

# Recent developments in étale cohomology

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# Introduction

Recall: Étale cohomology is **the** correct analogue of singular cohomology in algebraic geometry. Invented to prove the Weil conjectures, but now of central importance for many different reasons.

Key points: For any reasonable scheme, have a category  $D_c^b(X, \mathbf{Q}_\ell)$  which satisfies a six operations formalism; Lefschetz trace formula; theory of weights for varieties over finite fields.

## A misconception

In talking to graduate students, I've noticed a common misconception that étale cohomology is a “dead” / “static” / “ancient” subject, with nothing left to be done.



Figure: The creation of étale cohomology

It is true that many mathematicians can profitably use étale cohomology as a black box, never looking beyond Freitag-Kiehl or Milne. However, it is **not** a dead subject!

## A brief and subjective history

- First mentioned (without name) in Grothendieck's 1958 ICM report.
- Key foundations laid in SGA4 (M. Artin, Deligne, Grothendieck, Verdier): Basic definitions, construction of the six operations, smooth and proper base change, Poincaré duality, comparison with singular cohomology for complex varieties, affine vanishing, (conditional) finiteness and biduality theorems in char. 0. All with torsion coefficients.
- SGA 4 1/2 (Deligne): Unconditional finiteness and biduality theorems for schemes of finite type over regular bases of dimension  $\leq 1$ , complete proof of the Lefschetz trace formula for Frobenius. "This report should allow the user to forget about SGA 5, which can be considered as a series of digressions, some very interesting."
- Deligne's Weil I paper: Unconditional definition of  $D_c^b(X, \mathbf{Q}_\ell)$  w. six operations formalism for  $X$  a variety over any *finite or alg. closed* field. Enough to prove the Weil conjectures.

End of the initial period of development.

## History cont'd

Some key later developments:

- Beilinson-Bernstein-Deligne-Gabber '83: Perverse sheaves, t-structures, decomposition theorem, purity for intersection cohomology.
- Thomason '84, Gabber '94: Proof of Grothendieck's absolute purity conjecture, by infusion of ideas from  $K$ -theory.
- Ekedahl '90, Bhatt-Scholze '15: Proper development of the formalism with  $\mathbf{Q}_\ell$ -coefficients.
- Gabber, late '00s: New proof of absolute purity, optimal finiteness and biduality theorems for excellent schemes. Very sophisticated arguments.
- Laszlo-Olsson '05-'06, Liu-Zheng, '12: Flexible six operations formalism for sheaves on Artin stacks.

So much for being a dead subject. Is there still anything left to be done?

# Étale cohomology of rigid spaces

Natural question: Is there a good étale cohomology formalism for rigid analytic spaces? The answer should obviously be “yes”, but setting up the formalism presents some new challenges.

Foundations laid by Berkovich and Huber in the '90s: Construction of the six operations, smooth and proper base change, Poincaré duality, some comparison and finiteness theorems.

MISSING: A natural class of sheaves (with torsion or  $\mathbf{Z}_\ell$ -coefficients) stable under the six operations, admitting a perverse t-structure, satisfying affine vanishing, etc.

# Zariski-constructible sheaves

Let  $X$  be a rigid space,  $\Lambda$  a Noetherian ring. Key new definition:

## Definition

- An étale sheaf  $\mathcal{F} \in \text{Sh}(X, \Lambda)$  is **Zariski-constructible** if there is a locally finite stratification  $X = \cup_{i \in I} X_i$  for some Zariski locally closed subsets  $X_i \subset X$  such that  $\mathcal{F}|_{X_i}$  is finite locally constant for all  $i \in I$ .
- A complex  $A \in D(X, \Lambda)$  is Zariski-constructible if all cohomology sheaves are.

(In first part, can replace “locally finite” with “finite” unless  $\dim X = \infty$ .)

Key motivation: If  $X$  is an algebraic variety, the natural pullback  $\text{Sh}(X, \Lambda) \rightarrow \text{Sh}(X^{\text{an}}, \Lambda)$  carries constructible sheaves on  $X$  to Zariski-constructible sheaves on  $X^{\text{an}}$ .

