Minimal modularity lifting for GL_2 over an arbitrary number field

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Abstract

We prove a modularity lifting theorem for minimally ramified deformations of two-dimensional odd Galois representations, over an arbitrary number field. The main ingredient is a generalization of the Taylor-Wiles method in which we patch complexes rather than modules.

1 Introduction

Fix a number field $F/\mathbf{Q}$. The Taylor-Wiles method [TW95] is a technique for proving that a surjection $R_{\overline{\rho}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$ from a Galois deformation ring to a Hecke algebra at minimal level is an isomorphism. Essentially all incarnations of the Taylor-Wiles method have been limited to situations where F is totally real or CM, $R_{\overline{\rho}}$ parametrizes deformations satisfying strong regularity and self-duality assumptions, and $\mathbf{T}_{\mathfrak{m}}$ arises from the the middle-dimensional cohomology of a Shimura variety. In a recent and very striking breakthrough, Calegari and Geraghty [CG12] found a novel generalization of the Taylor-Wiles method which eliminates some of these restrictions. More precisely, their method applies when $\mathbf{T}_{\mathfrak{m}}$ acts on the cohomology of a locally symmetric space X such that $H^i(X)_{\mathfrak{m}}$ is nonvanishing in only two consecutive degrees. In this paper we develop a further generalization of the Taylor-Wiles method; in principle, our method requires no restriction whatsoever on the range of degrees for which $H^*(X)_{\mathfrak{m}}$ is nonzero.

As a sample application, we prove the following theorem, restricting ourselves to the simplest possible situation in which our technique yields a new result. Let F be an *arbitrary* number field; set $d = [F : \mathbf{Q}]$, and let r be the number of nonreal infinite places of F. Fix a finite field k of characteristic $p \geq 3$ with p unramified in F, and set $\mathcal{O} = W(k)$. Fix an absolutely irreducible Galois representation $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ unramified at all but finitely many primes, with no "vexing" primes of ramification. Suppose $\overline{\rho}$ has the following properties:

- $\overline{\rho}|D_v$ is ordinary or finite flat for all $v|p$,
- $\det \overline{\rho}(c_\sigma) = -1$ for all real infinite places σ and complex conjugations c_σ,
- $\overline{\rho}|G_{F(\zeta_p)}$ is absolutely irreducible.

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Let $\mathcal{D}$ denote the functor which assigns to an Artinian $\mathcal{O}$-algebra A the set of equivalence classes of deformations $\tilde{\rho} : G_F \to \mathrm{GL}_2(A)$ of $\overline{\rho}$ which are minimally ramified at all places $v \nmid p$ and ordinary or finite flat at all places $v|p$. This functor is represented by a complete local Noetherian $\mathcal{O}$-algebra $R^{\min}_{\overline{\rho}}$ together with a natural universal lifting $\rho^{\min} : G_F \to \mathrm{GL}_2(R^{\min}_{\overline{\rho}})$. Let $\mathbf{T}$ be the Hecke algebra defined in §3; this is defined as a subalgebra of the ring of endomorphisms of $H^*(Y, \mathcal{O})$ for Y a certain locally symmetric quotient of $\mathrm{GL}_2(F_\infty)$. We suppose there is a maximal ideal $\mathfrak{m} \subset \mathbf{T}$ with residue field k together with a surjection $\phi_\mathfrak{m} : R^{\min}_{\overline{\rho}} \twoheadrightarrow \mathbf{T}_\mathfrak{m}$ such that $\rho_\mathfrak{m} = \phi_\mathfrak{m} \circ \rho^{\min} : G_F \to \mathrm{GL}_2(\mathbf{T}_\mathfrak{m})$ has characteristic polynomial $X^2 - T_v X + \mathbf{N}v \langle v \rangle \in \mathbf{T}_\mathfrak{m}[X]$ on almost all Frobenius elements Frob_v. In order to apply our generalization of the Taylor-Wiles method we also need this in non-minimal situations, and we need to know something about the vanishing of cohomology after localizing at a "non-Eisenstein" prime; for a precise statement, see Conjecture 3.1.

