Abstracts

The categorical local Langlands program

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(joint work with Lucas Mann)

This talk is a report on joint work in progress with Lucas Mann. Our goal is to formulate a program to prove the categorical local Langlands conjecture (CLLC) of Fargues–Scholze [3] for many groups.

We begin by briefly recalling the setup for this conjecture. Fix a finite extension $E/\mathbf{Q}_p$ and a connected reductive quasisplit group G/E. Fix also a prime $\ell \neq p$. On the automorphic side, the main geometric player is the stack Bun_G of G-bundles on the Fargues–Fontaine curve. This behaves like a smooth Artin stack of dimension zero. Moreover, it has a stratification indexed by the Kottwitz set $B(G)$ whose strata Bun_G^b are essentially the classifying stacks of the locally profinite groups $G_b(E)$. Here G_b is an inner form of a Levi in G.

With specific technical effort, Fargues–Scholze defined a category $D(\mathrm{Bun}_G) := D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$ of constructible ℓ-adic sheaves on Bun_G. Similar categories are defined for each stratum, which satisfy equivalences $D(\mathrm{Bun}_G^b) \cong D(G_b(E), \overline{\mathbf{Q}_\ell})$ where the right-hand side denotes the derived category of the usual category of smooth $G_b(E)$-representations. There are then some obvious functors

$$D(G_b(E), \overline{\mathbf{Q}_\ell}) \underset{i_b^*}{\overset{i_{b!}}{\rightleftarrows}} D(\mathrm{Bun}_G)$$

relating sheaves on Bun_G with representations of the groups G_b, and in fact $D(\mathrm{Bun}_G)$ is semi-orthogonally decomposed into the categories $D(G_b(E), \overline{\mathbf{Q}_\ell})$. We note in particular that for $b = 1$, $G_1 = G$, and $i_{1!}$ embeds smooth $G(E)$-representations fully faithfully into sheaves on Bun_G.

On the spectral side, the main player is the stack Par_G of ℓ-adically continuous L-parameters $\phi : W_E \to {}^LG(\overline{\mathbf{Q}_\ell})$. It is a little subtle to make this notion precise, but after pinning down its meaning, this turns out to be a very reasonable space: by independent works of Fargues–Scholze, Zhu, Hellmann, and Dat–Helm–Kurinczuk–Moss, we know that Par_G is a reduced Artin stack which is a global lci of pure dimension zero over $\mathrm{Spec}\overline{\mathbf{Q}_\ell}$, and each connected component is the quotient of an affine variety by a reductive group action. Moreover, Par_G comes with a canonical map $\mathrm{Par}_G \to B\hat{G}$.

There are then two closely related sheaf categories on the spectral side: the usual quasicoherent derived category $\mathrm{QCoh}(\mathrm{Par}_G)$, and the slightly larger category of *ind-coherent* sheaves $\mathrm{IndCoh}(\mathrm{Par}_G)$. These are related by a pair of adjoint functors

$$\mathrm{QCoh}(\mathrm{Par}_G) \overset{\Xi}{\hookrightarrow} \mathrm{IndCoh}(\mathrm{Par}_G) \overset{\Psi}{\twoheadrightarrow} \mathrm{QCoh}(\mathrm{Par}_G).$$

We note that Ψ is an equivalence on the "obvious" copies of Coh contained in its source and target, but this fails very badly for Ξ.[1]

A priori, these two sides are unrelated. However, Fargues–Scholze constructed a canonical $\otimes$-action of $\mathrm{QCoh}(\mathrm{Par}_G)$ on $D(\mathrm{Bun}_G)$, usually called "the spectral action". Given $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Par}_G)$ and $A \in D(\mathrm{Bun}_G)$, we write $\mathcal{F} * A$ for the object obtained by acting via $\mathcal{F}$ on A. Very roughly, this action is normalized by the requirement that $V * (-) = T_V(-)$, where on the left $V \in \mathrm{Rep}\hat{G}$ is regarded as a vector bundle on $B\hat{G}$ and then pulled back to a vector bundle on Par_G, and on the right T_V denotes a Hecke operator acting on sheaves on Bun_G, constructed via a suitable form of geometric Satake.