However, NO interesting properties of these sheaves are obvious!

## Zariski-constructible sheaves cont'd

Until further notice:  $K$  a nonarchimedean field of **characteristic zero** and residue characteristic  $p \geq 0$ ,  $\Lambda = \mathbf{Z}/n\mathbf{Z}$ .

### Main theorem (Bhatt-H.)

*On rigid spaces  $X/K$ ,  $D_{zc}^{(b)}(X, \Lambda)$  is stable under the operations  $f^*$ ,  $Rf_*$  for proper  $f$ ,  $Rf_!$  and  $Rf_*$  on lisse sheaves for Zariski-compactifiable  $f$ ,  $Rf^!$  if  $p \nmid n$  or  $f$  is finite,  $\otimes$  and  $R\mathcal{H}om$  (under a finite tor-dimension assumption), and Verdier duality. Moreover,  $D_{zc}^{(b)}(X, \Lambda)$  carries a natural perverse  $t$ -structure whose abelian heart satisfies all expected properties. All of these statements are compatible with their schematic counterparts under analytification, and with extensions of the base field. Similar results hold with  $\mathbf{Z}_\ell$ -coefficients.*

Proof requires many auxiliary ingredients, possibly of independent interest.



First main ingredient:

### Algebraization theorem (Bhatt-H.)

*If  $A$  is a  $K$ -affinoid ring and  $\mathcal{X}$  is a scheme of finite type over  $\mathrm{Spec}A$ , the natural functor  $(-)^{\mathrm{an}} : D_c^b(\mathcal{X}, \Lambda) \rightarrow D_{z\text{-c}}^b(\mathcal{X}^{\mathrm{an}}, \Lambda)$  is fully faithful. If  $\mathcal{X} \rightarrow \mathrm{Spec}A$  is proper, this functor is an equivalence of categories.*

Most useful case:  $\mathcal{X} = \mathrm{Spec}A$ . This is a key input into the proof of the following result.

### Locality theorem (Bhatt-H.)

*Zariski-constructibility is an étale-local property.*

Upshot: In the proof of the main theorem, all claims can be checked locally in the analytic topology.

Proof of the algebraization theorem relies, in turn, on two separate ingredients.

### Comparison theorems (H.)

*Let  $A$  be a  $K$ -affinoid ring.*

*1) Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be any finite type map of locally finite type  $\text{Spec}A$ -schemes. Then for any  $F \in D_c^+(\mathcal{X}, \Lambda)$ , the natural map  $(Rf_*F)^{\text{an}} \rightarrow Rf_*^{\text{an}}F^{\text{an}}$  is an isomorphism.*

*2) For any  $F \in D^+(\text{Spec}A, \Lambda)$ , the natural map  $R\Gamma(\text{Spec}A, F) \rightarrow R\Gamma(\text{Spa}A, F^{\text{an}})$  is an isomorphism.*

1) proved by Berkovich and Huber if  $f$  proper or  $\mathcal{Y} = \text{Spec}K$ . General case uses Nagata compactification, Gabber's finiteness theorems, and resolution of singularities (many times) to reduce to a purity theorem proved by Huber.

2) proved by Huber if  $F$  has constant cohomology sheaves. General case follows from a trick.

1)+2) immediately give full faithfulness in the algebraization theorem. Essential surjectivity can then be checked on hearts. After stratifying, reduce to proving that lisse sheaves algebraize.

