Local Shimura varieties and their cohomology

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Shimura varieties - quick overview

- Constructed **axiomatically** from abstract group-theoretic input ("Shimura datum")
- Many (but not all) Shimura varieties can be interpreted as moduli spaces of abelian varieties w. extra structures.
- Cohomology should realize (instances of) **global Langlands correspondence** between automorphic forms and global Galois representations

Shimura varieties

Recall: A **Shimura datum** is (roughly) a pair (G, X) where $G/\mathbf{Q}$ is a connected reductive group and $X \simeq G(\mathbf{R})/K_\infty$ is a Hermitian symmetric domain.

Theorem (Baily-Borel, Shimura, Deligne, Borovoi, Milne)

Given (G, X) as above, and $K \subset G(\mathbf{A}_f)$ a sufficiently small open compact subgroup, the locally symmetric manifold

$$G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K$$

is $\cong \mathrm{Sh}(G, X)_K(\mathbf{C})$ for a certain smooth quasiprojective algebraic variety $\mathrm{Sh}(G, X)_K$ defined over a number field $E = E(G, X)$.

$\rightsquigarrow$ Get a tower $\{\mathrm{Sh}(G, X)_K\}_K$ of algebraic varieties with $G(\mathbf{A}_f)$-action.

Modular curves

Take $(G, X) = (\mathrm{GL}_2, \mathfrak{H}^{\pm} = \mathbf{C} - \mathbf{R})$. Then $E = \mathbf{Q}$ and $Y_K = \mathrm{Sh}(G, X)_K$ is the usual tower of modular curves.
When $K = K(N) = \{g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) | g = 1 \bmod N\}$, $Y_{K(N)}$ is the moduli space of elliptic curves E with a trivialization $(\mathbf{Z}/N\mathbf{Z})^2 \simeq E[N]$.
Note: Cohomology

$$\mathrm{colim}_{K \to \{1\}} H^1_{\mathrm{et}}(Y_{K,\overline{\mathbf{Q}}}, \overline{\mathbf{Q}_\ell})$$

has a natural action of $\Gamma_{\mathbf{Q}} \times \mathrm{GL}_2(\mathbf{A}_f)$.
What information does this action encode?

Modular curves cont'd

Recall: Let $f = \sum_{n \geq 1} a_n q^n \in S_2(N)$ be any normalized weight two cuspidal Hecke eigenform.
$\rightsquigarrow$ Eichler-Shimura: There is a (unique) Galois representation $\rho_f : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}_\ell})$ unramified outside $N\ell$ such that $\mathrm{tr}\rho_f(\mathrm{Fr}_p) = a_p$ for all $p \nmid N\ell$.
Example: If

$$f = q \prod_{n \geq 1} (1 - q^n)(1 - q^{3n})(1 - q^{5n})(1 - q^{15n}) \in S_2(15),$$

then ρ_f is realized in the ℓ-adic Tate module of the elliptic curve $y^2 + xy + y = x^3 + x^2$.

Modular curves cont'd

Theorem (Langlands, Piatetski-Shapiro, Deligne, Brylinski, ...)

As $\Gamma_{\mathbf{Q}} \times \mathrm{GL}_2(\mathbf{A}_f)$-representations, there is an isomorphism

$$\mathrm{colim}_{K \to \{1\}} H^1_{\mathrm{et}}(Y_{K,\overline{\mathbf{Q}}}, \overline{\mathbf{Q}_\ell}) = \oplus_f \rho_f \otimes \bigotimes_p{}' \pi_{f,p} + \dots,$$

where $\pi_{f,p} \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(\mathrm{GL}_2(\mathbf{Q}_p))$ matches $\rho_f|W_{\mathbf{Q}_p}$ under the **local Langlands correspondence***.*

Here "$\dots$" is some "boring" part (all $\Gamma_{\mathbf{Q}}$-irreps occurring there are one-dimensional).

Similar (largely conjectural) story for more general Shimura varieties (Langlands, Arthur, Kottwitz, ...).

Local Shimura varieties - quick overview

- Constructed **axiomatically** from abstract group-theoretic input ("local Shimura datum")
- Many (but not all) local Shimura varieties can be interpreted as moduli spaces of p-divisible groups w. extra structures.
- Cohomology should realize (instances of) **local Langlands correspondence** and Jacquet-Langlands correspondence

Local Shimura varieties

Fix a prime p, and set $\breve{\mathbf{Q}}_p = \widehat{\mathbf{Q}_p^{\mathrm{unr}}} \circlearrowleft \sigma =$ lift of $x \mapsto x^p$.

