zink uniformization, that $\mathcal{X}$ contains an open substack consisting of finitely many proper genus zero curves.

5 Period maps

5.1 Vector bundles and G-bundles on adic spaces

Let X be an adic space over $\mathbf{Q}_p$. We write $\mathrm{VB}(X)$ for the category of vector bundles on X, and we define $\mathrm{FilVB}(X)$ as the category of pairs $(\mathcal{V}, \mathrm{Fil}^\bullet\mathcal{V})$ where $\mathcal{V}$ is a vector bundle on X and $\mathrm{Fil}^\bullet\mathcal{V} \subset \mathcal{V}$ is an exhaustive separated decreasing filtration of $\mathcal{V}$ by sub-vector bundles which are $\mathcal{O}_X$-direct summands locally on X. We make $\mathrm{FilVB}(X)$ into a tensor category by setting

$$(\mathcal{V}, \mathrm{Fil}^\bullet\mathcal{V}) \otimes (\mathcal{V}', \mathrm{Fil}^\bullet\mathcal{V}') = \left(\mathcal{V} \otimes \mathcal{V}', \sum_{i+j=\bullet} \mathrm{Fil}^i\mathcal{V} \otimes_{\mathcal{O}_X} \mathrm{Fil}^j\mathcal{V}'\right),$$

i.e. by giving $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{V}'$ the convolution filtration.

Definition 5.1. Let X be any adic space over $\mathbf{Q}_p$, and let $\mathbf{G}$ be any reductive group over $\mathbf{Q}_p$. A $\mathbf{G}$-*bundle on* X is an exact additive tensor functor

$$\begin{aligned} \mathcal{V} : \mathrm{Rep}(\mathbf{G}) &\to \mathrm{VB}(X) \\ (W, \rho) &\mapsto \mathcal{V}_W. \end{aligned}$$

A *filtered* $\mathbf{G}$-*bundle on* X (with underlying $\mathbf{G}$-bundle $\mathcal{V}$) is an exact additive tensor functor

$$\begin{aligned} \mathcal{F} = (\mathcal{V}, \mathrm{Fil}^\bullet\mathcal{V}) : \mathrm{Rep}(\mathbf{G}) &\to \mathrm{FilVB}(X) \\ (W, \rho) &\mapsto (\mathcal{V}_W, \mathrm{Fil}^\bullet\mathcal{V}_W). \end{aligned}$$

Morphisms in these categories are natural isomorphisms of tensor functors (respecting the additional structures).

Of course, the trivial $\mathbf{G}$-bundle is the functor $\mathcal{V}^{\mathrm{triv}} = \mathcal{V}_X^{\mathrm{triv}} : (W, \rho) \to \mathcal{O}_X \otimes_{\mathbf{Q}_p} W$.

Proposition 5.2. *Any* $\mathbf{G}$*-bundle on any adic space* $X/\mathbf{Q}_p$ *is étale-locally trivial.*

Proof. Vector bundles, and hence $\mathbf{G}$-bundles, glue for the analytic topology on X [KL15, Theorem 2.7.7], so we may assume $X = \mathrm{Spa}(A, A^+)$ is affinoid. Then we have a tensor equivalence between the category $\mathrm{VB}(X)$ and the category $\mathrm{FP}(A)$ of finitely generated projective A-modules. But an additive exact tensor functor $\mathcal{V} : \mathrm{Rep}(\mathbf{G}) \to \mathrm{FP}(A)$ is the same thing as a $\mathbf{G}$-torsor $\mathcal{G}$ over $\mathrm{Spec}\, A$ [DMOS82, Theorem II.3.2], and any such $\mathcal{G} \to \mathrm{Spec}\, A$ is smooth and thus étale-locally split. Finally, any étale cover of $\mathrm{Spec}\, A$ can be refined to an étale cover of $\mathrm{Spa}(A, A^+)$. $\square$

