EXAMPLES HANDOUT (II)

1. The conjecture

Take $G = \mathrm{GL}_n$ over a finite extension $E/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Set $\Lambda = \overline{\mathbb{Q}}_\ell$ and fix $q^{1/2} \in \Lambda$.

- Let φ be the semisimple L-parameter such that $\varphi(I_F) = 1$ and maps the geometric Frobenius to $\delta^{1/2} := \mathrm{diag}(q^{(1-n)/2}, \dots, q^{(n-1)/2}) \in \hat{T}(\Lambda)$.
- For $G = \mathrm{GL}_n$, we have Vogan stack

$$\begin{array}{ccc} V_{\hat{G},\varphi} = \mathbb{A}^{n-1}/\hat{T} & \xhookrightarrow{\iota} & \mathrm{Par}^{\mathrm{unip}}_{\hat{G}}. \\ \downarrow & & \\ \mathbb{B}\hat{T} & & \end{array}$$

  Each $\boldsymbol{k} \in \mathbf{X}^*(\hat{T})$, viewed as a coherent sheaf on $\mathbb{B}\hat{T}$, yields $\mathcal{L}_{\boldsymbol{k}} := \iota_* \mathcal{O}(\boldsymbol{k}) \in \mathrm{Coh}(\mathrm{Par}^{\mathrm{unip}}_{\hat{G}})$, corresponding to an automorphic sheaf $\mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$ on Bun_G by assuming categorical local Langlands equivalence. (If we work at the unipotent level, this will be unconditional by [Zhu25].)
- Let $\mathrm{b}_{d/n} = \mathrm{b}_d$ be the basic isocrystal of slope d/n. Then $G_{\mathrm{b}_d} = \mathrm{GL}_{(d,n)}(D_{d'/n'})$ gives an inner form of G, which depends only on d mod n.

It turns out that the stalk $i^*_{\mathrm{b}_d} \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$ is identically zero unless $\boldsymbol{k}$ has degree d.

Conjecture 1.1 (Hansen)**.** *Suppose $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbf{X}^*(\hat{T})$ has degree $d \in \mathbb{Z}$. Then we expect*

$$i^*_{\mathrm{b}_d} \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}} = \pi_{\mathrm{I}_{\boldsymbol{k}}}\big[\textstyle\sum_{j \in \mathrm{J}_{\boldsymbol{k}}} (\delta_j - 2m_j)\big]$$

between derived complexes in $\mathrm{Rep}(G_{\mathrm{b}_d}(E), \Lambda)$ that are concentrated in degree 0. Here,

- $\mathrm{I}_{\boldsymbol{k}} := \{i \in \{1, \dots, (d,n) - 1\} \mid m_{n'i} \leqslant 0\}$ *and* $\mathrm{J}_{\boldsymbol{k}} := \{i \in \{1, \dots, n-1\} \mid m_i > 0\}$;
- $\pi_{\mathrm{I}_{\boldsymbol{k}}}$ *is the unique generalized Steinberg representation of $G_{\mathrm{b}_d}(E)$ corresponding to $\mathrm{I}_{\boldsymbol{k}}$;*
- $\delta_i \in \{0, 1\}$, *and $\delta_i = 1$ if and only if $n' = n/(d,n)$ divides i;*
- m_j*'s are integers determined by $\boldsymbol{k}$, and $m_j = k_1 + \dots + k_j$ if $d = 0$.*

2. The proof for $\mathrm{b} = 1$

When $\mathrm{b} = 1$ (achieved by taking $d = 0$), Conjecture 1.1 can be simplified into the following.

Theorem 2.1. *For $\boldsymbol{k} = (k_1, \dots, k_n)$ of degree 0, writing $m_j = k_1 + \dots + k_j$, we have*

$$i^*_1 \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}} = \pi_{\mathrm{I}_{\boldsymbol{k}}}\big[\textstyle\sum_{j \notin \mathrm{I}_{\boldsymbol{k}}} (1 - 2m_j)\big].$$

2.1. Jacquet module and coherent Springer sheaf.

Proposition 2.2 ([Zel80, §2])**.** *For any $\mathrm{I} \subset \{1, \dots, n-1\}$, the Jacquet module of π_{I} is*

$$\mathbf{r}^B_G(\pi_{\mathrm{I}}) = \bigoplus_{\sigma} \sigma(\delta^{1/2}),$$

where the direct sum runs over $\sigma \in S_n$ such that $\mathrm{I} = \{i \in \{1, \dots, n-1\} \mid \sigma^{-1}(i) < \sigma^{-1}(i+1)\}$.