Definition (Rapoport-Viehmann)

A **local Shimura datum** *is a triple* $(G, \{\mu\}, b)$ *where* $G/\mathbf{Q}_p$ *is a connected reductive group,* $\{\mu\}$ *is a conjugacy class of minuscule cocharacters* $\mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$*, and* $b \in G(\breve{\mathbf{Q}}_p)$ *is an element such that* $b \in B(G, \mu)$*.*

(Can and do fix $\mu \in \{\mu\}$ defined over a minimal fin. extension $E = E(G, \{\mu\})/\mathbf{Q}_p$; ignore difference between μ and $\{\mu\}$. Set $\breve{E} = E.\breve{\mathbf{Q}}_p$.)

Example (Key example)

Take $G = \mathrm{GL}_n$, $\mu(z) = \mathrm{diag}(\underbrace{z, \dots, z}_{d}, 1, \dots, 1)$ *for some* $1 \leq d < n$,

$b \in \mathrm{GL}_n(\mathbf{Q}_p)$ *any element with characteristic polynomial* $X^n - p^d$.

Conjecture (Rapoport-Viehmann)

Given (G, μ, b) as above, should have a natural tower of smooth rigid analytic spaces $\{\mathcal{M}_K = \mathcal{M}(G, \mu, b)_K\}_K$ over $\breve{E}$, indexed by open compact $K \subset G(\mathbf{Q}_p)$, equipped with commuting actions of $G(\mathbf{Q}_p)$ (on the whole tower) and $G_b(\mathbf{Q}_p) := \{j \in G(\breve{\mathbf{Q}}_p) \mid b\sigma(j)b^{-1} = j\}$ (on each $\mathcal{M}_K$). Moreover:

- *$\mathcal{M}(G, \mu, b)_K$ should often (but not always!) coincide with the generic fiber of a formal scheme (a* **Rapoport-Zink space***) parametrizing p-divisible groups w./w.o. extra structures. Works for $G = \mathrm{GL}_n$, GSp_{2n}, GSpin_n... Doesn't work if e.g. $G = \mathrm{PGL}_n$, E_7.*
- *Should have a canonical (non-effective) descent datum to $E \rightsquigarrow$ $\lim_{K\to 1} H^*_{\text{ét}}(\mathcal{M}_{K,\overline{E}}, \overline{\mathbf{Q}_\ell})$ has commuting $G(\mathbf{Q}_p)$-, $G_b(\mathbf{Q}_p)$-, and W_E-actions.*
- *(Many more.)*

Local Shimura varieties (cont'd)

Theorem (Scholze ∼late 2014, via Caraiani-Scholze, Fargues & Fargues-Fontaine, Kedlaya-Liu, Scholze-Weinstein)

Local Shimura varieties exist with all expected properties, as the solution to a natural moduli problem determined by the datum (G, μ, b). In fact, they exist in the larger category of diamonds for any μ, and even for more general input data $(G, \{\mu_i\}_{1\leq i\leq n}, b)$, in which case the resulting spaces live over "n copies of $\mathrm{Spa}(\breve{E})$" in a precise sense.

This last construction is in parallel with the moduli spaces of local/global shtukas defined in equal characteristic p settings (Drinfeld, Varshavsky, Lafforgue, ...), which fiber over some finite self-product of copies of the base (which is either $\mathrm{Spec}\,\mathbf{F}_q((t))$ or a projective curve over $\mathbf{F}_q$).
$\rightsquigarrow$ Fargues-Scholze: Combining this with ideas of V. Lafforgue, get a construction associating a semisimple L-parameter with any irreducible smooth representation of any $G(\mathbf{Q}_p)$, giving a candidate for the local Langlands correspondence.

Rough idea of construction

Fix (G, μ, b) as before, with E the field of def. of μ. Let $C/\breve{E}$ be any complete algebraically closed extension. Let's unwind $\mathcal{M}_\infty(C) \overset{def}{=} \lim_{\overleftarrow{K}} \mathcal{M}_K(C)$.

Fargues-Fontaine: Any such $C \rightsquigarrow$ a certain magic **curve** $X = X_{C^\flat}$ (a connected reg. Noeth. 1-dim'l scheme over $\mathrm{Spec}\,\mathbf{Q}_p$) together with a distinguished closed point $\infty \in X$ such that $\kappa(\infty) \cong C$.