Theorem 1.1. *Suppose Conjecture 3.1 is true. Then $\phi_\mathfrak{m} : R^{\min}_{\overline{\rho}} \to \mathbf{T}_\mathfrak{m}$ is an isomorphism and $H^d(Y, \mathcal{O})_\mathfrak{m}$ is free over $\mathbf{T}_\mathfrak{m}$.*

When $r = 1$ this theorem follows from the method of [CG12]. Note that $\mathbf{T}_\mathfrak{m}$ often contains $\mathcal{O}$-torsion elements when $r \geq 1$, whereas the classical Taylor-Wiles method (as streamlined by Diamond [Dia97] and Fujiwara [Fuj06]) requires an *a priori* assumption that $\mathbf{T}_\mathfrak{m}$ be $\mathcal{O}$-flat.

Let us briefly explain the proof of Theorem 1.1. Set $q = \dim_k H^1_\emptyset(F, \mathrm{ad}^0\overline{\rho}(1))$, and write $R_\infty = \mathcal{O}[[x_1, \ldots, x_{q-r}]]$ and $S_\infty = \mathcal{O}[[T_1, \ldots, T_q]]$. Let us abbreviate $H = H^d(Y, \mathcal{O})_\mathfrak{m}$ and $R = R^{\min}_{\overline{\rho}}$; we regard H as an R-module via $\phi_\mathfrak{m}$. By a patching technique (Theorem 2.2.1), we construct an algebra homomorphism $i_\infty : S_\infty \to R_\infty$ and a finite R_∞-module H_∞, together with a surjection $\phi_\infty : R_\infty \twoheadrightarrow R$ and an ideal $\mathfrak{a} \subset S_\infty$ with $(\phi_\infty \circ i_\infty)(\mathfrak{a}) = 0$ such that $H \simeq H_\infty/\mathfrak{a}H_\infty$ as R_∞-modules, where R_∞ acts on H through ϕ_∞. Suppose we could show the S_∞-depth of H_∞ was at least $1+q-r$. Then via i_∞ the R_∞-depth would be at least $1 + q - r = \dim R_\infty$, so H_∞ would be free over R_∞ by the Auslander-Buchsbaum formula. We would then easily conclude that H is free over $R_\infty/i_\infty(\mathfrak{a})$, whence the surjection $R_\infty/i_\infty(\mathfrak{a}) \twoheadrightarrow R$ would be an isomorphism and H would be free over R.

In order to carry this out, we appeal crucially to the construction of H_∞: it is the top degree cohomology of a complex $F^\bullet_\infty$ of free finite rank S_∞-modules concentrated in a range of degrees of length $\leq r$. By a general theorem in commutative algebra (Theorem 2.1.1), this forces every irreducible component of the S_∞-support of $H^*(F^\bullet_\infty)$ to have dimension $\geq 1 + q - r$. However, the patching construction yields an R_∞-module structure on H^*_∞ which implies the opposite inequality, from whence we deduce (by Theorem 2.1.1 again) that $H^i(F^\bullet_\infty)$ vanishes for all degrees i except the top degree. As such, $F^\bullet_\infty$ yields a free resolution of H_∞ of length r, so $\mathrm{projdim}_{S_\infty}(H_\infty) = r$. But then $\mathrm{depth}_{S_\infty}(H_\infty) = 1 + q - r$ by another application of Auslander-Buchsbaum.

The numerical coincidence driving this argument persists far beyond GL_2. Roughly speaking, when considering a Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(k)$ for G some (F-split) reductive algebraic group, we require the equality

$$[F : \mathbf{Q}](\dim G - \dim B) + l(G) = \sum_{v|\infty} H^0(F_v, \mathrm{ad}^0\overline{\rho})$$

where $l(G)$ denotes the length of the range of degrees for which deformations of $\overline{\rho}$ contribute to the Betti cohomology of locally symmetric quotients of $G(F_\infty)$; the reader may wish to compare this with the numerical condition given in [CHT08]. At the very least, our method generalizes to the case when $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$ is odd (i.e. $|\mathrm{tr}\overline{\rho}(c_\sigma)| \leq 1$ for all real places σ and complex conjugations c_σ) and absolutely irreducible with big image, and $\mathcal{D}$ parametrizes minimally ramified regular crystalline deformations in the Fontaine-Laffaille range. Note the absence of any restrictions on F or any self-duality hypothesis on $\overline{\rho}$. This, again, is contingent on assuming the existence of various surjections $R_{\overline{\rho}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$.

Acknowledgments

The great debt of inspiration this work owes to the beautiful ideas of Calegari and Geraghty [CG12] will be evident to the reader; the idea of patching complexes grew naturally out of their success at patching presentations. In addition, I'm grateful to Calegari and Geraghty for pointing out a mistake in the initial public version of this paper. I'm also grateful to Avner Ash, Hailong Dao, Michael Harris, and Jack Thorne for some helpful remarks on earlier drafts of this paper. Finally, I thank the anonymous referee for a careful reading.

