Local Shimura varieties - past, present, future

David Hansen

National University of Singapore

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Shimura varieties

Recall: A **Shimura datum** is (roughly) a pair (G, X) where $G/\mathbf{Q}$ is a connected reductive group and $X \simeq G(\mathbf{R})/K_\infty$ is a Hermitian symmetric domain.

Theorem (Baily-Borel, Shimura, Deligne, Borovoi, Milne)

Given (G, X) as above, and $K \subset G(\mathbf{A}_f)$ a sufficiently small open compact subgroup, the locally symmetric manifold

$$G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K$$

is $\cong \mathrm{Sh}(G, X)_K(\mathbf{C})$ for a certain smooth quasiprojective algebraic variety $\mathrm{Sh}(G, X)_K$ defined over a number field $E = E(G, X)$.

$\rightsquigarrow$ Get a tower $\{\mathrm{Sh}(G, X)_K\}_K$ with $G(\mathbf{A}_f)$-action.

Modular curves

Take $(G, X) = (\mathrm{GL}_2, \mathfrak{H}^\pm = \mathbf{C} - \mathbf{R})$. Then $E = \mathbf{Q}$ and $Y_K = \mathrm{Sh}(G, X)_K$ is the usual tower of modular curves.

When $K = K(N) = \{g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) | g = 1 \bmod N\}$, $Y_{K(N)}$ is the moduli space of elliptic curves E with a trivialization $(\mathbf{Z}/N\mathbf{Z})^2 \simeq E[N]$.

Note: Cohomology

$$\mathrm{colim}_{K \to \{1\}} H^1_{\mathrm{et}}(Y_{K,\overline{\mathbf{Q}}}, \overline{\mathbf{Q}_\ell})$$

has a natural action of $\Gamma_{\mathbf{Q}} \times \mathrm{GL}_2(\mathbf{A}_f)$.

What information does this action encode?

Modular curves cont'd

Recall: Let $f = \sum_{n\geq 1} a_n q^n \in S_2(N)$ be any normalized weight two cuspidal Hecke eigenform.
$\rightsquigarrow$ Eichler-Shimura: There is a (unique) Galois representation $\rho_f : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ unramified outside $N\ell$ such that $\mathrm{tr}\rho_f(\mathrm{Fr}_p) = a_p$ for all $p \nmid N\ell$.
Example: If

$$f = q \prod_{n\geq 1}(1-q^n)(1-q^{3n})(1-q^{5n})(1-q^{15n}) \in S_2(15),$$

then ρ_f is realized in the ℓ-adic Tate module of the elliptic curve $y^2 + xy + y = x^3 + x^2$.

Modular curves cont'd

Theorem (Langlands, Piatetski-Shapiro, Deligne, Brylinski, ...)

As $\Gamma_{\mathbf{Q}} \times \mathrm{GL}_2(\mathbf{A}_f)$-representations, there is an isomorphism

$$\mathrm{colim}_{K \to \{1\}} H^1_{\mathrm{et}}(Y_{K,\overline{\mathbf{Q}}}, \overline{\mathbf{Q}_\ell}) = \oplus_f \rho_f \otimes \bigotimes_p \pi_{f,p} + \dots,$$

where $\pi_{f,p} \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(\mathrm{GL}_2(\mathbf{Q}_p))$ matches $\rho_f | W_{\mathbf{Q}_p}$ via local Langlands correspondence.

Here "$\dots$" is some "boring" part (all $\Gamma_{\mathbf{Q}}$-irreps occurring there are one-dimensional).

Back to general Shimura varieties

General expectations (Langlands, Arthur, Kottwitz, ...) Given a Shimura datum (G, X), can extract an algebraic representation $r : {}^L G \to \mathrm{GL}_m$. Then expect that

$$\mathrm{colim}_{K \to \{1\}} H_{\mathrm{et}}^{\dim X}(\mathrm{Sh}(G, X)_{K,\overline{E}}, \overline{\mathbf{Q}_\ell}) = \oplus_\pi r \circ \rho_\pi|_{\Gamma_E} \otimes \bigotimes_p \pi_p + \dots.$$

Here the sum runs over some suitable set of cuspidal aut. reps. of G, and $\rho_\pi : \Gamma_{\mathbf{Q}} \to {}^L G(\overline{\mathbf{Q}_\ell})$ is a Galois representation such that $\pi_p \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G(\mathbf{Q}_p))$ matches $\rho_\pi | W_{\mathbf{Q}_p}$ via LLC.
Here "..." is some "less interesting" part.

Local Shimura varieties

Fix a prime p, and set $\breve{\mathbf{Q}}_p = \widehat{\mathbf{Q}_p^{\mathrm{unr}}} \circlearrowleft \sigma =$ lift of $x \mapsto x^p$.