We now use the second ingredient:

### Extension theorem (H.)

*Let  $X$  be a normal rigid space,  $U \subset X$  the complement of a nowhere-dense closed analytic subset. Then any finite étale map  $V \rightarrow U$  extends uniquely to a finite map  $V' \rightarrow X$ .*

Essential case of  $X$  smooth and  $X - U$  an snc divisor treated by Lütkebohmert. General case can be deduced by resolution of singularities.  $\rightsquigarrow$  If  $\mathcal{X}$  is a scheme of finite type over  $\text{Spec}A$  - e.g. an open subscheme of  $\text{Spec}A$  - finite étale covers of  $\mathcal{X}^{\text{an}}$  are uniquely algebraizable.  $\rightsquigarrow$  Same statement for lcc sheaves of  $\Lambda$ -modules. Algebraization theorem follows. Also get the following very useful result:

### Generation lemma (Bhatt-H.)

*If  $X$  is a quasicompact rigid space,  $D_{\text{zc}}^b(X, \Lambda)$  is the thick triangulated subcategory of  $D(X, \Lambda)$  generated by  $f_*M$  for all finite maps  $f : X' \rightarrow X$  and finite  $\Lambda$ -modules  $M$ .*

Upshot: In the proof of the main theorem, we can (usually) reduce to checking claims in the special case of constant sheaves.

The proof of the main theorem combines all these materials with a few additional ingredients:

- 1 Gabber's results on étale cohomology of excellent schemes: finiteness theorems, regular base change, existence and good properties of dualizing complexes, etc.
- 2 The known preservation of lisse complexes under  $Rf_*$  for proper smooth  $f$ . (Huber for  $p \nmid n$ , Gabber/Scholze-Weinstein for  $n = p^t$ .)
- 3 A key new "generic smoothness" theorem.

All claims now follow by judiciously combining everything.

Sample argument i.: stability under  $Rf_*$  for proper  $f$  follows from the locality theorem + generation lemma + 2) + 3).

Sample argument ii.: compatibility of six operations with extension of base field  $L/K$  follows from locality theorem + algebraization theorem + regular base change and consequences thereof + regularity of  $A \rightarrow A \widehat{\otimes}_K L$ .

# Open problems

One key consequence of all this: now have a good theory of IC sheaves on (char. 0) rigid spaces. In particular, for any proper rigid space  $X/K$ , get intersection cohomology groups  $IH^*(X_{\overline{K}}, \mathbf{Q}_\ell)$ . Finite-dimensional; Poincaré duality holds for  $\ell \neq p$ .

- Poincaré duality for  $\ell = p$ ?
- Conjecture: If  $K/\mathbf{Q}_p$  finite and  $\ell \neq p$ , Frobenius eigenvalues on  $IH^*(X_{\overline{K}}, \mathbf{Q}_\ell)$  are  $q$ -Weil numbers of weights  $\in [0, 2 \dim X]$ .
- Conjecture: If  $K/\mathbf{Q}_p$  finite,  $IH^*(X_{\overline{K}}, \mathbf{Q}_p)$  is a de Rham  $G_K$ -representation.
- Conjecture: Some form of the decomposition theorem holds for **projective** morphisms of rigid spaces.

Lastly: How much of this theory extends to the situation where  $\text{char} K > 0$ ?? Major difficulties: resolution of singularities unknown, extension theorem and essential surjectivity part of the algebraization theorem **both fail**. New ideas needed.

## Reminder on perverse sheaves

Let  $X$  be an algebraic variety over a field  $K$ ,  $\ell$  a prime invertible in  $K$ . Recall:  $A \in D_c^b(X, \mathbf{Q}_\ell)$  is **perverse** if  $\dim \operatorname{supp} \mathcal{H}^n(A) \leq -n$  for all  $n \in \mathbf{Z}$ , and likewise with  $\mathbf{D}A$  in place of  $A$ .

First (resp. second) condition defines the left half  ${}^p D^{\leq 0}$  (resp. right half  ${}^p D^{\geq 0}$ ) of a t-structure on  $D_c^b(X, \mathbf{Q}_\ell)$ .

If  $X$  is smooth and  $F$  is lisse, then  $F[\dim X]$  is perverse. Generally, perverse sheaves are the “right” generalization of lisse sheaves, with excellent categorical properties. They are enormously useful in geometric representation theory, and are fascinating in their own right.

In 2018, I began (publicly) asking: is there a “relative” / “in families” version of perverse sheaves?