2 Commutative algebra

2.1 The height-amplitude theorem

Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $\mathbf{D}^-_{\mathrm{fg}}(R)$ denote the derived category of bounded-above R-module complexes with finitely generated cohomology in each degree. Given $C^\bullet \in \mathbf{D}^-_{\mathrm{fg}}(R)$, we set

$$\tau^i(C^\bullet) = \dim_k H^i(C^\bullet \otimes^{\mathbf{L}}_R k);$$

the hypertor spectral sequence shows that $H^i(C^\bullet \otimes^{\mathbf{L}}_R k)$, as a k-vector space, is isomorphic to a direct sum of subquotients of $\mathrm{Tor}_j^R(H^{i+j}(C^\bullet), k)$, from which the finiteness of the τ^i's follows easily. Any complex $C^\bullet \in \mathbf{D}^-_{\mathrm{fg}}(R)$ has a unique *minimal resolution:* a bounded-above complex $F^\bullet$ of free R-modules quasi-isomorphic to $C^\bullet$ and such that $\mathrm{im} d^i_{F^\bullet} \subseteq \mathfrak{m}F^{i+1}$ for all i. For the existence of minimal resolutions, see [Rob80]. A simple calculation shows that $\mathrm{rank}_R F^i = \tau^i(C^\bullet)$. By Nakayama's lemma, the greatest integer i such that $\tau^i(C^\bullet) \neq 0$ coincides with the greatest integer j such that $H^j(C^\bullet) \neq 0$; we denote their common value by $d^+(C^\bullet)$ or simply by d^+ if $C^\bullet$ is clear.

Given $C^\bullet \in \mathbf{D}^-_{\mathrm{fg}}(R)$, we define the *amplitude* of $C^\bullet$ as the difference

$$\mathrm{am}(C^\bullet) = \sup\left\{i | \tau^i(C^\bullet) \neq 0\right\} - \inf\left\{i | \tau^i(C^\bullet) \neq 0\right\}.$$

In general the amplitude need not be finite: if M is a finite R-module, viewed as a complex concentrated in degree zero, then $\mathrm{am}(M) = \mathrm{projdim}(M)$. Note that the amplitude is finite if and only if the minimal resolution of $C^\bullet$ is a bounded complex.

The first two parts of the following theorem and their proofs are implicit in James Newton's appendix to [Han12].

Theorem 2.1.1. *Suppose R is Cohen-Macaulay and $C^\bullet \in \mathbf{D}^-_{\mathrm{fg}}(R)$ is a complex of finite amplitude.*

i. *Any minimal prime $\mathfrak{p}$ in the R-support of $H^*(C^\bullet)$ satisfies*

$$\mathrm{ht}\,\mathfrak{p} \leq \mathrm{am}(C^\bullet).$$

ii. *If $\mathfrak{p}$ is a minimal prime in the R-support of $H^*(C^\bullet)$ with* $\operatorname{ht}\mathfrak{p} = \operatorname{am}(C^\bullet)$, *then $H^j(C^\bullet)_\mathfrak{p} = 0$ for $j \neq d^+$.*

iii. *If* $\operatorname{ht}\mathfrak{p} = \operatorname{am}(C^\bullet)$ *for* every *minimal prime in the R-support of $H^*(C^\bullet)$, then $H^j(C^\bullet) = 0$ for $j \neq d^+$, and $H^{d^+}(C^\bullet)$ is a perfect R-module.*

Proof of i. and ii. Replacing $C^\bullet$ by its minimal resolution, we may assume $C^\bullet$ is a bounded complex of free R-modules of finite rank (and as such, we may write derived tensor products of $C^\bullet$ as ordinary tensor products). Let d^- be the least integer i for which $\tau^i(C^\bullet) \neq 0$. Let $\mathfrak{p}$ be a minimal element of $\operatorname{Supp}H^*(C^\bullet)$, and let r be the least degree with $\mathfrak{p} \in \operatorname{Supp}H^r(C^\bullet)$. Let $h = \operatorname{ht}\mathfrak{p}$, and choose a system of parameters $x_1, \dots, x_h \in \mathfrak{p}$ for $R_\mathfrak{p}$. Set $J_n = (x_1, \dots, x_n)$. We will show inductively that $H^{r-n}(C^\bullet \otimes R/J_n) \neq 0$ for $1 \leq n \leq h$. Granted this inductive step, the theorem follows from the following observation: letting $\check{C}_\bullet = \operatorname{Hom}_R(C^\bullet, R)$ denote the dual complex, there is a natural spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_R(H_j(\check{C}_\bullet), R/I) \Rightarrow H^{i+j}(C^\bullet \otimes R/I).$$