Applying [HHS24, Corollary 2.2.1], we get

$$\mathbf{r}^B_G\, i^*_1 \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}} = i^{*,T}_1 \mathrm{CT}_{B,!} \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}.$$

On spectral side, we need $\mathrm{CT}^{\mathrm{Spec}}_B \mathcal{L}_{\boldsymbol{k}} := \mathfrak{p}^{\mathrm{Spec}}_* \mathfrak{q}^{\mathrm{Spec},!} \mathcal{L}_{\boldsymbol{k}}$.

$$\begin{array}{ccccc} & & \mathrm{Par}^{\mathrm{unip}}_{\hat{B}} & & \\ & \swarrow^{\mathfrak{p}^{\mathrm{Spec}}} & & \searrow^{\mathfrak{q}^{\mathrm{Spec}}} & \\ \hat{T} \times \mathbb{B}\hat{T} = \mathrm{Par}^{\mathrm{unip}}_{\hat{T}} & & & & \mathrm{Par}^{\mathrm{unip}}_{\hat{G}} \xhookleftarrow{\iota} V_{\hat{G},\varphi} \end{array} \qquad \begin{array}{ccc} \mathrm{Par}^{\mathrm{unip}}_{\hat{B}} & \longleftarrow & V^{\wedge}_{\hat{B},\varphi} \\ \searrow^{\hat{\mathfrak{q}}^{\mathrm{Spec}}} & & \searrow \\ & \mathrm{Par}^{\mathrm{unip},\wedge}_{\hat{G}} & \xhookleftarrow{\hat{\iota}} V^{\wedge}_{\hat{G},\varphi} \end{array}$$

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Proposition 2.3 (Xiangqian Yang, see [Yan25, Proposition 3.12])**.** *For each $\sigma \in S_n$, denote $\mathrm{P}_\sigma \coloneqq \{i \in \{1,\dots,n-1\} \mid \sigma^{-1}(i) < \sigma^{-1}(i+1)\}$ and let Q_σ be its complement in $\{1,\dots,n-1\}$. Then*

$$V^\wedge_{\hat{B},\varphi} \simeq \coprod_{\sigma \in S_n} (\operatorname{Spf} \Lambda[v_i]_{i\in \mathrm{P}_\sigma}[\![u_1,\dots,u_n]\!]/(u_i v_i)_{i\in\mathrm{P}_\sigma})/\hat{T}.$$

In particular, taking $\sigma = \mathrm{id}$ gives the closed substack $V^\wedge_{\hat{B},\varphi} \to V^\wedge_{\hat{G},\varphi}$ corresponding to $\mathrm{Q}_\sigma = \emptyset$.

To further simplify the notations, write

- $X_\sigma \coloneqq (\operatorname{Spf} \Lambda[v_i]_{i\in \mathrm{P}_\sigma}[\![u_1,\dots,u_n]\!]/(u_i v_i)_{i\in\mathrm{P}_\sigma})/\hat{T}$ (in particular, $X \coloneqq X_{\mathrm{id}} = V^\wedge_{\hat{G},\varphi}$);
- $\mathcal{O}_X/\boldsymbol{u} \coloneqq \mathcal{O}_{X_{\mathrm{id}}}/(u_1,\dots,u_n) = \Lambda[v_1,\dots,v_{n-1}]$.

Then we have

◇ $\mathbb{L}^{\mathrm{unip}}(\mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}) = \hat{\iota}_*(\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}) \in \operatorname{Coh}(\operatorname{Par}^{\mathrm{unip},\wedge}_{\hat{G}})$, and

◇ $\mathbb{L}^{\mathrm{unip}}(\text{c-Ind}^G_I \mathbb{1}) = \operatorname{CohSpr}^{\mathrm{unip}} \coloneqq \hat{\mathfrak{q}}^{\mathrm{Spec}}_* \mathcal{O}_{\operatorname{Par}^{\mathrm{unip}}_{\hat{B}}} \in \operatorname{Coh}(\operatorname{Par}^{\mathrm{unip},\wedge}_{\hat{G}})$.