Theorem 5.3. *Let X be a connected locally Noetherian adic space over $\mathbf{Q}_p$, and let $\mathcal{F}$ be a filtered $\mathbf{G}$-bundle on X with underlying vector bundle $\mathcal{V}$. Then there is a unique $\mathbf{G}(\overline{\mathbf{Q}_p})$-conjugacy class of cocharacters $\{\mu_{\mathcal{F}}\} : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to \mathbf{G}_{\overline{\mathbf{Q}_p}}$ which induces the filtration on $\mathcal{V}$ étale-locally on X, in the following precise sense: we can find an étale covering $\{f_i : U_i \to X\}_{i\in I}$ by some affinoid adic spaces U_i which each sit in a commutative diagram*

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spa}\, L_i \arrow[r] & \mathrm{Spa}\, \mathbf{Q}_p \end{tikzcd}$$

for some finite extensions $L_i/\mathbf{Q}_p$, $L_i \subset \overline{\mathbf{Q}_p}$, together with cocharacters $\mu_i \in \{\mu_{\mathcal{F}}\}$ defined over L_i and trivializations $\alpha_i : f_i^*\mathcal{V} \simeq \mathcal{V}_{U_i}^{\mathrm{triv}}$, such that μ_i defines (and splits) the filtration induced on $\mathcal{V}_{U_i}^{\mathrm{triv}}$ by α_i for each $i \in I$.

The final clause here means that for any fixed i, then for all (W,ρ) and all $n \in \mathbf{Z}$ we have

$$\alpha_i\left(f_i^*\mathrm{Fil}^n_{\mathcal{F}}\mathcal{V}_W\right) = \oplus_{m\geq n}\mathcal{V}^{\mathrm{triv}}_{U_i,W}(m)$$

as sub-$\mathcal{O}_{U_i}$-modules of $\mathcal{V}^{\mathrm{triv}}_{U_i,W}$, where

$$\mathcal{V}^{\mathrm{triv}}_{U_i,W}(m) = \left\{ s \in \mathcal{V}^{\mathrm{triv}}_{U_i,W} \cong \mathcal{O}_{U_i} \otimes_{\mathbf{Q}_p} W \mid \rho(\mu_i(u)) \cdot s = u^m s \,\forall u \in \mathcal{O}(U_i)^\times \right\}.$$

Proof. Arguing as in the proof of Proposition 5.2 reduces us to proving the analogous result for filtered $\mathbf{G}$-bundles over a Noetherian affine $\mathbf{Q}_p$-scheme $\mathrm{Spec}\, A$. In this case the claim follows from the results in [Lev13, §6.1]. □

5.2 The de Rham period map

Fix a complete discretely valued extension $E/\mathbf{Q}_p$ with perfect residue field, and suppose X is a smooth rigid analytic space over $\mathrm{Spa}\, E$. Let $\mathbf{V}$ be a $\mathbf{Q}_p$-local system on X. Then we set

$$\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) = \lambda_*\left(\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}\right),$$

where $\lambda : X_{\mathrm{proet}} \to X_{\mathrm{an}}$ denotes the natural projection of sites. This is canonically a vector bundle on X, with filtration and integrable connection induced from the filtration and connection on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$, and there is a natural injective $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$-linear comparison map

$$\alpha_{\mathrm{dR}} : \lambda^*\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) \otimes_{\lambda^*\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}.$$

As in the introduction, we say $\mathbf{V}$ is *de Rham* if α_{dR} is an isomorphism. It's easy to check that $\mathbf{D}_{\mathrm{dR}}(-)$ defines an exact additive tensor functor from de Rham $\mathbf{Q}_p$-local systems on X to filtered vector bundles with integrable connection on X.

Suppose more generally that $\mathbf{V}$ is a $G = \mathbf{G}(\mathbf{Q}_p)$-local system on X, for $\mathbf{G}/\mathbf{Q}_p$ as before. As in the introduction, we say $\mathbf{V}$ is de Rham if each $\mathbf{V}_W$ is de Rham. Then for $\mathbf{V}$ a de Rham G-local system, the functor

$$\begin{aligned} \mathbf{D}_{\mathrm{dR}}(\mathbf{V}) : \mathrm{Rep}(\mathbf{G}) &\to \mathrm{FilVB}(X) \\ (W,\rho) &\mapsto (\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_W), \mathrm{Fil}^\bullet\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_W)) \end{aligned}$$

defines a filtered $\mathbf{G}$-bundle on X.