Since $H^{r-h}(C^\bullet \otimes R/J_h) \neq 0$, the least j with $H_j(\check{C}_\bullet) \neq 0$, say $j_{\min}$, satisfies $j_{\min} \leq r-h$. Taking $I = \mathfrak{m}$, the entry $E_2^{0,j_{\min}}$ is stable and nonzero, so $d^- = j_{\min} \leq r - h$. Putting things together gives

$$d^- + h \leq r \leq d^+,$$

so $h \leq d^+ - d^- = \operatorname{am}(C^\bullet)$, as desired. If equality holds then $r \geq d^- + \operatorname{am}(C^\bullet) = d^+$, but $r \leq d^+$ was the *least* degree with $\mathfrak{p} \in \operatorname{Supp}H^r(C^\bullet)$.

It remains to carry out the inductive step. Let $\mathfrak{p}_n$ denote the image of $\mathfrak{p}$ under $R_\mathfrak{p} \to R_\mathfrak{p}/J_n$. For $0 \leq n \leq h-1$, suppose $H^{r-n}(C^\bullet \otimes R/J_n)_{\mathfrak{p}_n}$ is nonzero with $\mathfrak{p}_n$ an associated prime, and $H^i(C^\bullet \otimes R/J_n)_{\mathfrak{p}_n} = 0$ for $i < r-n$. Then $H^{r-n-1}(C^\bullet \otimes R/J_{n+1})_{\mathfrak{p}_{n+1}}$ is nonzero with $\mathfrak{p}_{n+1}$ an associated prime, and $H^i(C^\bullet \otimes R)_{\mathfrak{p}_{n+1}} = 0$ for $i < r-n-1$. The supposition is true for $n = 0$ by our assumptions and the fact that minimal primes are associated primes. To prove the induction, we proceed as follows. For each $0 \leq n \leq h-1$ we have a spectral sequence

$$E_2^{i,j} = \operatorname{Tor}_{-i}^{R/J_n}(H^j(C^\bullet \otimes R/J_n), R/J_{n+1}) \Rightarrow H^{i+j}(C^\bullet \otimes R/J_{n+1})$$

of R/J_n-modules. Localize this spectral sequence at $\mathfrak{p}$. Since $R_\mathfrak{p}$ is Cohen-Macaulay, any system of parameters is a regular sequence on $R_\mathfrak{p}$. As such, caculating $\operatorname{Tor}^{R_\mathfrak{p}/J_n}(-, R_\mathfrak{p}/J_{n+1})$ via the resolution

$$0 \to R_\mathfrak{p}/J_n \xrightarrow{\cdot x_{n+1}} R_\mathfrak{p}/J_n \to R_\mathfrak{p}/J_{n+1} \to 0$$

implies that the entries of the spectral sequence vanish for $i \neq 0, 1$, with $E_2^{-1,j} \simeq H^j(C^\bullet \otimes R/J_n)_{\mathfrak{p}_n}[x_{n+1}]$. The vanishing claim follows easily, and we get an isomorphism

$$H^{r-n-1}(C^\bullet \otimes R/J_{n+1})_{\mathfrak{p}_{n+1}} \simeq H^{r-n}(C^\bullet \otimes R/J_n)_{\mathfrak{p}_n}[x_{n+1}],$$

of $R_\mathfrak{p}/J_{n+1}$-modules; by our inductive hypothesis the right-hand side is easily seen to be nonzero with $\mathfrak{p}_{n+1}$ an associated prime. This completes the proof of i. and ii.