To state the categorical conjecture, we need one more piece of data, namely a choice of Whittaker datum. This is a pair (B, ψ) where $B = TN \subset G$ is a Borel and $\psi : N(E) \to \overline{\mathbf{Q}_\ell}^\times$ is a nondegenerate character. From this we form the Whittaker representation $W_\psi = c - \mathrm{ind}_{N(E)}^{G(E)} \psi$, where $c - \mathrm{ind}$ denotes smooth induction with compact support. Via the functor $i_{1!}$, we extend this to a sheaf $i_{1!}W_\psi$ on Bun_G, and then consider the functor

$$\begin{aligned} a_\psi : \mathrm{QCoh}(\mathrm{Par}_G) &\to D(\mathrm{Bun}_G) \\ \mathcal{F} &\mapsto \mathcal{F} * i_{1!}W_\psi \end{aligned}$$

given by acting spectrally on this sheaf. We can now formulate the categorical local Langlands conjecture after Fargues–Scholze.

Conjecture 0.1. *The functor a_ψ is fully faithful, and extends to an equivalence of categories $\mathbf{L}_\psi : D(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}(\mathrm{Par}_G)$ such that the diagram*

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{Par}_G) \arrow[r, "a_\psi"] \arrow[dr, "\Xi"'] & D(\mathrm{Bun}_G) \arrow[d, "\mathbf{L}_\psi", "\wr"'] \\ & \mathrm{IndCoh}(\mathrm{Par}_G) \end{tikzcd}$$

commutes.

In fact, this conjecture can be sharpened quite a bit.

Proposition 0.2. *The equivalence $\mathbf{L}_\psi$ is unique if it exists, and it exists if and only if the functor c_ψ - the right adjoint of a_ψ - restricts to give an equivalence*

$$c_\psi : D(\mathrm{Bun}_G)^{\mathrm{cpct}} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G),$$

in which case $\mathbf{L}_\psi$ is simply the ind-completion of this equivalence.

Our goal is to prove this sharpened form of CLLC for many groups. For general groups, this is a hopeless task at present, because the conjecture simply carries too much information. However, for groups where we have a solid understanding of *classical* local Langlands, we are in much better shape.

[1] We write Coh where many people would write $D^{b,\mathrm{qc}}_{\mathrm{coh}}$. With this notation, it is literally true that $\mathrm{IndCoh} = \mathrm{Ind}(\mathrm{Coh})$ for Par_G.

Definition 0.3. A quasisplit group G is *well-understood* if there is a known $B(G)_{\mathrm{basic}}$ local Langlands correspondence for G and all its Levi subgroups, which satisfies some standard expected properties (finite fibers, Whittaker-normalized, expected parametrization of discrete L-packets and the endoscopic character identites for them), and which agrees up to semisimplification with the Fargues–Scholze construction of L-parameters.

This is quite a lot to demand, but actually many groups are well-understood at present, including GL_n (for any E), as well as GSp_4, SO_{2n+1}, and the unramified form of U_{2n+1} (all with some restrictions on E).

To make progress for this class of groups, we also need one more piece of control: we *assume* that the functor c_ψ is compatible with Eisenstein series (in a precise sense). In "classical" geometric Langlands, this is a recent result of Faergeman–Hayash. In the present setting, the case of $G = \mathrm{GL}_2$ was proved by Hamann, and the general case is work in progress of Hamann–DH–Mann. We will assume this compatibility in what follows, but we only need it as a black box. We can now state our first result.

Theorem 0.4. *The categorical local Langlands conjecture is true for* GL_2*.*

This is a specialization from much more general results.

Theorem 0.5. *Let G be any well-understood group, with a fixed choice of Whittaker datum.*

i. *There is a unique continuous functor* $\mathbf{L}_\psi : D(\mathrm{Bun}_G) \to \mathrm{IndCoh}(\mathrm{Par}_G)$ *preserving compact objects and making the diagram*

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\mathbf{L}_\psi"] \arrow[dr, "c_\psi"'] & \mathrm{IndCoh}(\mathrm{Par}_G) \arrow[d, "\Psi"] \\ & \mathrm{QCoh}(\mathrm{Par}_G) \end{tikzcd}$$

commute. The functor $\mathbf{L}_\psi$ is $\mathrm{QCoh}(\mathrm{Par}_G)$*-linear and compatible with Eisenstein series.*

ii. *The functor $\mathbf{L}_\psi$ has a* $\mathrm{QCoh}(\mathrm{Par}_G)$*-linear continuous right adjoint* $\mathbf{R}_\psi : \mathrm{IndCoh}(\mathrm{Par}_G) \to D(\mathrm{Bun}_G)$ *compatible with constant terms, which also preserves compact objects.*

iii. *If a_ψ is fully faithful, then $\mathbf{R}_\psi$ is fully faithful.*

For GL_n, we can say much more.