To determine the stalk $i_1^* \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$ (for $\mathrm{b} = 1$ and $G_{\mathrm{b}} = G$), it suffices to compute $\operatorname{RHom}(\text{c-Ind}^G_I\mathbb{1}, \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}) \cong \operatorname{RHom}(\operatorname{CohSpr}^{\mathrm{unip}}, \hat{\iota}_*(\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))$. But $\hat{\iota}^!(\operatorname{CohSpr}^{\mathrm{unip}}) \cong \bigoplus_{\sigma\in S_n} \mathcal{O}_{X_\sigma}$ as ind-coherent sheaves on X by Proposition 2.3. So it reduces to

$$\bigoplus_{\sigma\in S_n} \operatorname{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}})). \tag{†}$$

2.2. **A projective resolution.** Continue with the computation of cohomology of (†). We construct a free projective resolution $\mathcal{P}_\bullet \to \mathcal{O}_{X_\sigma} = \mathcal{O}_X/(v_j)_{j\in\mathrm{Q}_\sigma} \to 0$ of $\mathcal{O}_X$-modules that is $\hat{T}$-equivariant.

Construction 2.4. Fix an arbitrary subset $\mathrm{Q} \subset \{1,\dots,n-1\}$.

- Define $\mathbb{N}^{\mathrm{Q}} \coloneqq \{\boldsymbol{d} = (d_i)_{i\in\mathrm{Q}} \mid d_i \in \mathbb{Z}_{\geqslant 0}\}$;
- For each $\boldsymbol{d} \in \mathbb{N}^{\mathrm{Q}}$, write $|\boldsymbol{d}| \coloneqq \sum_{i\in\mathrm{Q}} d_i$ and define the weight $\chi_{\boldsymbol{d}} \coloneqq \sum_{i\in\mathrm{Q}} \lceil d_i/2\rceil \alpha_i$.
- For $t \geqslant 0$, set
$$\mathcal{P}_t \coloneqq \bigoplus_{\boldsymbol{d}\in\mathbb{N}^{\mathrm{Q}},\, |\boldsymbol{d}|=t} \mathcal{O}_X(\chi_{\boldsymbol{d}}),$$
with the differential maps $\partial\colon \mathcal{P}_{t+1} \to \mathcal{P}_t$ given by
$$\mathsf{e}_{\boldsymbol{d}} \longmapsto \sum_{i\in\mathrm{Q},\, d_i>0} (-1)^{\varepsilon(i,\boldsymbol{d})} \vartheta(d_i)\cdot \mathsf{e}_{\boldsymbol{d}-\boldsymbol{e}_i}.$$

The notations in the formula of $\partial(\mathsf{e}_{\boldsymbol{d}})$ are explained below.

- For v_i, u_i in $\mathcal{O}_X$, we set $\vartheta_i(r) = v_i$ when r is odd, and $\vartheta_i(r) = u_i$ when r is even.
- Define $\varepsilon(i,\boldsymbol{d}) \coloneqq \sum_{j\in\mathrm{Q}_{<i}} d_j$, where $\mathrm{Q}_{<i} = \mathrm{Q} \cap \{1,\dots,i-1\}$; declare $\varepsilon(1,\boldsymbol{d}) = 0$.
- The $\boldsymbol{e}_i$ is the $|\mathrm{Q}|$-tuple with 1 on its i-th coordinate and 0 elsewhere; set $\mathsf{e}_{\boldsymbol{d}-\boldsymbol{e}_i} = 0$ if any coordinate of $\boldsymbol{d}-\boldsymbol{e}_i$ is negative.

This resolution depends on Q, but we omit Q from the notation. One can check that $\mathcal{P}_\bullet$ is automatically $\hat{T}$-equivariant, by using the setting that $\hat{T}$ acts on u_i and v_i respectively by weights α_i and 0.

Example 2.5. (1) Take $G = \mathrm{GL}_2$ and $\mathrm{Q} = \{1\}$. Then $\mathcal{P}_\bullet$ looks like

$$\begin{array}{ccccccccccc} \cdots \xrightarrow{u_1} & \mathcal{O}_X(2\alpha_1) & \xrightarrow{v_1} & \mathcal{O}_X(\alpha_1) & \xrightarrow{u_1} & \mathcal{O}_X(\alpha_1) & \xrightarrow{v_1} & \mathcal{O}_X & \twoheadrightarrow & \mathcal{O}_{X_{(12)}} & \longrightarrow 0. \\ & \parallel & & \parallel & & \parallel & & \parallel & & & \\ & \mathcal{P}_3 & & \mathcal{P}_2 & & \mathcal{P}_1 & & \mathcal{P}_0 & & & \end{array}$$

This recovers the resolution in [BM23, §2] for PGL_2 (essentially the same as the GL_2 case).