Theorem 5.4. *Suppose $\mathbf{V} \in G\mathrm{Loc}(X)$ is de Rham with constant Hodge cocharacter μ. Then we have a G-equivariant period morphism*

$$\pi_{\mathrm{dR}} : \mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to \mathrm{Gr}_{\mathbf{G},\mu}$$

of diamonds over $\mathrm{Spd}\,\mathbf{Q}_p$.

Proof. Let $T = \mathrm{Spa}(R, R^+) \to \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ be an (R, R^+) point of $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ for some perfectoid affinoid $(E, \mathcal{O}_E)$-algebra (R, R^+). Specifying such a point amounts to specifying a pair (f, β) where $f : T \to X$ is a T-point of X and $\beta : \mathbf{V}^{\mathrm{triv}} \xrightarrow{\sim} f^*\mathbf{V}$ is a trivialization. We need to describe an (R, R^+)-point of $\mathrm{Gr}_{\mathbf{G},\mu}$.

Recall that $\mathrm{Gr}_{\mathbf{G},\mu}$ is the functor on perfectoid $(E, \mathcal{O}_E)$-algebras sending (R, R^+) to the set of associations

$$\Lambda : (\rho, W) \in \mathrm{Rep}(\mathbf{G}) \to \{\Lambda_W \subset W \otimes_{\mathbf{Q}_p} \mathbb{B}_{\mathrm{dR}}(R) \,\mathrm{a}\, \mathbb{B}^+_{\mathrm{dR}}(R)\text{−lattice}\}$$

compatible with tensor products and short exact sequences, such that Λ_W is $\rho\circ\mu$-positioned relative to the lattice $W \otimes_{\mathbf{Q}_p} \mathbb{B}^+_{\mathrm{dR}}(R)$. For any $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$, let $\mathbf{M}_{0,W} \subset \mathbf{V}_W \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}$ be the $\mathbb{B}^+_{\mathrm{dR}}$-local system given by the image of

$$\left(\lambda^*\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_W) \otimes_{\lambda^*\mathcal{O}_X} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}\right)^{\nabla=0}$$

under the isomorphism

$$\alpha^{\nabla=0}_{\mathrm{dR}} : (\lambda^*\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_W) \otimes_{\lambda^*\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}})^{\nabla=0} \xrightarrow{\sim} \mathbf{V}_W \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}.$$

The assignment $(W, \rho) \mapsto \mathbf{M}_{0,W} \subset \mathbf{V}_W \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}$ is a tensor functor in the obvious sense: it describes a "$\mathbb{B}^+_{\mathrm{dR}}$-lattice with G-structure inside $\mathbf{V} \otimes \mathbb{B}_{\mathrm{dR}}$". Pulling back under f, composing with the inverse of the isomorphism $\beta_W : \underline{W} \xrightarrow{\sim} f^*\mathbf{V}_W$ induced by β, and passing to global sections on T gives us a family of $\mathbb{B}^+_{\mathrm{dR}}(R)$-lattices

$$\Lambda_W = H^0_{\mathrm{proet}}\left(T, (\beta_W \otimes 1)^{-1}(f^*\mathbf{M}_{0,W})\right) \subset W \otimes_{\mathbf{Q}_p} \mathbb{B}_{\mathrm{dR}}(R)$$

compatible with tensor products and short exact sequences. By [Sch13, Proposition 7.9], each $\mathbf{M}_{0,W}$ is $\rho\circ\mu$-positioned relative to the lattice $\mathbf{V}_W \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}^+_{\mathrm{dR}}$. Thus we conclude that the association $(W, \rho) \mapsto \Lambda_W \subset W \otimes_{\mathbf{Q}_p} \mathbb{B}_{\mathrm{dR}}(R)$ defines an (R, R^+)-point of $\mathrm{Gr}_{\mathbf{G},\mu}$ as desired. The G-equivariance of the map π_{dR} is clear from the construction. ∎