# Why this question isn't so unreasonable

There are **hints** towards a positive answer in the geometric Langlands literature. For instance, one can extract from various geometric Langlands papers the following theorem:

## Theorem

*Let  $f : X \rightarrow S$  be any morphism of varieties with  $S$  smooth (of pure dimension  $d$ ). If  $A \in D_c^b(X, \mathbf{Q}_\ell)$  is perverse and  $f$ -ULA, then  $(A|_{X_s})[-d]$  is perverse for all points  $s \rightarrow S$ . More generally, for any  $g : T \rightarrow S$  with  $T$  smooth,  $f^*A[\dim T - \dim S]$  is perverse and  $f_T$ -ULA.*

This is good enough for constructing the fusion product in geometric Satake. Suggests that (for a smooth base  $S$ ) one should consider the category  $\text{Perv}^{\text{ULA}}(X/S)$  of objects  $A \in D_c^b(X, \mathbf{Q}_\ell)$  which are  $f$ -ULA and with  $A[\dim S]$  perverse. By previous theorem, this is stable under any base change and gives usual perverse sheaves after pullback to a point. However, the ULA condition is very restrictive. Moreover, it is not clear whether  $\text{Perv}^{\text{ULA}}(X/S)$  is an abelian category.

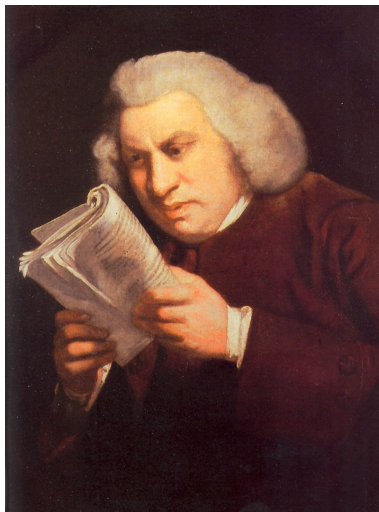
The hoped-for abelian property of  $\text{Perv}^{\text{ULA}}(X/S)$  basically reduces to:

### Theorem (Gaitsgory)

*Let  $f : X \rightarrow S$  be any morphism of varieties with  $S$  smooth. If  $A \in D_c^b(X, \mathbf{Q}_\ell)$  is  $f$ -ULA, then all perverse cohomologies  ${}^p\mathcal{H}^n(A)$  are  $f$ -ULA, and moreover any perverse subquotient of any  ${}^p\mathcal{H}^n(A)$  is  $f$ -ULA.*

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This theorem is... not well-documented. Moreover, one expert's reaction when I told it to them is that it is “obviously false”, because the perverse cohomologies of  $A$  have nothing to do with  $f$ !

What is going on??

# Relative perversity

## Main theorem (H.-Scholze)

*Let  $f : X \rightarrow S$  be a finite type map of reasonable schemes,  $\ell$  a prime invertible on  $S$ . There is a natural t-structure  $({}^{p/S}D^{\leq 0}, {}^{p/S}D^{\geq 0})$  on  $D_c^b(X, \mathbf{Q}_\ell)$  such that  $A$  lies in  ${}^{p/S}D^{\leq 0}$  resp.  ${}^{p/S}D^{\geq 0}$  iff  $A|_{X_s}$  lies in  ${}^pD^{\leq 0}$  resp.  ${}^pD^{\geq 0}$  for all geometric points  $s \rightarrow S$ . Moreover, the truncation functors  ${}^{p/S}\tau^{\leq n}, {}^{p/S}\tau^{\geq n}$  preserve  $f$ -ULA objects.*

The heart of this t-structure is exactly the objects in  $D_c^b(X, \mathbf{Q}_\ell)$  which restrict to a perverse sheaf on each geometric fiber of  $f$ . In particular, objects of this type naturally form an abelian category  $\text{Perv}(X/S)$ . No idea how to see this directly!