(2) Take $G = \mathrm{GL}_3$ and $\mathrm{Q} = \{1,2\}$. Then we have

$$\mathcal{P}_t = \bigoplus_{d_1+d_2=t} \mathcal{O}_X(\lceil d_1/2\rceil\alpha_1 + \lceil d_2/2\rceil\alpha_2),$$

and that $\partial(\mathsf{e}_{(d_1,d_2)}) = \vartheta_1(d_1)\mathsf{e}_{(d_1-1,d_2)} + (-1)^{d_1}\vartheta_2(d_2)\mathsf{e}_{(d_1,d_2-1)}$.

2.3. **Proof of Theorem 2.1.** We proceed to compute the cohomology

$$\mathrm{H}^*(\mathrm{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))).$$

In $\mathrm{IndCoh}(\mathrm{Par}_{\hat{G}}^{\mathrm{unip},\wedge})$, from the resolution $\mathcal{P}_\bullet$ we get a cochain complex

$$(\mathcal{C}^\bullet)^{\hat{T}} := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_\bullet, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))^{\hat{T}} \cong \bigoplus_{\boldsymbol{d}\in\mathbb{N}^{\mathrm{Q}},\, |\boldsymbol{d}|=t} (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}} - \chi_{\boldsymbol{d}})^{\hat{T}}.$$

Lemma 2.6. *Let $\nu = r_1\alpha_1 + \cdots + r_{n-1}\alpha_{n-1}$ for $r_1, \ldots, r_{n-1} \in \mathbb{Z}$. Then we have*

$$\mathcal{O}_X(\nu)^{\hat{T}} = \begin{cases} 0, & \text{if } r_j > 0 \text{ for some } j; \\ v_1^{-r_1} \cdots v_{n-1}^{-r_{n-1}} \Lambda[\![u_n]\!][\![u_j]\!]_{r_j=0}, & \text{otherwise.} \end{cases}$$

In particular, $(\mathcal{O}_X/\boldsymbol{u})(\nu)^{\hat{T}}$ is either isomorphic to Λ or 0, determined by whether all $r_i \leqslant 0$ or not.

Corollary 2.7. *Fix $\mathrm{Q} \subset \{1, \ldots, n-1\}$ as before. If $(\mathcal{C}^\bullet)^{\hat{T}} \neq 0$, then there exists $\boldsymbol{d} \in \mathbb{N}^{\mathrm{Q}}$ such that for all $i \in \mathrm{Q}$, we have $\lceil d_i/2 \rceil \geqslant m_i = k_1 + \cdots + k_i$.*

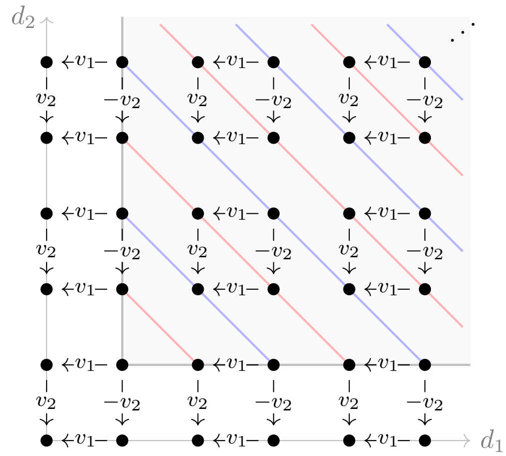
To prove Theorem 2.1, the case of GL_3 already exhibits all phenomena present for GL_n.

(i) For $\mathrm{Q}_\sigma = \emptyset$, no resolution is in need and apply Lemma 2.6 directly.

(ii) For $\mathrm{Q}_\sigma = \{1\}$, the resolution of $\mathcal{O}_{X_\sigma}$ is essentially the same as Example 2.5(1) for the GL_2 case, namely $\mathcal{P}_\bullet \to \mathcal{O}_{X_\sigma} \to 0$ with $\mathcal{P}_t = \mathcal{O}_X(\lceil t/2 \rceil \alpha_1)$. Then Lemma 2.6 computes $(\mathcal{C}^\bullet)^{\hat{T}}$ and Corollary 2.7 determines the degree shift.

(iii) For $\mathrm{Q}_\sigma = \{2\}$, this is the same as in (ii).

(iv) For $\mathrm{Q}_\sigma = \{1, 2\}$, we are in the case of Example 2.5(2). The resolution is illustrated below.



It turns out that the cohomology is non-vanishing only at the corner $(d_1, d_2) = (2m_1 - 1, 2m_2 - 1)$ by Corollary 2.7.

The computation above finishes the proof of Theorem 2.1 for GL_3.

References

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