p-adic period maps and variations of p-adic Hodge structure

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Let X be a connected complex manifold, and let $\mathbf{V} = (V_{\mathbf{Z}}, \mathrm{Fil}^\bullet \subset V_{\mathbf{Z}} \otimes_{\underline{Z}} \mathcal{O}_X, \psi)$ be a polarized $\mathbf{Z}$-variation of Hodge structure on X; for example, if $X = \mathcal{X}^{an}$ for some smooth quasiprojective variety $\mathcal{X}/\mathbf{C}$ and $f : \mathcal{Y} \to \mathcal{X}$ is some smooth projective family equipped with a choice of relatively ample line bundle, then the $\mathbf{Z}$-local system given by the primitive part of $R^i f_*^{an} \underline{\mathbf{Z}}_{/\mathrm{torsion}}$ canonically underlies a polarized $\mathbf{Z}$-VHS on X. Writing G for the automorphism group of the (bilinear form associated with the) polarization, there is a natural $G(\mathbf{Z})$-covering $\tilde{X} \to X$ parametrizing trivializations of $V_{\mathbf{Z}}$ compatible with the polarization, and a natural $G(\mathbf{Z})$-equivariant *period morphism* $\pi : \tilde{X} \to \mathcal{F}l$, where $\mathcal{F}l = G(\mathbf{C})/Q$ is a certain generalized flag variety. Much of classical Hodge theory amounts to the study of these period maps and their properties.

What is the p-adic analogue of this story? Until recently this question seemed rather murky; however, the situation changed dramatically with Scholze's discovery of the *Hodge-Tate period map* out of a Siegel Shimura variety with infinite level at p [5]. This construction was then generalized by Caraiani-Scholze (resp. Shen) to arbitrary perfectoid Shimura varieties of Hodge type (resp. abelian type) [1, 6]. In this report we sketch a general framework for constructing p-adic period maps; in particular, we get a uniform and functorial construction of a Hodge-Tate period morphism for *any* Shimura variety,[1] including those with no known interpretation as moduli of motives. Our main guide is the following slogan: the p-adic analogue of $\mathbf{Z}$- or $\mathbf{Q}$-VHS over a smooth complex manifold is a *de Rham* $\mathbf{Q}_p$*-local system on a smooth rigid analytic space.*

Given such a $\mathbf{Q}_p$-local system on a rigid space X, maybe with some G-structure, we'd like to have some covering space $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to X$ parametrizing "frames of $\mathbf{V}$" analogous to the $G(\mathbf{Z})$-covering from in the archimedean story. One quickly observes that this space must be a bit wild: the map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to X$ should have infinite (but locally profinite) geometric fibers, and in particular will not be étale. It's unclear if this space should be representable by an adic space in general; our main observation is that if one is willing to work with *diamonds*, then such a space exists.

In order to state these results more precisely, let us quickly fix some notation and terminology. In what follows, we fix a prime p, and use the term "adic space" in reference to (honest) analytic adic spaces over $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$; we also write Perf for the category of perfectoid spaces in characteristic p.

Definition 1.1.

(1) Let $\mathcal{S}$ be any site whose objects U have a "natural underlying topological space $|U|$", and let T be any topological group or ring (or just any topological space). Then we can define a sheaf $\underline{T}$ of groups (or rings) on $\mathcal{C}$ by sheafifying the presheaf $U \mapsto \mathcal{C}(|U|, T)$. If A is a topological ring and

[1] At the expense of passing to its diamond when the Shimura data is not of abelian type, cf. below.

$X \in \mathcal{S}$ is any given object, we define an A-*local system on* X as a sheaf M of $\underline{A}$-modules on $\mathcal{S}_{/X}$ such that $M|_{U_i} \simeq \underline{A}^{n_i}|_{U_i}$ locally on some covering $\{U_i \to X\}$.

(2) (Kedlaya-Liu, Scholze) Let X be any adic space or diamond; then X has a well-behaved pro-étale site X_{proet}, and every $Y \in X_{\text{proet}}$ has an underlying topological space $|Y|$ with a natural map to $|X|$. In particular, given any topological ring A, we have an associated category $A\text{Loc}(X)$ of A-local systems on (the pro-étale site of) X.

Now, given a reductive group $\mathbf{G}/\mathbf{Q}_p$ with $G = \mathbf{G}(\mathbf{Q}_p)$ its (locally profinite) group of $\mathbf{Q}_p$-points, we consider the following category.