Proof of iii. Let

$$F^\bullet : 0 \to F^0 \to F^1 \to \cdots \to F^d \to 0$$

be a complex of free finite rank R-modules such that every minimal prime in the R-support of $H^*(F^\bullet)$ has height exactly d. By parts i. and ii., every minimal prime in the support of $H^i(F^\bullet)$

for $0 \le i \le d-1$ has height $\ge d+1$. Consider the dual complex $\check{F}^\bullet = \mathrm{Hom}_R(F^{-\bullet}, R)$. A priori the cohomology of $\check{F}^\bullet$ is concentrated in degrees $-d$ through 0, and we have a convergent spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R(H^{-j}(F^\bullet), R) \Rightarrow H^{i+j}(\check{F}^\bullet).$$

Since R is Cohen-Macaulay, we have $\mathrm{grade}M + \dim M = \dim R$ for any R-module M, and thus the entries $E_2^{i,j}$ vanish for $j < -d$, for $j = -d$ with $i < d$, and for $j > -d$ with $i < d+1$. Thus the spectral sequence yields isomorphisms $H^i(\check{F}^\bullet) = 0$ for $i \neq 0$ and $H^0(\check{F}^\bullet) \simeq \mathrm{Ext}^d_R(H^d(F^\bullet), R)$. Applying the adjunction isomorphism

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(F^\bullet, R), R) \cong F^\bullet$$

yields a dual spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R(H^{-j}(\check{F}^\bullet), R) \Rightarrow H^{i+j}(F^\bullet),$$

which correspondingly degenerates to an isomorphism $H^i(F^\bullet) \simeq \mathrm{Ext}^i_R(H^0(\check{F}^\bullet), R)$ for any i. Since $\mathrm{Ann}H^0(\check{F}^\bullet) \supseteq \mathrm{Ann}H^d(F^\bullet)$, we have $\mathrm{grade}H^0(\check{F}^\bullet) \ge \mathrm{grade}H^d(F^\bullet) = d$, so $H^i(F^\bullet)$ vanishes for $i < d$. Thus $F^\bullet$ yields a free resolution of $H^d(F^\bullet)$, so $\mathrm{projdim}H^d(F^\bullet) \le d$. Quite generally we have $\mathrm{grade}M \le \mathrm{projdim}M$, so perfection follows. □

2.2 A patching theorem for complexes

Fix a complete discrete valuation ring $\mathcal{O}$. Set $R_\infty = \mathcal{O}[[x_1, \dots, x_{q-r}]]$ and $S_\infty = \mathcal{O}[[T_1, \dots, T_q]]$. Write S_n for the quotient $S_\infty / \big((1+T_1)^{p^n} - 1, \dots, (1+T_q)^{p^n} - 1\big)$, with $\overline{S}_n = S_n/\varpi^n$. We write $\mathfrak{a}$ for the ideal $(T_1, \dots, T_q)$ in S_∞ and in S_n, and we abusively write k for the common residue field of all these local rings.

Theorem 2.2.1. *Let R be a complete local Noetherian $\mathcal{O}$-algebra, and let H be an R-module which is $\mathcal{O}$-module finite. Suppose for each integer $n \ge 1$ we have a surjection $\phi_n : R_\infty \to R$ and a complex $C_n^\bullet \in \mathbf{D}^-_{\mathrm{fg}}(S_n)$ of S_n-modules with the following properties:*

i. *$\tau^i(C_n^\bullet)$ is independent of n for $i \in [d, d-r]$ and zero for $i \notin [d, d-r]$, where d is some fixed integer.*

ii. *There is a degree-preseving R_∞-module structure on $H_n^* = H^*(C_n^\bullet)$ such that the image of S_n in $\mathrm{End}_{\mathcal{O}}(H_n^*)$ arises from an algebra homomorphism $i_n : S_\infty \to R_\infty$ with $(\phi_n \circ i_n)(\mathfrak{a}) = 0$.*

iii. *Writing $H_n = H^d(C_n^\bullet)$, there is an isomorphism $H_n/\mathfrak{a}H_n \simeq H$ of R_∞-modules where R_∞ acts on H via ϕ_n.*

Then H is free over R.

Proof. Let $F_n^\bullet$ be the minimal resolution of $C_n^\bullet$, and set $\overline{F}^\bullet_{n,m} = F_n^\bullet \otimes_{S_n} \overline{S}_m$ for $m \le n$. By our assumptions $\dim_{\overline{S}_m} \overline{F}^i_{n,m} = \tau^i(C_n^\bullet) = \tau^i$ is independent of n and m. Choosing bases we can and do represent the differentials $d^i_{n,m}$ of $\overline{F}^\bullet_{n,m}$ by matrices $\delta^i_{n,m} \in M_{\tau^i \times \tau^{i+1}}(\overline{S}_m)$. By the usual pigeonhole argument we may find integers $j_n, n \ge 1$ such that $\delta^i_{j_n,n}$ is the reduction of $\delta^i_{j_{n+1},n+1}$. Let δ^i_∞ be the limit of the sequence $\delta^i_{j_n,n}$ as $n \to \infty$, and let $F_\infty^\bullet$ be the bounded complex of free finite rank S_∞-modules whose differentials are given by the δ^i_∞'s. Set $H^*_\infty = H^*(F^\bullet_\infty)$ and $H_\infty = H^d(F^\bullet_\infty)$. Passing to a further subsequence if necessary, the maps i_n and ϕ_n converge to algebra homomorphisms i_∞