Fargues: Any $b \in G(\breve{\mathbf{Q}}_p) \rightsquigarrow$ a G-bundle $\mathcal{E}_b$ over X, with $\mathrm{Aut}(\mathcal{E}_b) \supset G_b(\mathbf{Q}_p)$ (and sometimes with equality; in particular, $\mathrm{Aut}(\mathcal{E}_1) \cong G(\mathbf{Q}_p)$).

Idea

$\mathcal{M}_\infty(C) = \{$*modifications* $\mathcal{E}_1 \to \mathcal{E}_b$ *supported at* ∞, *of type* $\mu\}$, *with the natural group actions on the RHS.*

Note: In the case of RZ spaces, Scholze-Weinstein had already reinterpreted their generic fibers in this language of modified vector bundles in 2012.

Expectations for cohomology

Fix (G, μ, b). Want to decompose the cohomology $R\Gamma_c(\mathcal{M}(G, \mu, b)_{\infty,\overline{E}}, \overline{\mathbf{Q}_\ell})$ representation-theoretically under the natural $G(\mathbf{Q}_p) \times G_b(\mathbf{Q}_p) \times W_E$-action.
First problem: This object is too big to be studied meaningfully. (Too big = not admissible as a $G(\mathbf{Q}_p) \times G_b(\mathbf{Q}_p)$-representation.)
Solution: Given (G, μ, b), *fix* an admissible smooth $G_b(\mathbf{Q}_p)$-representation ρ, and consider

$$H^i_c(G, \mu, b)[\rho] = H^i(R\Gamma_c(\mathcal{M}_{\infty,\overline{E}}, \overline{\mathbf{Q}_\ell}(\tfrac{\dim}{2})) \otimes^{\mathbf{L}}_{\mathcal{H}(G_b(\mathbf{Q}_p))} \rho).$$

"Derived ρ-isotypic part of the cohomology." Still a representation of $G(\mathbf{Q}_p) \times W_E$.

Theorem (Fargues-Scholze)

If ρ is admissible, then $H^i_c(G, \mu, b)[\rho]$ is an admissible $G(\mathbf{Q}_p)$-representation, and $H^i_c(G, \mu, b)[\rho] = 0$ unless $0 \leq i \leq 2\dim \mathcal{M}(G, \mu, b)_K$.

Can we describe these groups?

Expectations for cohomology, cont'd

Assume b is **basic** ($\leftrightsquigarrow$ G_b is an inner form of G). Two key conjectures (stated roughly):

- **Kottwitz conjecture ('90s)**: Let ρ be an irreducible smooth rep. of $G_b(\mathbf{Q}_p)$ which lies in a **supercuspidal L-packet**. Then

  $$\rho \rightsquigarrow \mathcal{H}(G,\mu,b)[\rho] \overset{def}{=} \sum_{i\geq 0}(-1)^i H^i_c(G,\mu,b)[\rho]$$

  can be described explicitly in terms of the local Langlands correspondence.
- **Vanishing conjecture (Ito '12, Scholze '17)**: Let ρ be an irreducible smooth rep. of $G_b(\mathbf{Q}_p)$ which lies in a supercuspidal L-packet. Then $H^i_c(G,\mu,b)[\rho]=0$ for all $i \neq \dim \mathcal{M}(G,\mu,b)_K$. $\rightsquigarrow$ **No cancellation** in the sum defining $\mathcal{H}(G,\mu,b)[\rho]$.

Today: First general results on these conjectures.

Key example of both conjectures

Take $G = \mathrm{GL}_n$, $\mu(z) = \mathrm{diag}(\underbrace{z, \ldots, z}_{d}, 1, \ldots, 1)$ for some $1 \leq d < n$, b any element with characteristic polynomial $X^n - p^d$. Then $G_b(\mathbf{Q}_p) = A^\times$, $A/\mathbf{Q}_p$ central simple algebra of rank n^2 and Hasse invariant d/n.
Jacquet-Langlands, Deligne-Kazhdan-Vignéras, Rogawski: Any supercuspidal representation π of $\mathrm{GL}_n(\mathbf{Q}_p)$ has a canonical transfer to a representation ρ_π of $A^\times$.

Conjecture

In this setup, expect the following:

- *(Kottwitz)* $\mathcal{H}(G, \mu, b)[\rho_\pi] \simeq \pm\pi \boxtimes \mathrm{sym}^d(\varphi_\pi)$.
- *(Vanishing)* $H^i_c(G, \mu, b)[\rho_\pi] = 0$ *for all* $i \neq d(n-d)$.