5.3 The Hodge-Tate period map

Let X be as in the previous section. Following Hyodo [Hyo89], the relative analogue of the period ring $\mathbf{C}_p$ over X is the sheaf of rings $\mathcal{O}\mathbf{C} = \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ on X_{proet}. For any $\mathbf{V} \in \mathbf{Q}_p\mathrm{Loc}(X)$, we set

$$\mathbf{D}^i_{\mathrm{HT}}(\mathbf{V}) = \lambda_*\left(\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbf{C}(i)\right),$$

which turns out to be a vector bundle on X. We have a natural map

$$\oplus_{i\in\mathbf{Z}}\lambda^*\mathbf{D}^i_{\mathrm{HT}}(\mathbf{V}) \otimes_{\lambda^*\mathcal{O}_X} \mathcal{O}\mathbf{C}(-i) \to \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbf{C}$$

which is always injective; we say $\mathbf{V}$ is *Hodge-Tate* if this map is an isomorphism. These constructions are again compatible with tensor product, direct sum and subquotient.

Proposition 5.5. *If* $\mathbf{V} \in \mathbf{Q}_p\mathrm{Loc}(X)$ *is Hodge-Tate, there is a natural ascending filtration on* $\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \widehat{\mathcal{O}}_X$ *by* $\widehat{\mathcal{O}}_X$*-submodules* Fil_i *such that*

$$\mathrm{Fil}_i/\mathrm{Fil}_{i-1} \cong \lambda^*\mathbf{D}^i_{\mathrm{HT}}(\mathbf{V}) \otimes_{\lambda^*\mathcal{O}_X} \widehat{\mathcal{O}}_X(-i).$$

This is the relative Hodge-Tate filtration.

Proof. Set $\mathcal{O}\mathbb{B}_{\mathrm{HT}} = \mathrm{gr}(\mathcal{O}\mathbb{B}_{\mathrm{dR}})$, so $\mathcal{O}\mathbf{C}$ is the zeroth graded of this ring and α_{HT} is the zeroth graded of the isomorphism

$$\alpha_{\mathrm{HT}} : \lambda^*\mathbf{D}_{\mathrm{HT}}(\mathbf{V}) \otimes_{\lambda^*\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{HT}} \xrightarrow{\sim} \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{HT}}.$$

We have a natural long exact sequence

$$0 \to \oplus_{i\in\mathbf{Z}}\widehat{\mathcal{O}}_X(i) \to \mathcal{O}\mathbb{B}_{\mathrm{HT}} \xrightarrow{\vartheta} \mathcal{O}\mathbb{B}_{\mathrm{HT}} \otimes_{\lambda^*\mathcal{O}_X} \lambda^*\Omega^1_X(-1) \to \cdots$$

compatible with the grading, which induces a linear endomorphism $\vartheta : \mathbf{D}_{\mathrm{HT}}(\mathbf{V}) \to \mathbf{D}_{\mathrm{HT}}(\mathbf{V}) \otimes_{\mathcal{O}_X} \Omega^1_X$ compatible with the grading if we put Ω^1_X in degree one. Now consider the ascending filtration on $\mathcal{O}\mathbb{B}_{\mathrm{HT}}$ with $\mathrm{Fil}_j\mathcal{O}\mathbb{B}_{\mathrm{HT}}$ given as the associated graded of the submodule $t^{-j}\mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \subset \mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Then we define

$$\mathrm{Fil}_j(\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \widehat{\mathcal{O}}_X) \subset \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \widehat{\mathcal{O}}_X = \mathrm{gr}^0(\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{HT}})^{\vartheta=0}$$

as the image of

$$\mathrm{gr}^0\left(\lambda^*\mathbf{D}_{\mathrm{HT}}(\mathbf{V}) \otimes_{\lambda^*\mathcal{O}_X} \mathrm{Fil}_j\mathcal{O}\mathbb{B}_{\mathrm{HT}}\right)^{\vartheta=0}$$

under $\mathrm{gr}^0(\alpha_{\mathrm{HT}})$. □

Again we have a notion of a Hodge-Tate G-local system $\mathbf{V}$, and we get a Hodge cocharacter $\mu_{\mathbf{V}}$ such that $\mathbf{D}^i_{\mathrm{HT}}(\mathbf{V}_W)$ has rank equal to the dimension of the ith weight space of the cocharacter $\rho \circ \mu_{\mathbf{V}} : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to \mathbf{GL}(W)_{\overline{\mathbf{Q}_p}}$ for any $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$. The previous proposition then gives $\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \widehat{\mathcal{O}}_X$ the structure of a filtered $\mathbf{G}$-bundle on X_{proet}, in the evident sense.