Theorem 0.6. *Assume $G = \mathrm{GL}_n$.*

i. *The functor $\mathbf{L}_\psi \circ i_{1!}$ coincides with the fully faithful embedding* $D(\mathrm{GL}_n(E), \overline{\mathbf{Q}_\ell}) \to \mathrm{IndCoh}(\mathrm{Par}_{\mathrm{GL}_n})$ *constructed by Ben-Zvi–Chen–Helm–Nadler* [1].

ii. *The functors a_ψ and $\mathbf{R}_\psi$ are fully faithful.*

iii. *We have* $\mathbf{R}_\psi \circ \Xi = a_\psi$.

iv. *On compact sheaves, we have the duality compatibility* $\mathbf{D}_{\mathrm{tw.GS}} \circ \mathbf{L}_\psi = \mathbf{L}_{\psi^{-1}} \circ \mathbf{D}_{\mathrm{BZ}}$*, where $\mathbf{D}_{\mathrm{tw.GS}}$ denotes Chevally-twisted Grothendieck–Serre duality, and $\mathbf{D}_{\mathrm{BZ}}$ is the Bernstein–Zelevinsky ("miraculous") duality on* Bun_G*.*

We note that parts ii.-iv. depend crucially on part i.

For GL_n, this reduces the whole conjecture to the conservativity of $\mathbf{L}_\psi$. In "classical" geometric Langlands, this conservativity was a recent breakthrough of Faergeman–Raskin [2], but their microlocal techniques do not seem to adapt to our setting.

To go further, we import some idea from geometric Langlands theory "with restricted variation". Let $D(\mathrm{Bun}_G)_{\mathrm{fin}} \subset D(\mathrm{Bun}_G)$ denote the full subcategory of sheaves A such that

$$\sum_{b \in B(G), n \in \mathbf{Z}} \mathrm{length} H^n(i_b^* A) < +\infty.$$

Let $\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} \subset \mathrm{Coh}(\mathrm{Par}_G)$ denote the full subcategory of objects which are supported set-theoretically on finitely many closed fibers of the map from Par_G to its GIT quotient. It is easy to see that if the full CLLC is true, then it restricts to an equivalence $D(\mathrm{Bun}_G)_{\mathrm{fin}} \simeq \mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}$. This also has a strong converse.

Theorem 0.7. *If G is well-understood, $\mathbf{L}_\psi$ and $\mathbf{R}_\psi$ restrict to an adjoint pair of functors between $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ and $\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}$, and if either of those restricted functors is an equivalence, then the full CLLC is true for G.*

For GL_n, this reduces the whole conjecture to showing that the (fully faithful!) functor $\mathbf{R}_\psi : \mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} \to D(\mathrm{Bun}_G)_{\mathrm{fin}}$ is essentially surjective. For GL_2, we are (barely) able to check this by hand, taking advantage of the compatible gradings on the source and target by semisimple L-parameters. Up to twist, the only parameters which cause difficulty are the trivial L-parameter, where we make use of Bezrukavnikov's theory of perverse coherent sheaves on the nilpotent cone, and the semisimplification of the Steinberg parameter, where we make heavy use of an exhaustive table of RHom's between explicit generating sheaves on the coherent side, which was computed independently by Bertoloni Meli and Koshikawa.

In the talk I had almost no time to discuss the proofs. Let me briefly mention some key new ideas here:

- Very strong finiteness theorems for spectral constant term functors.
- A new theory of "admissible" ind-coherent sheaves, which comes with its own intrinsic stability properties and duality functor.
- New duality theorems for the spectral action.
- A spectral analogue of the fact that "$\mathrm{ps} - \mathrm{id}_*$ annihilates antitempered D-modules".

Using these ingredients, we are able to give an *explicit formula* for $\mathbf{R}_\psi|_{\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}}$ purely in terms of the spectral action and various dualities on both sides. This is the crucial source of control in many of our results.

References

[1] D. Ben-Zvi, H. Chen, D. Helm and D. Nadler, *Coherent Springer theory and the categorical Deligne-Langlands correspondence*, Invent. Math. 235 (2024).

[2] J. Faergeman and S. Raskin, *Nonvanishing of geometric Whittaker coefficients for reductive groups*, preprint (2022).

[3] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, preprint (2021).

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