Definition 1.2. Given X any adic space or diamond, a G-*local system* on X is a faithful exact tensor functor $\mathbf{V} : \text{Rep}(\mathbf{G}) \to \mathbf{Q}_p\text{Loc}(X)$ with the following property[2]: for each connected component of X, there is a geometric point $\overline{x} \in X$ lying in that component such that the composite tensor functor

$$\mathbf{V}_{\overline{x}} : \text{Rep}(\mathbf{G}) \to \mathbf{Q}_p\text{Loc}(X) \xrightarrow{\overline{x}^*} \mathbf{Q}_p\text{Loc}(\overline{x}) \cong \text{Vect}_{\mathbf{Q}_p}$$

is tensor-isomorphic to the trivial fiber functor

$$\begin{array}{rcl} \text{Rep}(\mathbf{G}) & \to & \text{Vect}_{\mathbf{Q}_p} \\ (W, \rho) & \mapsto & W. \end{array}$$

These form a category $G\text{Loc}(X)$ in a natural way, and this category turns out to enjoy a number of good properties:

Theorem 1.3.

(1) *If X is any adic space, with associated diamond $X^\diamond$, there is a natural equivalence $G\text{Loc}(X) \cong G\text{Loc}(X^\diamond)$.*

(2) *If X is any adic space or diamond and we take $\mathbf{G} = \text{GL}_n$, there is a natural equivalence $G\text{Loc}(X) \cong \mathbf{Q}_p\text{Loc}(X)^{\text{rank}=n}$.*

(3) *If $\mathcal{D}$ is any diamond and $\mathbf{V}$ is any G-local system on $\mathcal{D}$, then the presheaf with G-action*

$$\begin{array}{rcl} \mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}} : \text{Perf}_{/\mathcal{D}} & \to & \text{Sets} \\ (f : T^\diamond \to \mathcal{D}) & \mapsto & \text{Isom}_{G\text{Loc}(T)}(\mathbf{V}_{\text{triv}}, f^*\mathbf{V}) \end{array}$$

defines a pro-étale sheaf on the site $\text{Perf}_{/\mathcal{D}}$. Moreover, the morphism $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ is representable, surjective, and pro-étale, and $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}}$ is naturally a pro-étale $\underline{G}$-torsor over $\mathcal{D}$. In particular, $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}}$ is a diamond.

(4) *For any open subgroup $K \subset G$, the sheaf $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}}/\underline{K}$ of "K-level structures on $\mathbf{V}$" is a diamond, and its structure map to $\mathcal{D}$ is separated, étale, partially proper, and surjective. If $\mathcal{D} = X^\diamond$ is the diamond of a locally*

[2]This property turns out to be automatic for some groups (for example, for $\mathbf{G}$ which are $\mathbf{Q}_p$-split with simply connected derived group) but not for all groups. At the time of my talk in the workshop, I didn't properly appreciate the necessity of this condition, and I'm very grateful to several workshop participants, especially B. Bhatt and P. Scholze, for conversations which clarified this point.

Noetherian adic space, then $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond}/\underline{K}$ *is naturally the diamond of a locally Noetherian adic space over* X*.*

(5) *The association* $\mathbf{V} \mapsto \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$ *defines an equivalence from the category* $G\mathrm{Loc}(\mathcal{D})$ *to the category* $G\mathrm{Tor}(\mathcal{D})$ *of pro-étale* $\underline{G}$ *torsors over* $\mathcal{D}$*, with an explicit essential inverse.*

We briefly mention some of the ideas. (1) reduces immediately to the tensor equivalence $\mathbf{Q}_p\mathrm{Loc}(X) \cong \mathbf{Q}_p\mathrm{Loc}(X^\diamondsuit)$, which can be deduced from some descent results of Kedlaya-Liu [2, 3]. The proofs of (2-5) are intertwined; aside from various "reduction to GL_n"-type tricks familiar from the theory of Shimura varieties, the most crucial inputs are:

i. The fact (announced by Fargues) that the Kottwitz map is constant on connected components of the stack $\mathrm{Bun}_{\mathbf{G}}$ of $\mathbf{G}$-bundles on the Fargues-Fontaine curve; this allows one to boostrap from knowing that the map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ hits every connected component of the target (which is guaranteed by the definition of a G-local system) to the surjectivity of this map.

ii. The fact that every pro-étale $\underline{G}$-torsor $\tilde{\mathcal{D}}$ over a given diamond $\mathcal{D}$ is a well-behaved diamond; in particular, this guarantees that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$ is a diamond and has an associated topological space, which is prerequisite to writing down a candidate for the essential inverse mentioned in (5). This fact is deduced in turn from the following result, which may be of independent utility:

Theorem 1.4. *Let* G *be any locally profinite group, and let* $f : \tilde{\mathcal{F}} \to \mathcal{F}$ *be any* $\underline{G}$*-torsor in pro-étale sheaves on* Perf*. Then the morphism* f *is representable and pro-étale, and* $\tilde{\mathcal{F}}/\underline{K} \to \mathcal{F}$ *is representable and étale for any open subgroup* $K \subset G$*.*

This relies on some technical pro-étale descent results for finite étale and separated étale morphisms to perfectoid spaces, due to Weinstein and to Scholze (respectively).