Theorem 5.6. *Suppose* $\mathbf{V} \in G\mathrm{Loc}(X)$ *is Hodge-Tate with constant Hodge cocharacter* μ*. Then we have a* G*-equivariant period morphism*

$$\pi_{\mathrm{HT}} : \mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to \mathscr{F}\ell^{\lozenge}_{\mathbf{G},\mu}$$

of diamonds over $\mathrm{Spd}\,\mathbf{Q}_p$*.*

Proof. Let $\hat{\mathcal{V}} = \mathbf{V}^{\mathrm{triv}} \otimes \widehat{\mathcal{O}}_X$ be the trivial $\mathbf{G}$-bundle on X_{proet}, i.e. the tensor functor $\hat{\mathcal{V}} : (W, \rho) \mapsto \widehat{\mathcal{O}}_X \otimes_{\mathbf{Q}_p} W$. The character μ induces an increasing filtration $\mathrm{Fil}^{\mu}_{\bullet}$ on $\hat{\mathcal{V}}$. Following [CS15, §2.3], consider the $\mathbf{P}^{\mathrm{std}}_{\mu}$-torsor over X_{proet} with sections over a fixed $f : U \to X$ given by the set

$$\mathcal{P}(U) = \left\{ i : \hat{\mathcal{V}}|_U \xrightarrow{\sim} f^*\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \widehat{\mathcal{O}}_U \mid i\left(\mathrm{Fil}^{\mu}_{\bullet}\right) = f^*\mathrm{Fil}_{\bullet} \right\},$$

where of course $f^*\mathrm{Fil}_{\bullet}$ denotes the pullback of the relative Hodge-Tate filtration on $\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \widehat{\mathcal{O}}_X$. Pushing out $\mathcal{P}$ along $- \times_{\mathbf{P}^{\mathrm{std}}_{\mu}} \mathbf{G}$ gives a $\mathbf{G}$-torsor $\mathcal{G}$ over X_{proet} with sections over $f : U \to \overline{X}$ given by

$$\mathcal{G}(U) = \left\{ i : \hat{\mathcal{V}}|_U \xrightarrow{\sim} f^*\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \widehat{\mathcal{O}}_U \right\}.$$

We can equally well regard $\mathcal{G}$ as a presheaf on $\widetilde{\mathrm{Perf}}_{/X}$. By construction, we have a natural G-equivariant map

$$\mathcal{G} \to \mathscr{F}\ell^{\lozenge}_{\mathbf{G},\mu}$$

of (pre)sheaves. On the other hand, we have a natural G-equivariant map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to \mathcal{G}$ sending a T-point

$$(f : T \to X, \beta \in \mathrm{Isom}(\mathbf{V}^{\mathrm{triv}}, f^*\mathbf{V}))$$

of $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ to the section

$$\beta \otimes 1 \in \mathcal{G}(T).$$

We define π_{HT} as the composite of these two maps. $\square$

Remark 5.7. This argument also shows that $\mathcal{G}$ is a diamond: the above construction gives a natural surjective and pro-étale map

$$\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^{\lozenge}} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathbf{G}^{\lozenge} \to \mathcal{G}.$$

5.4 The Hodge-Tate period map for Shimura varieties

Let $(\mathbf{G}, X)$, μ and $E_{\mathfrak{p}}$ be as before. Fix a tame level $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$, so we have the rigid analytic Shimura variety $\mathcal{S}_{K_p} = \mathcal{S}_{K^pK_p}$ over $\mathrm{Spa}\, E_{\mathfrak{p}}$ for any open compact $K_p \subset \mathbf{G}(\mathbf{Q}_p) = G$, with associated diamond $\mathscr{S}_{K_p}$ over $\mathrm{Spd}\, E_{\mathfrak{p}}$. For simplicity, we assume that $(\mathbf{G}, X)$ satisfies Milne's axiom SV5: this guarantees that the symmetry group of the tower $\{\mathcal{S}_{K_p}\}_{K_p}$ is the full group G and not a quotient thereof [Mil04, Theorem 5.28]. For the purposes of constructing π_{HT}, one can certainly replace $(\mathbf{G}, X)$ at the outset by $(\mathbf{G}, X)^{\mathrm{ad}}$, so there's no harm in imposing this condition.