Here $\varphi_\pi : W_{\mathbf{Q}_p} \to \mathrm{GL}_n(\overline{\mathbf{Q}_\ell})$ is the L-parameter associated with π by the local Langlands correspondence.
Both conjectures proved for $d = 1$ by Harris-Taylor in their work on local Langlands. Kottwitz conjecture proved for arbitrary d by Fargues and Shin.

Interlude: Local Langlands correspondence

Setup: Fix $G/\mathbf{Q}_p$ a connected reductive group, C an algebraically closed field of characteristic zero.
Expectation: There should be a "natural" finite-to-one map

$$\mathrm{Irr}_C(G(\mathbf{Q}_p)) \to L-\mathrm{parameters}\,\varphi : W_{\mathbf{Q}_p} \to {}^L G(C)$$

with many good properties. Write $\Pi_\varphi(G)$ for the fiber of this map over a given φ.
The precise structure of the fibers should be governed by the group $S_\varphi = \mathrm{Cent}_{\hat{G}}(\varphi)$ (or variants thereof).

Local Langlands correspondence cont'd

The group S_φ is a (possibly disconnected) reductive group, containing $Z(\hat{G})^\Gamma$ as a central subgroup.
Vague idea, redux: The structure of the packet $\Pi_\varphi(G)$ should be governed by the algebraic representations of S_φ.
Better idea (Vogan): Should try to parametrize **all** the packets $\Pi_\varphi(H)$ as H varies over inner forms of G.
Our next goal is to make this more precise.

Isocrystal local Langlands correspondence

Set $B(G) = G(\breve{\mathbf{Q}}_p)/(b \sim gb\sigma(g)^{-1})$. Kottwitz's set of isocrystals with G-structure.
This comes with a natural subset $B(G)_{\mathrm{bas}}$ of basic elements, and there is a natural bijection

$$\kappa : B(G)_{\mathrm{bas}} \to X^*(Z(\hat{G})^{\Gamma}).$$

For any $b \in B(G)_{\mathrm{bas}}$, get an inner form G_b as before.
Refined local Langlands conjecture, "isocrystal form" (Kottwitz, Kaletha). If G is quasisplit and φ is **supercuspidal**, then for every $b \in B(G)_{\mathrm{bas}}$, should have a natural bijection

$$\iota : \Pi_{\varphi}(G_b) \to \mathrm{Irr}(S_{\varphi}, \kappa(b))$$

satisfying various properties. Here $\mathrm{Irr}(S_{\varphi}, \kappa(b))$ denotes the set of irreducible **algebraic** representations of S_{φ} whose restriction to $Z(\hat{G})^{\Gamma}$ is $\kappa(b)$-isotypic.

Example one: $G = \mathrm{GL}_n$

Let's take $G = \mathrm{GL}_n$. Then $\hat{G} = \mathrm{GL}_n$, and φ is supercuspidal iff it is irreducible. In this situation, get $S_\varphi = Z(\hat{G})^\Gamma = \mathbf{G}_m$. Hence

$$B(G)_{\mathrm{bas}} \cong X^*(Z(\hat{G})^\Gamma) \cong \mathbf{Z}.$$

Concretely, if $\kappa(b) = d$, then $G_b(\mathbf{Q}_p) = A^\times$, $A/\mathbf{Q}_p$ the central simple algebra of rank n^2 and Hasse invariant d/n mod 1. (So G_b depends only on d mod n.) Refined LLC then boils down to the expectation that each $\Pi_\varphi(G_b)$ is a singleton. This is known; moreover, the elements of these packets are "Jacquet-Langlands transfers" of each other (Jacquet-Langlands, Deligne-Kazhdan-Vignéras, Rogawski).

Example two: $G = \mathrm{GSp}_4$

Now take $G = \mathrm{GSp}_4$. Then $\hat{G} = \mathrm{GSp}_4$ (accidentally!) and $Z(\hat{G})^\Gamma = \mathbf{G}_m$, so again

$$B(G)_{\mathrm{bas}} \cong X^*(Z(\hat{G})^\Gamma) \cong \mathbf{Z}.$$

Concretely, if $\kappa(b) = d$, then G_b depends only on the parity of d: get GSp_4 if d is even, and $J = \mathrm{GU}_2(D)$ (unique inner form!) if d is odd.
But now it gets interesting: if φ is supercuspidal, can look at $\mathrm{std} \circ \varphi : W_{\mathbf{Q}_p} \to \mathrm{GL}_4$. It turns out that one of two things will happen:

- $\mathrm{std} \circ \varphi$ is irreducible. Then $S_\varphi = \mathbf{G}_m$, and $\Pi_\varphi(G)$ and $\Pi_\varphi(J)$ are singletons.
- $\mathrm{std} \circ \varphi \simeq \varphi_1 \oplus \varphi_2$ with φ_i two-dimensional, distinct and irreducible (and with equal determinants). Then S_φ is disconnected with neutral component $\mathbf{G}_m$ of index two. Both $\Pi_\varphi(G)$ and $\Pi_\varphi(J)$ contain **two** elements.