Returning to the situation of interest, let X be a smooth connected rigid analytic space over a p-adic field E. Given $\mathbf{G}$ as before, we say a G-local system $\mathbf{V}$ on X is *de Rham* if for some (equivalently, any) faithful representation $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$, the $\mathbf{Q}_p$-local system $\mathbf{V}_{W,\rho}$ is de Rham in the sense of relative p-adic Hodge theory. With these preparations we can state our main result:

Theorem 1.5. *Maintain the above assumptions on* $\mathbf{V}$ *and* X*. Then:*

(1) *There is a unique geometric conjugacy class of Hodge cocharacters* $\mu : \mathbf{G}_m \to \mathbf{G}$ *such that the weights of* $\rho \circ \mu$ *record the Hodge filtration on the vector bundle* $\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_{W,\rho})$ *for every* $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$*.*

(2) *There is a natural* G*-equivariant Hodge-Tate period morphism*

$$\pi_{\mathrm{HT}} : \mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond} \to \mathscr{F}\!\ell^{\diamondsuit}_{\mathbf{G},\mu}$$

of diamonds over $\mathrm{Spd}\, E$*, where* $\mathscr{F}\!\ell_{\mathbf{G},\mu}$ *is a certain generalized flag variety for* $\mathbf{G}$*.*

Finally, we apply these ideas to Shimura varieties. Let $(\mathbf{G}, X)$ be a Shimura datum, with reflex field E and Hodge cocharacter μ. Let $\{S_K\}_{K \subset \mathbf{G}(\mathbf{A}_f)}$ be the associated tower of smooth quasiprojective Shimura varieties over E. Choose a prime $\mathfrak{p}$ of E over p and a tame level group $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$; then for any open compact subgroup $K_p \subset \mathbf{G}(\mathbf{Q}_p)$, we get a smooth rigid analytic space $\mathcal{S}_{K^pK_p} = (S_{K^pK_p} \times_E E_{\mathfrak{p}})^{ad}$ over $\operatorname{Spa} E_{\mathfrak{p}}$.

Theorem 1.6. *There is a natural* $\mathbf{G}(\mathbf{Q}_p)$*-equivariant Hodge-Tate period map*

$$\pi_{\mathrm{HT}} : \mathcal{S}^{\Diamond}_{K^p} := \varprojlim_{K_p} \mathcal{S}^{\Diamond}_{K^pK_p} \to \mathscr{F}\ell^{\Diamond}_{\mathbf{G},\mu}$$

of diamonds over $\operatorname{Spd} E_{\mathfrak{p}}$*, compatible with morphisms of arbitrary Shimura data. When* $(\mathbf{G}, X)$ *is of Hodge or abelian type, this is the "diamondization" of the Hodge-Tate period morphism constructed by Caraiani-Scholze and Shen.*

The rough idea here is that (under a mild condition on the Shimura datum) the pushout $\widetilde{\mathcal{S}^{\Diamond}_{K^pK_p}} = \mathcal{S}^{\Diamond}_{K^p} \times^{K_p} \underline{\mathbf{G}(\mathbf{Q}_p)}$ defines a pro-étale $\underline{\mathbf{G}(\mathbf{Q}_p)}$-torsor over $\mathcal{S}^{\Diamond}_{K^pK_p}$, which gives rise to an associated $\mathbf{G}(\mathbf{Q}_p)$-local system $\mathbf{V}$ over $\mathcal{S}_{K^pK_p}$ by Theorem 1.3. By a remarkable result of Liu-Zhu [4], $\mathbf{V}$ is de Rham with Hodge cocharacter μ, so Theorem 1.5 applies.

References

[1] A. Caraiani, P. Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, preprint (2015)
[2] K. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations*, Asterisque 371 (2015)
[3] K. Kedlaya, R. Liu, *Relative p-adic Hodge theory, II*, preprint (2016)
[4] R. Liu, X. Zhu, *Rigidity and a Riemann-Hilbert correspondence for p-adic local systems*, Inventiones math. (to appear)
[5] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Ann. Math. (2015).
[6] X. Shen, *Perfectoid Shimura varieties of abelian type*, preprint (2015)

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