Consider the diamond

$$\mathscr{S}_{\infty} = \lim_{\leftarrow K_p} \mathscr{S}_{K_p}$$

with its natural action of G, so $\mathscr{S}_{\infty}$ is a pro-étale K_p-torsor over any $\mathscr{S}_{K_p}$. Pushing out along $- \times_{K_p} G$ gives a G-torsor $\widetilde{\mathscr{S}_{K_p}} \to \mathscr{S}_{K_p}$ and thus by Theorem 4.13 a G-local system

$$\mathbf{V}_{K_p} \in G\mathrm{Loc}(\mathscr{S}_{K_p}) \cong G\mathrm{Loc}(\mathcal{S}_{K_p}).$$

By [LZ16], this local system is de Rham with Hodge cocharacter μ, so Theorem 5.6 gives a G-equivariant Hodge-Tate period map

$$\pi_{\mathrm{HT},K_p} : \mathcal{T}\mathrm{riv}_{\mathbf{V}_{K_p}/\mathscr{S}_{K_p}} \cong \mathscr{S}_{\infty} \times_{\underline{K_p}} \underline{G} \to \mathscr{F}\ell^{\lozenge}_{\mathbf{G},\mu}$$

which depends *a priori* on our choice of K_p.

Theorem 5.8. *There is a unique G-equivariant morphism*

$$\pi_{\mathrm{HT}} : \mathscr{S}_{\infty} \to \mathscr{F}\ell^{\lozenge}_{\mathbf{G},\mu}$$

such that the diagram

$$\begin{tikzcd} \mathscr{S}_{\infty} \times_{\underline{K_p}} \underline{G} \arrow[rr, "\pi_{\mathrm{HT},K_p}"] \arrow[dr] & & \mathscr{F}\ell^{\lozenge}_{\mathbf{G},\mu} \\ & \mathscr{S}_{\infty} \arrow[ur, "\pi_{\mathrm{HT}}"'] & \end{tikzcd}$$

commutes G-equivariantly for any choice of K_p.

Proof. The best way to think about this is probably through the following statement: *there is a de Rham G-local system $\mathbf{V}_0$ on the stack $\mathscr{S}_0 = [\mathscr{S}_\infty/G]$ such that each $\mathbf{V}_{K_p}$ is obtained by pulling back $\mathbf{V}_0$ under the natural map $\mathscr{S}_{K_p} \to \mathscr{S}_0$, and such that*

$$\mathscr{S}_\infty \cong \mathcal{T}\mathrm{riv}_{\mathbf{V}_0/\mathscr{S}_0}.$$

With a little work, one can make literal sense out of this statement, and then simply *define* π_{HT} by applying Theorem 5.6 directly to the space $\mathcal{T}\mathrm{riv}_{\mathbf{V}_0/\mathscr{S}_0}$. However, the existence of $\mathbf{V}_0$ is merely a slick repackaging of the compatibility of the $\mathbf{V}_{K_p}$'s with the symmetries of the tower $\{\mathscr{S}_{K_p}\}_{K_p}$: for any $K'_p \subset K_p$ (resp. any K_p and $\gamma \in G$) the associated map $\mathrm{pr} : \mathscr{S}_{K'_p} \to \mathscr{S}_{K_p}$ (resp. $\gamma : \mathscr{S}_{\gamma K_p \gamma^{-1}} \xrightarrow{\sim} \mathscr{S}_{K_p}$) induces a canonical isomorphism $\mathrm{pr}^*\mathbf{V}_{K_p} \cong \mathbf{V}_{K'_p}$ (resp. $\gamma^*\mathbf{V}_{K_p} \cong \mathbf{V}_{\gamma K_p \gamma^{-1}}$). $\square$