and ϕ_∞, and the R_∞-module structures on $H^*_{j_n}$ patch together into an R_∞-module structure on H^*_∞ such that S_∞ acts through i_∞. Since H^*_∞ is finite over S_∞, it is finite over R_∞, so in particular

$$\dim_{S_\infty} H^*_\infty = \dim_{R_\infty} H^*_\infty \leq \dim R_\infty = 1 + q - r.$$

On the other hand, the first part of the height-amplitude theorem implies the *opposite* inequality, so every minimal prime in the S_∞-support of H^*_∞ has height exactly r. Therefore, $H_\infty \simeq H^*_\infty$ by the third part of the height-amplitude theorem, and $F^\bullet_\infty$ is a free resolution of H_∞ of length r. This shows that $\mathrm{projdim}_{S_\infty}(H_\infty) = r$, so $\mathrm{depth}_{S_\infty}(H_\infty) = 1 + q - r$ by the Auslander-Buchsbaum formula. But then $\mathrm{depth}_{R_\infty}(H_\infty) = 1 + q - r = \dim R_\infty$ via i_∞, so H_∞ is a free module over R_∞ by a second application of Auslander-Buchsbaum. Therefore $H_\infty/\mathfrak{a}H_\infty$ is a free module over $R_\infty/i_\infty(\mathfrak{a})$. But $H_\infty/\mathfrak{a}H_\infty \simeq H$ as $R_\infty/i_\infty(\mathfrak{a})$-modules, where $R_\infty/i_\infty(\mathfrak{a})$ acts on H through the surjection $R_\infty/i_\infty(\mathfrak{a}) \twoheadrightarrow R$ induced by ϕ_∞. $\square$

3 Modularity lifting

We return to the notation of the introduction. Let $S(\overline{\rho})$ be the ramification set of $\overline{\rho}$, let Q be any finite set of primes disjoint from $S(\overline{\rho}) \cup \{v|p\}$, and let S_Q denote the set of places $Q \cup S(\overline{\rho}) \cup \{v|p\}$. For any such Q, let R_Q denote the deformation ring defined in §4.1 of [CG12]; this is a complete local Noetherian $\mathcal{O}$-algebra. Let $H^1_Q(F, \mathrm{ad}^0\overline{\rho})$ be the Selmer group defined as the kernel of the map

$$H^1(F, \mathrm{ad}^0\overline{\rho}) \to \prod_v H^1(F_v, \mathrm{ad}^0\overline{\rho})/L_v$$

where $L_v = H^1_{\mathrm{ur}}(F_v, \mathrm{ad}^0\overline{\rho})$ if $v \notin Q \cup \{v|p\}$, $L_v = H^1(F_v, \mathrm{ad}^0\overline{\rho})$ if $v \in Q$, and $L_v = H^1_{\mathrm{f}}(F_v, \mathrm{ad}^0\overline{\rho})$ if $v|p$ (here H^1_{f} is as in §2.4 of [DDT94]). Modifying the proof of Corollary 2.43 of [DDT94] via Corollary 2.4.3 of [CHT08], we find that the reduced tangent space of R_Q has dimension at most

$$\dim_k H^1_Q(F, \mathrm{ad}^0\overline{\rho}(1)) - r + \sum_{v \in Q} \dim_k H^0(F_v, \mathrm{ad}^0\overline{\rho}(1)).$$