We also show that for regular  $S$ , perverse and relative perverse t-structures agree up to (explicit) shift on ULA objects.  $\rightsquigarrow$  New proof of Gaitsgory's theorem.

# Outline of proof

Key steps in the proof:

- 1 Reduction to a similar statement with  $\mathbf{Z}/n\mathbf{Z}$ -coefficients.
  - 2 Reduce by general descent arguments to the special case where  $S = \text{Spec} V$ ,  $V$  a rank one valuation ring with algebraically closed fraction field (i.e. a rank one “aic” valuation ring).
  - 3 Over rank one aic valuation rings, make a direct argument using the perverse t-exactness of nearby cycles.
1. is “boring” and I won’t talk about it. Remainder of the talk: sketch of 2. and 3. (in reverse order).

## Sketch of argument over rank one aic valuation rings

Let  $S = \text{Spec} V$  be the spectrum of a rank one aic valuation ring, with generic point  $\eta$  and special point  $s$ . For any finite type  $S$ -scheme  $X$ , get  $j : X_\eta \rightarrow X$  and  $i : X_s \rightarrow X$  as usual.

Set  ${}^{p/S}D^{\leq 0}(X, \mathbf{Z}/n) =$  objects  $A$  in  $D(X, \mathbf{Z}/n)$  such that  $i^*A$  and  $j^*A$  both lie in  ${}^pD^{\leq 0}$ . This defines the left half of a t-structure by general nonsense. Want to identify the right half.

Right half characterized a priori by condition that  $Ri^!A$  and  $j^*A$  both lie in  ${}^pD^{\geq 0}$ . Need to see that this is equivalent to the same containment for  $i^*A$  and  $j^*A$ .

Key point: Look at the triangle

$$Ri^!A \rightarrow i^*A \rightarrow i^*Rj_*j^*A \rightarrow,$$

and use the fact that  $i^*Rj_* : D(X_\eta, \Lambda) \rightarrow D(X_s, \Lambda)$  is perverse t-exact (Gabber). This + condition on  $j^*A$  implies that  $Ri^!A$  and  $i^*A$  have same perverse cohomology in negative degrees. Done.

From the case where  $S$  is the spectrum of a rank one aic valuation ring, some small arguments extend the result first to the case where  $S$  is the spectrum of any aic valuation ring, and then to the case where  $S$  is qcqs and **all connected components of  $S$  are spectra of aic valuation rings**. Such schemes are far from Noetherian, but in other ways they are not so bad. The spectrum of an aic valuation ring is basically a “spike”, so a scheme like this is some profinite collection of spikes.



Figure: A scheme of this flavor

Since the t-structure we are seeking is supposed to behave well with respect to any base change on  $S$ , we're now in a position to define it in the general case by descent from this funny case.

# Descent

Two key points:

- ① Any qcqs scheme  $S$  has a  $v$ -hypercover  $S_\bullet \rightarrow S$  by qcqs schemes all of whose connected components are spectra of aic valuation rings.
- ② (Bhatt-Mathew, Gabber) The association  $X \mapsto \mathcal{D}^+(X_{\text{ét}}, \mathbf{Z}/n)$  is a hypercomplete  $v$ -sheaf, and in fact a hypercomplete sheaf for the topology of universal submersions.

Back to a general  $X \rightarrow S$  as before. Can pick a  $v$ -hypercover  $S_\bullet \rightarrow S$  as in 1. Then 2. gives  $\mathcal{D}^+(X, \mathbf{Z}/n) \simeq \lim_m \mathcal{D}^+(X \times_S S_m, \mathbf{Z}/n)$ , and we can now descend the t-structure as desired since all pullbacks

$$\mathcal{D}^+(X \times_S S_m, \mathbf{Z}/n) \rightarrow \mathcal{D}^+(X \times_S S_{m'}, \mathbf{Z}/n)$$

are t-exact.

Thank you for listening!

Featured art:

- *A Young Man Writing at a Cloth Covered Table* by Christian van Donck (circa 1653)
- *Portrait of Samuel Johnson* by Joshua Reynolds (1775)