5.5 Generic fiber Newton strata

In this section we sketch the proof of Theorem 1.10. Let $\mathbf{f} : \mathfrak{Y} \to \mathfrak{X}$, $f : Y \to X$, $\mathcal{M}$ and $\mathbf{V}$ be as in the theorem. Fix any rank one point $x = \mathrm{Spa}(K, \mathcal{O}_K) \to X$, and choose a geometric point $\overline{x} = \mathrm{Spa}(C, \mathcal{O}_C)$ lying over it, with corresponding point $\overline{x} : \mathrm{Spf}\, \mathcal{O}_C \to \mathfrak{X}$. Let

$$k = \mathcal{O}_C/\mathfrak{m}_C = \mathcal{O}_{C^\flat}/\mathfrak{m}_{C^\flat}$$

be the residue field of $\mathcal{O}_C$, so we get a natural geometric point $\overline{s} : \mathrm{Spec}\, k \to \mathfrak{X}$. Note that $|\overline{s}| = |\overline{x}|$ as points in the topological space $|\mathfrak{X}|$, and that the specialization map $\mathbf{s} : |X| \to |\mathfrak{X}|$ sends $|x| = |\overline{x}|$ to this point. Set $L = W(k)[\frac{1}{p}]$ and $\mathbb{A} = W(\mathcal{O}_{C^\flat})$, so the surjection $\mathcal{O}_{C^\flat} \twoheadrightarrow k$ induces a surjection $\mathbb{A}[\frac{1}{p}] \twoheadrightarrow L$.

By [CS15, Lemma 4.4.1], we have $\mathbf{V}_{\overline{x}} \cong H^i_{\mathrm{et}}(Y_{\overline{x}}, \mathbf{Q}_p)$, so $T = H^i_{\mathrm{et}}(Y_{\overline{x}}, \mathbf{Z}_p)_{/\mathrm{tors}}$ gives a canonical $\mathbf{Z}_p$-lattice in $\mathbf{V}_{\overline{x}}$. On the other hand, $\mathbf{V}$ is de Rham, so we get $\mathbb{B}^+_{\mathrm{dR}}$-local systems $\mathbf{M}, \mathbf{M}_0 \subset \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}$ associated with $\mathbf{V}$ as in the introduction. Specializing the isomorphism $\mathbf{M}_0 \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathbb{B}_{\mathrm{dR}} \cong \mathbf{M} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathbb{B}_{\mathrm{dR}}$ at $\overline{x}$ and noting that $\mathbf{M}_{\overline{x}} \cong T \otimes_{\mathbf{Z}_p} \mathbb{B}^+_{\mathrm{dR}}(C)$, we get a pair (T, Ξ) consisting of a finite free $\mathbf{Z}_p$-module T together with a $\mathbb{B}^+_{\mathrm{dR}}(C)$-lattice

$$\Xi = \mathbf{M}_{0,\overline{x}} \subset T \otimes_{\mathbf{Z}_p} \mathbb{B}_{\mathrm{dR}}(C).$$

(When x is a classical rigid analytic point, we have $\Xi = H^i_{\mathrm{dR}}(Y_x) \otimes_K \mathbb{B}^+_{\mathrm{dR}}(C)$, but in general this expression doesn't make sense.) Note that in fact $\Xi \subset T \otimes_{\mathbf{Z}_p} \mathbb{B}^+_{\mathrm{dR}}(C)$. We call such pairs *Fargues pairs,* in recognition of the following theorem:

Theorem 5.9 (Fargues)**.** *The following categories are naturally equivalent:*
i) *Pairs (T, Ξ) as above.*
ii) *Breuil-Kisin modules over $\mathbb{A} := W(\mathcal{O}_{C^\flat})$.*
iii) *Shtukas over $\mathrm{Spa}\, C^\flat$ with one paw at C.*

Applying this theorem to the particular Fargues pair we associated with $\overline{x}$ above, we get a Breuil-Kisin module M over $\mathbb{A}$; specializing $M[\frac{1}{p}]$ along $\mathbb{A}[\frac{1}{p}] \twoheadrightarrow L$ gives a φ-isocrystal M_0 over L.