We define L_Q and K_Q be the open compact subgroups as in [CG12]. We denote by $Y_0(Q)$ the arithmetic quotient $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}_F)/L_Q K^\circ_\infty Z_\infty$, and by $Y_1(Q)$ the quotient $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}_F)/K_Q K^\circ_\infty Z_\infty$. For $Q = \emptyset$ we simply write Y. Let

$$T_v = L_Q \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} L_Q$$

and

$$\langle v \rangle = L_Q \begin{pmatrix} \varpi_v & \\ & \varpi_v \end{pmatrix} L_Q$$

denote the usual Hecke operators; when $v \in Q$ we write U_v for T_v as is customary. Let $\mathscr{T}^{\mathrm{an}}_Q$ denote the abstract polynomial algebra over $\mathcal{O}$ in the operators T_v and $\langle v \rangle$ for all places $v \notin S_Q$, and let $\mathscr{T}_Q$ denote the algebra generated by these operators together with the operators U_v for $v \in Q$. Write $\mathbf{T}^{\mathrm{an}}_Q$ and $\mathbf{T}_Q$ for the images of $\mathscr{T}^{\mathrm{an}}_Q$ and $\mathscr{T}_Q$ in $\mathrm{End}_{\mathcal{O}}(H^*(Y_1(Q), \mathcal{O}))$. When $Q = \emptyset$ we write $\mathbf{T} = \mathbf{T}_\emptyset$. By assumption $\overline{\rho}$ is associated with a maximal ideal of $\mathbf{T}$ which we denote by $\mathfrak{m}_\emptyset$.

Let $\mathfrak{m}$ be any maximal ideal of $\mathbf{T}_Q$ containing the preimage of $\mathfrak{m}_\emptyset$ under the natural map $\mathbf{T}_Q^{\mathrm{an}} \to \mathbf{T}$. We make the following conjecture:

Conjecture 3.1 (Existence Conjecture): *For any Q there is a surjection $\phi_Q : R_Q \twoheadrightarrow \mathbf{T}_{Q,\mathfrak{m}}$ such that the associated Galois representation $\rho_Q : G_F \to \mathrm{GL}_2(\mathbf{T}_{Q,\mathfrak{m}})$ has the following properties:*

i. *For any $v \notin S_Q$,*

$$\det(X - \rho_Q(\mathrm{Frob}_v)) = X^2 - T_v X + \mathbf{N}v \cdot \langle v \rangle \in \mathbf{T}_{Q,\mathfrak{m}}[X].$$

ii. *For any $v \in Q$, $\rho_Q | D_v \simeq \eta_1 \oplus \eta_2$ with η_1 unramified and $\eta_1(\mathrm{Frob}_v) = U_v$.*

Furthermore, $H^i(Y_1(Q), \mathcal{O}) \otimes_{\mathbf{T}_Q} \mathbf{T}_{Q,\mathfrak{m}}$ vanishes for $i \notin [d-r, d]$, where $d = [F : \mathbf{Q}]$.

Suppose now for each $n \geq 1$ that Q_n is a set of Taylor-Wiles primes of cardinality $q = \dim_k H^1_\emptyset(F, \mathrm{ad}^0 \overline{\rho}(1))$, such that each $v \in Q_n$ has $\mathbf{N}v \equiv 1 \bmod p^n$. The reduced tangent space of R_{Q_n} has dimension at most $q - r$. Let $\mathfrak{m}_n$ denote the maximal ideal of $\mathbf{T}_{Q_n}$ generated by the preimage of $\mathfrak{m}$ under the map $\mathbf{T}_{Q_n}^{\mathrm{an}} \to \mathbf{T}$ and by $U_v - \alpha_v$ for all $v \in Q_n$, where α_v is a fixed choice of one of the eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$.

Proposition 3.2. *There is an isomorphism*

$$H^*(Y_0(Q_n), \mathcal{O})_{\mathfrak{m}_n} \simeq H^*(Y, \mathcal{O})_{\mathfrak{m}}.$$

This is proved exactly as in Lemmas 3.4 and 4.6 of [CG12], working one degree at a time.

By design there is a natural surjection

$$\prod_{v \in Q_n} (\mathcal{O}_{F_v}/\varpi_v)^\times \twoheadrightarrow (\mathbf{Z}/p^n)^q.$$