Definition (Rapoport-Viehmann)

A **local Shimura datum** *is a triple* $(G, \{\mu\}, b)$ *where* $G/\mathbf{Q}_p$ *is a connected reductive group,* $\{\mu\}$ *is a conjugacy class of minuscule cocharacters* $\mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$*, and* $b \in G(\breve{\mathbf{Q}}_p)$ *is an element such that* $b \in B(G, \mu)$*.*

(Can and do fix $\mu \in \{\mu\}$ defined over a minimal fin. extension $E = E(G, \{\mu\})/\mathbf{Q}_p$; ignore difference between μ and $\{\mu\}$. Set $\breve{E} = E.\breve{\mathbf{Q}}_p$.)
Key Example: Take $G = \mathrm{GL}_n$. Then μ can be identified with a tuple $(k_1 \geq k_2 \geq \cdots \geq k_n) \in \mathbf{Z}^n$. On the other hand, $b \in \mathrm{GL}_n(\breve{\mathbf{Q}}_p) \rightsquigarrow$ rank n F-isocrystal $(\breve{\mathbf{Q}}_p^n, b\sigma) \overset{\mathrm{Dieudonne-Manin}}{\rightsquigarrow}$ slopes $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbf{Q}^n$.
Then $b \in B(G, \mu) \Leftrightarrow \sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} k_i$ with equality for $j = n$.

Conjecture (Rapoport-Viehmann)

Given (G, μ, b) as above, should have a natural tower of smooth rigid analytic spaces $\{\mathcal{M}_K = \mathcal{M}(G, \mu, b)_K\}_K$ over $\breve{E}$, indexed by open compact $K \subset G(\mathbf{Q}_p)$, equipped with commuting actions of $G(\mathbf{Q}_p)$ (on the whole tower) and $G_b(\mathbf{Q}_p) := \{j \in G(\breve{\mathbf{Q}}_p) \mid b\sigma(j)b^{-1} = j\}$ (on each $\mathcal{M}_K$). Moreover,...

- *$\mathcal{M}(G, \mu, b)_K$ should often (but not always!) coincide with the generic fiber of a formal scheme (a Rapoport-Zink space) parametrizing p-divisible groups w./w.o. extra structures. Works for $G = \mathrm{GL}_n$, GSp_{2n}, GSpin_n... Doesn't work if e.g. $G = \mathrm{PGL}_n$, E_7.*
- *Should have a canonical (non-effective) descent datum to E $\rightsquigarrow$ $\lim_{K\to 1} H^*_{\text{ét}}(\mathcal{M}_{K,\overline{E}}, \overline{\mathbf{Q}_\ell})$ has commuting $G(\mathbf{Q}_p)$-, $G_b(\mathbf{Q}_p)$-, and W_E-actions.*
- *(Many more.)*

Local Shimura varieties (cont'd)

Theorem (Scholze ∼late 2014, via Caraiani-Scholze, Fargues & Fargues-Fontaine, Kedlaya-Liu, Scholze-Weinstein)

Local Shimura varieties exist with all expected properties, as the solution to a natural moduli problem determined by the datum (G, μ, b). In fact, they exist in the larger category of diamonds for any μ, and even for more general input data $(G, \{\mu_i\}_{1\leq i\leq n}, b)$, in which case the resulting spaces live over "n copies of $\breve{E}$" in a precise sense.

This last construction is in parallel with the moduli spaces of local/global shtukas defined in equal characteristic p settings (Drinfeld, Varshavsky, Lafforgue(s), ...), which fiber over some finite self-product of copies of the base (which is either $\mathrm{Spec}\,\mathbf{F}_q((t))$ or a projective curve over $\mathbf{F}_q$).
Construction in one sentence: moduli of type μ modifications of vector bundles $\mathcal{E}_1 \dashrightarrow \mathcal{E}_b$ on the Fargues-Fontaine curve.

Expectations for cohomology

Fix (G, μ, b). Want to decompose the cohomology $R\Gamma_c(\mathcal{M}(G, \mu, b), \overline{\mathbf{Q}_\ell})$ of the tower representation-theoretically under the natural $G(\mathbf{Q}_p) \times G_b(\mathbf{Q}_p) \times W_E$-action.

Key issue: This object is huge, because local Shimura varieties are of highly infinite type. One cannot write down equations for them, they are not quasicompact, they usually do not admit explicit coverings by quasicompact pieces, etc. So how to attack their cohomology?

- (Geo) Local Shimura varieties can be realized inside some Hecke stacks which are proper ($\approx$ of finite type) over Bun_G, and their cohomology can be described in terms of Hecke operators acting on sheaves on Bun_G. $\Rightarrow$ Can use techniques from geometric Langlands. (Fargues-Scholze)
- (LTF) Can try to use Lefschetz trace formula techniques. Idea is not new, but technically difficult to implement.
- (Unif) Local Shimura varieties often related to global Shimura varieties by p-adic uniformization $\Rightarrow$ get a direct relationship between their cohomologies. This technique has been heavily exploited for many years.