Kottwitz conjecture: setup

Fix a local Shimura datum (G, μ, b) as before, with b **basic**. For any $\rho \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G_b(\mathbf{Q}_p))$, consider the virtual $G(\mathbf{Q}_p) \times W_E$ representation

$$\mathcal{H}(G, \mu, b)[\rho] \overset{def}{=} \sum_{i \geq 0} (-1)^i H^i_c(G, \mu, b)[\rho].$$

How to describe this explicitly?
First observation: From μ, get an algebraic representation $r_\mu : {}^L G \to \mathrm{GL}_m$.
For any L-parameter φ, the composition $r_\mu \circ \varphi|_{W_E}$ is a representation of $W_E \times S_\varphi$ (think about def. of S_φ).
Second observation: From refined LLC, given any $\pi \in \Pi_\varphi(G)$ and $\rho \in \Pi_\varphi(G_b)$, can extract an algebraic representation $\delta_{\pi,\rho}$ of S_φ which measures the "relative positions" of π and ρ. For G quasisplit this is given by $\iota(\pi) \otimes \iota(\rho)^\vee$, but there is a general recipe.

Kottwitz conjecture: statement

Conjecture (Kottwitz)

Fix a basic local Shimura datum (G, μ, b) and a supercuspidal L-parameter $\varphi : W_{\mathbf{Q}_p} \to {}^L G(\overline{\mathbf{Q}}_\ell)$. Let $\rho \in \Pi_\varphi(G_b)$ be any element. Then

$$\mathcal{H}(G, \mu, b)[\rho] = (-1)^{\langle 2\rho_G, \mu \rangle} \sum_{\pi \in \Pi_\varphi(G)} \pi \boxtimes \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu \circ \varphi|_{W_E})$$

as virtual $G(\mathbf{Q}_p) \times W_E$ representations.

Weakened Kottwitz conjecture: statement

If we ignore the Weil group action on the cohomology, the statement becomes simpler.

Conjecture (Kottwitz)

Fix a basic local Shimura datum (G, μ, b) *and a supercuspidal L-parameter* $\varphi : W_{\mathbf{Q}_p} \to {}^L G(\overline{\mathbf{Q}_\ell})$. *Let* $\rho \in \Pi_\varphi(G_b)$ *be any element. Then*

$$\mathcal{H}(G, \mu, b)[\rho] = (-1)^{\langle 2\rho_G, \mu \rangle} \sum_{\pi \in \Pi_\varphi(G)} [\dim \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu)]\pi$$

as virtual $G(\mathbf{Q}_p)$ *representations.*

The full Kottwitz conjecture is only known in a handful of situations:

- The Lubin-Tate/Drinfeld towers (Harris-Taylor)
- Local Shimura varieties of "unramified EL" type (Fargues, Shin)
- Some unitary local Shimura varieties of "unramified PEL" type (Bertoloni-Meli–Nguyen)

All of these works rely crucially on **global** methods (comparison with cohomology of global Shimura varieties). For the weakened Kottwitz conjecture, purely **local** methods can be brought to bear.

Kottwitz conjecture: main result

Theorem (H.-Kaletha-Weinstein '21)

Fix a basic local Shimura datum (G, μ, b) and a supercuspidal L-parameter $\varphi : W_{\mathbf{Q}_p} \to {}^L G(\overline{\mathbf{Q}_\ell})$. Let $\rho \in \Pi_\varphi(G_b)$ be any element. Then

$$\mathcal{H}(G, \mu, b)[\rho] = (-1)^{\langle 2\rho_G, \mu \rangle} \sum_{\pi \in \Pi_\varphi(G)} [\dim \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu)]\pi + \mathrm{err}$$

as virtual $G(F)$ representations, where err *is a* **non-elliptic** *virtual representation.*

Recall that a regular semisimple element $g \in G(\mathbf{Q}_p)$ is elliptic if the torus $\mathrm{Cent}_G(g)/Z(G)$ is anisotropic. A virtual representation is non-elliptic if its Harish-Chandra character vanishes on all elliptic elements.
Here we assume a certain refined form of LLC due to Kaletha. **Assuming** a certain compatibility between this and the Fargues-Scholze construction of L-parameters, we can also show that $\mathrm{err} = 0$ as expected.