Composing this with the natural reduction map $L_{Q_n} \to \prod_{v \in Q_n} (\mathcal{O}_{F_v}/\varpi_v)^\times$ gives $S_n = \mathcal{O}[(\mathbf{Z}/p^n)^q]$ the structure of a local system over $Y_0(Q_n)$. Let $X_\mathbf{A}$ be the quotient $\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}_F)/K_\infty^\circ Z_\infty$, and let $C_\bullet(X_\mathbf{A})$ be the complex of singular chains on $X_\mathbf{A}$ with $\mathbf{Z}$-coefficients. Set $C^\bullet(Q_n) = \mathrm{Hom}_\mathbf{Z}(C_\bullet(X_\mathbf{A}), S_n)^{L_{Q_n}}$, so there is a canonical isomorphism $H^*(C^\bullet(Q_n)) \simeq H^*(Y_0(Q_n), S_n)$. The canonical action of $\mathrm{GL}_2(\mathbf{A}_F^f)$ on $C_\bullet(X_\mathbf{A})$ induces a canonical action of the algebra $\mathscr{T}_{Q_n}$ on the complex $C^\bullet(Q_n)$ lifting the Hecke action on cohomology: precisely, given $\phi \in \mathrm{Hom}_\mathbf{Z}(C_\bullet(X_\mathbf{A}), S_n)^{L_{Q_n}}$ regarded as a function on chains, $T_g = [L_{Q_n} g L_{Q_n}] \in \mathscr{T}_{Q_n}$ acts on ϕ by

$$(T_g \cdot \phi)(\sigma) = \sum_{g_i \in L_{Q_n} g L_{Q_n}/L_{Q_n}} g_i \cdot \phi(\sigma g_i).$$

(For a more thorough discussion of this idea, which the author learned from Glenn Stevens, see [Han12].)

Let $\mathfrak{M}_n$ be the preimage of $\mathfrak{m}_n$ under the structure map $\mathscr{T}_{Q_n} \twoheadrightarrow \mathbf{T}_{Q_n}$, and set

$$C_n^\bullet = C^\bullet(Q_n) \otimes_{\mathscr{T}_{Q_n}} \mathscr{T}_{Q_n, \mathfrak{M}_n}.$$

Since $\mathscr{T}_{Q_n, \mathfrak{M}_n}$ is flat over $\mathscr{T}_{Q_n}$, the functor $- \otimes_{\mathscr{T}_{Q_n}} \mathscr{T}_{Q_n, \mathfrak{M}_n}$ commutes with taking cohomology, so we have canonical isomorphisms

$$\begin{aligned} H^*(C_n^\bullet) &\simeq H^*(C^\bullet(Q_n)) \otimes_{\mathscr{T}_{Q_n}} \mathscr{T}_{Q_n, \mathfrak{M}_n} \\ &\simeq H^*(Y_0(Q_n), S_n) \otimes_{\mathbf{T}_{Q_n}} \mathbf{T}_{Q_n} \otimes_{\mathscr{T}_{Q_n}} \mathscr{T}_{Q_n, \mathfrak{M}_n} \\ &\simeq H^*(Y_0(Q_n), S_n)_{\mathfrak{m}_n}. \end{aligned}$$

If $\mathfrak{a}$ denotes the augmentation ideal of S_n then

$$\begin{aligned} H^i(C_n^\bullet \otimes^{\mathbf{L}}_{S_n} S_n/\mathfrak{a}S_n) &\simeq H^i\left(Y_0(Q_n), \mathcal{O}\right)_{\mathfrak{m}_n} \\ &\simeq H^i(Y, \mathcal{O})_{\mathfrak{m}} \end{aligned}$$

by Proposition 3.2. This shows that the complexes $C_n^\bullet$ satisfy assumption i. of Theorem 2.2.1. We are ready to verify the rest of the assumptions of Theorem 2.2.1. For each n, fix a choice of a surjection $\sigma_n : R_\infty \twoheadrightarrow R_{Q_n}$. The composite $\phi_{Q_n} \circ \sigma_n$ where ϕ_{Q_n} is the map provided by the existence conjecture gives $H^*(C_n^\bullet)$ a degree-preserving R_∞-module structure. Define ϕ_n as the composite of our chosen surjection $R_\infty \twoheadrightarrow R_{Q_n}$ with the natural surjection $R_{Q_n} \twoheadrightarrow R^{\min}$. As in §2.8 of [DDT94], R_{Q_n} is naturally an S_n-algebra, with $R_{Q_n}/\mathfrak{a}R_{Q_n} \simeq R^{\min}$, and the map ϕ_{Q_n} is equivariant for the S_n-actions on its source and target. Let $i_n : S_\infty \to R_\infty$ be any fixed lift of the composite $S_\infty \to S_n \to R_{Q_n}$ compatible with σ_n, so $(\phi_n \circ i_n)(\mathfrak{a}) = 0$ by construction. We've now verified assumption ii. Assumption iii. is immediate from Hochschild-Serre, so Theorem 2.2.1 applies and modularity lifting at minimal level follows.

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