Proposition 5.10. *The Newton polygon of M_0 coincides with the Newton polygon of $H^i_{\mathrm{crys}}(\mathfrak{Y}_{\overline{s}}/W(k))[\frac{1}{p}]$, and thus determines the Newton stratum of $|\mathfrak{X}|$ containing $|s|$.*

Proof. In fact, much more is true. Bhatt-Morrow-Scholze have recently defined [BMS16] a remarkable cohomology functor $H^i_{\mathbb{A}}(-)$ on smooth proper formal schemes over $\mathrm{Spf}\,\mathcal{O}_C$ taking values in Breuil-Kisin modules over $\mathbb{A}$; combining their work with Fargues's theorem, we get an identification $M[\frac{1}{p}] \cong H^i_{\mathbb{A}}(\mathfrak{Y}_{\overline{x}})[\frac{1}{p}]$. Since Bhatt-Morrow-Scholze's work also gives a canonical identification $H^i_{\mathbb{A}}(\mathfrak{Y}_{\overline{x}}) \otimes_{\mathbb{A}} L \cong H^i_{\mathrm{crys}}(\mathfrak{Y}_{\overline{s}}/W(k))[\frac{1}{p}]$ compatible with Frobenius, we deduce that the φ-isocrystal M_0 coincides φ-equivariantly with the ith rational crystalline cohomology of $\mathfrak{Y}_{\overline{s}}$, and this certainly implies the claim. □

To finish the proof, we make use of the following compatibility. Note that we freely use some notation and terminology from [Sch14, §12-14] in what follows. In particular, we make use of the space $\mathcal{Y}_{(0,\infty)} = \mathrm{Spa}\,W(\mathcal{O}_{C^\flat}) \smallsetminus \{x \mid |p\,[\varpi]\,|_x = 0\}$ and the Fargues-Fontaine curve $\mathcal{X} = \mathcal{X}_{C^\flat} = \mathcal{Y}_{(0,\infty)}/\varphi^{\mathbf{Z}}$. Let $\pi : \mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ denote the natural map of adic spaces.

Proposition 5.11. *Notation and assumptions as above, let $\mathcal{E}$ be the vector bundle on the Fargues-Fontaine curve $\mathcal{X}$ obtained by modifying the trivial bundle $T \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathcal{X}}$ at the point ∞ in such a way that the sequence*

$$0 \to \mathcal{E} \to T \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathcal{X}} \to i_{\infty *}\left((T \otimes_{\mathbf{Z}_p} \mathbb{B}^+_{\mathrm{dR}}(C))/\Xi\right) \to 0$$

of coherent sheaves is exact. Let $\tilde{\mathcal{E}}$ be the extension of the associated φ-equivariant vector bundle $\pi^*\mathcal{E}$ on $\mathcal{Y}_{(0,\infty)}$ to a φ-equivariant vector bundle on $\mathcal{Y}_{(0,\infty]}$. Then:

i. *The specialization $\tilde{\mathcal{E}}_{x_L}$ is φ-equivariantly isomorphic to M_0.*

ii. *After choosing a splitting $k \hookrightarrow \mathcal{O}_{C^\flat}$, we get a φ-equivariant isomorphism*

$$\tilde{\mathcal{E}}_{x_L} \otimes_{W(k)[\frac{1}{p}]} \mathcal{O}_{\mathcal{Y}_{(0,\infty)}} \cong \pi^*\mathcal{E}.$$

Proof of Theorem 1.10. Unwinding the definition of Caraiani-Scholze shows that the stratum of $\mathrm{Gr}_{\mathrm{GL}_n,\mu}$ containing $\pi_{\mathrm{dR}}(x)$ simply records the Harder-Narasimhan polygon of the vector bundle $\mathcal{E}$. By Proposition 5.11.ii, the Harder-Narasimhan polygon of $\mathcal{E}$ coincides with the Newton polygon of $\tilde{\mathcal{E}}_{x_L}$. By Proposition 5.11.i, the latter polygon coincides with the Newton polygon of M_0, but we've already seen that

$$M_0 \cong H^i_{\mathrm{crys}}(\mathfrak{Y}_{\overline{s}}/W(k))[\tfrac{1}{p}],$$

so we're done. □

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