Recent results on cohomology

Fix (G, μ, b). How to cut the cohomology down?
"Symmetry breaking:" *Fix* a smooth $G_b(\mathbf{Q}_p)$-representation ρ, and consider

$$H^i_c(G, \mu, b)[\rho] = H^i(R\Gamma_c(\mathcal{M}(G, \mu, b), \overline{\mathbf{Q}_\ell}) \otimes_{\mathcal{H}(G_b(\mathbf{Q}_p))} \rho).$$

"Derived ρ-isotypic part of the cohomology." Still a $G(\mathbf{Q}_p) \times W_E$-representation.

Theorem (Fargues-Scholze '21)

If ρ is admissible, then $H^i_c(G, \mu, b)[\rho]$ is an admissible $G(\mathbf{Q}_p)$-representation, and $H^i_c(G, \mu, b)[\rho] = 0$ unless $0 \leq i \leq 2 \dim \mathcal{M}(G, \mu, b)_K$. If ρ is irreducible, each of these cohomologies is of finite length.

Proof via (Geo).
Can we describe these groups in terms of local Langlands correspondence?
Slightly easier: same question for the virtual representation

$$\mathcal{H}(G, \mu, b)[\rho] \overset{def}{=} \sum_{i \geq 0} (-1)^i H^i_c(G, \mu, b)[\rho]$$

.

Recent results on cohomology, cont'd

Fix (G, μ, b) with b basic ($\leftrightsquigarrow$ G_b is an inner form of G).

Theorem (H.-Kaletha-Weinstein '21)

If G and G_b satisfy "refined supercuspidal LLC" and ρ lives in a supercuspidal L-packet $\Pi_\phi(G_b)$, then

$$\mathcal{H}(G, \mu, b)[\rho] = \sum_{\pi \in \Pi_\phi(G)} m_\pi \pi + \mathit{error}$$

for some explicit multiplicities m_π.

Here *error* is a **Q**-linear combination of parabolic inductions. Proof uses (LTF), implemented via ideas from (Geo). Result is unconditional (with $\mathit{error} = 0$) for G any inner form of GL_n.

Recent results on cohomology, cont'd

Fix (G, μ, b) with b basic.

Theorem (H. '21)

If ρ has supercuspidal L-parameter and $\mathcal{M}_K$ related to global Shimura varieties via p-adic uniformization, then $H^i_c(G, \mu, b)[\rho] = 0$ unless $i =$ middle degree.

Proof is an unexpected combination of ideas from (Geo) and (Unif). Would prefer to deduce this from

Conjecture (H.-Scholze)

Local Shimura varieties are Stein spaces.

If you can prove this for the datum with $G = \mathrm{GL}_5$, $\mu = (1, 1, 0, 0, 0)$, b basic, I will donate €1000 to a charity of your choice. (If you are a graduate student, your bank account is a valid charity.)

Future results on cohomology

Beyond basic b? Harris-Viehmann conjecture: For non-basic b, should have

$$\mathcal{H}(G, \mu, b)[\rho] \simeq \pm \oplus_i \mathrm{Ind}_{P_i(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)} \mathcal{H}(M_i, \mu_i, b)[\rho]$$

for some explicit finite list of local Shimura data (M_i, μ_i, b) assoc. with Levi subgroups $M_i \subset G$.

Should follow from a detailed study of geometric Eisenstein series. Hamann '22 in many cases, Hamann-H.-Scholze (work in progress, ready by 80th birthday?...).

Beyond supercuspidal L-packets? Best understood in terms of categorical local Langlands. I hope for a clean answer at least for discrete L-parameters. (Hellmann, Bertoloni Meli, Koshikawa, H., ongoing dialogues...)

Conjecture (H., Koshikawa)

For any (G, μ, b), $R\Gamma_c(\mathcal{M}(G, \mu, b), \overline{\mathbf{Q}_\ell})$ is concentrated in degrees $\geq \langle 2\rho, \mu + \nu_b \rangle$.

Stein property alone would give concentration in degrees $\geq \langle 2\rho, \mu \rangle$, so this is some supervanishing property. Predicted by categorical local Langlands!

The lecturer's ignorance laid bare

Some open problems!

- Vanishing conjectures from earlier.
- Is $\mathcal{M}_\infty$ always a perfectoid space? True for (G, μ, b) of local abelian type.
- Fix (G, μ, b) plus $\mathcal{G}$ a parahoric model of G. Is the v-sheaf integral model $\mathcal{M}^{\mathrm{int}}(\mathcal{G}, \mu, b)$ of $\mathcal{M}(G, \mu, b)_{\mathcal{G}(\mathbf{Z}_p)}$ always representable by a nice formal scheme? Known in (most) local abelian type cases.
- Do local Shimura varieties with "deeper than parahoric" level admit natural integral models? Cf. Bieker's thesis.
- Does cohomology with $\mathbf{F}_p$-coefficients have any good finiteness properties?

Thank you for listening!