Some commentary

- We actually prove a more general result, for general moduli of local shtukas (no restriction on μ). Here one essentially needs to take **intersection cohomology** in the definition of $\mathcal{H}(G, \mu, b)[\rho]$.
- We also prove a result for more general **discrete** L-parameters. Here the error term can definitely be nonzero!
- The key idea, of using the Lefschetz trace formula in some form, goes back to a literal dream of Harris in the early '90s.
- There are previous results of Faltings (for GL_2), Strauch (for GL_n), and Mieda (for GSp_4), also using trace formula methods. However, our implementation of this idea is totally different, and uses the full force of modern p-adic geometry a la Scholze.

Comments on the proof

Key idea: Define an **explicit** transfer operator $T_{b,\mu}^{G_b \to G}$ from conjugation-invariant functions on $G_b(\mathbf{Q}_p)_{\mathrm{ell}}$ towards similar functions on $G(\mathbf{Q}_p)_{\mathrm{ell}}$, and then prove that $T_{b,\mu}^{G_b \to G}$ encodes both the LHS and RHS of weakened Kottwitz conjecture. Connection with RHS: direct calculation using endoscopic character identities in the LLC. Connection with LHS: Lefschetz trace formula plus a subtle continuity argument.
Some essential ingredients:

- Lu-Zheng's new point of view on the Lefschetz trace formula, via the **symmetric monoidal 2-category of cohomological correspondences**.
- The monumental work of Fargues-Scholze: $\mathcal{H}(G, \mu, b)[\rho]$ in terms of Hecke operators on the stack Bun_G, ULA sheaves in p-adic geometry, geometric Satake for the B_{dR}-affine Grassmannian,...
- Recent work of Varshavsky on local terms in the (classical) Lefschetz trace formula.

Vanishing conjecture: main result

Theorem (H. 2020)

Fix (G, μ, b) as before, with b basic. Let ρ be an irreducible admissible representation of $G_b(\mathbf{Q}_p)$. Suppose that

1. *The local Shimura varieties $\mathcal{M}(G, \mu, b)_K$ are related to some global Shimura varieties via p-adic uniformization.*
2. *The L-parameter $\varphi_\rho : W_{\mathbf{Q}_p} \to {}^L G(\overline{\mathbf{Q}_\ell})$ associated with ρ by Fargues-Scholze is supercuspidal.*

Then $H^i_c(G, \mu, b)[\rho] = 0$ for all $i \neq \dim \mathcal{M}(G, \mu, b)_K$.

Some commentary

- The theorem is new in essentially all cases beyond the classical Lubin-Tate and Drinfeld towers.
- Condition 1. is crucial for our argument, but the theorem should hold without it. Can check this condition for most (but not all!) local Shimura data of classical interest. Probably true for every local Shimura datum.
- Condition 2. seems to be **necessary** for the conclusion of the theorem to hold!
- Condition 2. is not easy to check in practice: related to the difficult problem of comparing the FS construction with other constructions of local Langlands correspondence. *However*, for inner forms of GL_n, Condition 2. can be unwound to something more explicit. $\rightsquigarrow$ Get a proof of the vanishing conjecture in the "key example".

Sketch of the argument

We know that global Shimura varieties satisfy vanishing theorems like this.
Want to make a **global-to-local** argument.
Use geometry of the uniformization isomorphism to build a map

$$\Theta_\rho : R\Gamma_c(G, \mu, b)[\rho] \to R\Gamma(\mathrm{Sh}(\mathbf{G}, X), \mathcal{L}_\xi)$$

where $(\mathbf{G}, X)$ is a well-chosen global Shimura datum and $\mathcal{L}_\xi$ is an algebraic local system. This map depends (only) on a globalization of ρ.
Li-Schwermer: If ξ regular, RHS vanishes in degrees $>$ dim (and in all degrees $\neq$ dim if the global Shimura varieties are compact).
Key new idea: Use condition 2. to show that Θ_ρ is a split injection. Argue at the level of sheaves on the flag variety $\mathcal{F}\ell_{G,\mu}$, using the geometry of the Hodge-Tate period map together with the Fargues-Scholze machinery.

Thank you for listening!