Iwasawa theory of overconvergent modular forms, I: Critical-slope p-adic L-functions

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Abstract

We construct an Euler system of p-adic zeta elements over the eigencurve which interpolates Kato's zeta elements over all classical points. Applying a big regulator map gives rise to a purely algebraic construction of a two-variable p-adic L-function over the eigencurve. As a first application of these ideas, we prove the equality of the p-adic L-functions associated with a critical-slope refinement of a modular form by the works of Bellaïche/Pollack-Stevens and Kato/Perrin-Riou.

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1 Introduction

1.1 Euler systems and zeta elements

The Birch—Swinnerton-Dyer conjecture and its generalizations relate special values of L-functions with various arithmetically defined groups. These conjectures are among the most fascinating and

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difficult problems in modern number theory. One of the most powerful techniques for attacking these problems is the theory of *Euler systems,* families of Galois cohomology classes with very special properties [Kol90, Rub00]. Euler systems forge a strong link between special values of L-functions and Galois cohomology, and every known Euler system has led to very deep results: among other successes, we mention Rubin's proof of the Iwasawa main conjectures for imaginary quadratic fields [Rub91], Kolyvagin's results on the Birch—Swinnerton-Dyer conjecture [Kol88], and Kato's work on the Bloch-Kato conjecture for elliptic modular forms [BK90, Kat04].

In this article, we construct Euler systems associated with finite-slope overconvergent modular forms, and with rigid analytic families of such forms. Since many classical modular forms are finite-slope overconvergent, our constructions have consequences for the Iwasawa theory of classical modular forms and elliptic curves. In particular, Theorem 1.2.1 below settles the long-standing question of comparing the analytic and algebraic p-adic L-functions attached to a critical-slope refinement of a modular form by Pollack-Stevens and Kato. The proof of this theorem follows naturally from our construction of a "two-variable algebraic p-adic L-function" over the eigencurve. These ideas also lead to some partial results towards a two-variable main conjecture, which we defer to a second article [Han16]; it seems likely that our constructions will have further applications. In the context of Hida families, many of the results in this paper have been established by Ochiai in a beautiful series of articles [Och03, Och05, Och06], which gave us the confidence to attempt this project. We also mention the related work of Wang [Wan12], who has proven a lovely two-variable interpolation of Kato's explicit reciprocity law.

Our point of departure is the monumental work of Kato [Kat04]. Speaking very roughly, we are able to interpolate Kato's constructions of p-adic zeta elements and Euler systems over the Coleman-Mazur-Buzzard eigencurve. In the course of doing this, we are led to several intermediate results which may be of interest: in particular, we put canonical integral structures on Coleman families and on the universal Galois representations over them.

Before stating our results, we recall the eigencurve and define some distinguished classes of points on it. Fix an odd prime p, an algebraic closure $\overline{\mathbf{Q}_p}$, and an isomorphism $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}_p}$. Fix an integer $N \geq 1$ prime to p, and let $\mathbf{T}$ be the polynomial algebra over $\mathbf{Z}$ generated by the operators $T_\ell, \ell \nmid Np$, U_p and $\langle d \rangle, d \in (\mathbf{Z}/N\mathbf{Z})^\times$. Set $\mathfrak{W} = \mathrm{Spf}(\mathbf{Z}_p[[\mathbf{Z}_p^\times]])$, and let $\mathscr{W} = \mathfrak{W}^{\mathrm{rig}}$ be the rigid analytic space of characters of $\mathbf{Z}_p^\times$ together with its universal character $\chi_{\mathscr{W}} : \mathbf{Z}_p^\times \to \mathscr{O}(\mathscr{W})^\times$; we embed $\mathbf{Z}$ in $\mathscr{W}(\mathbf{Q}_p)$ by mapping k to the character $t \mapsto t^{k-2}$. For any $\lambda \in \mathscr{W}(\overline{\mathbf{Q}_p})$ we (slightly abusively) write

$$M_\lambda^\dagger(\Gamma_1(N)) \subset \overline{\mathbf{Q}_p}[[q]]$$

for the space of q-expansions of overconvergent modular forms of weight λ and tame level N. Coleman-Mazur [CM98] and Buzzard [Buz07] constructed a rigid analytic curve $\mathscr{C}(N)$ equipped with a flat, locally finite morphism $w : \mathscr{C}(N) \to \mathscr{W}$ and an algebra homomorphism $\phi : \mathbf{T} \to \mathscr{O}(\mathscr{C}(N))$ such that $\mathscr{C}(N)(\overline{\mathbf{Q}_p})$ parametrizes the set of overconvergent eigenforms of finite slope and tame level N, with $x \in \mathscr{C}(N)(\overline{\mathbf{Q}_p})$ corresponding to the generalized eigenspace

$$\begin{aligned} M_{w(x)}^\dagger(\Gamma_1(N))_{\ker \phi_x} &= \left\{ f \in M_{w(x)}^\dagger(\Gamma_1(N)) \mid (T - \phi_x(T))^n f = 0 \,\forall T \in \mathbf{T} \,\text{and}\, n \gg 0 \right\} \\ &\subset M_{w(x)}^\dagger(\Gamma_1(N)). \end{aligned}$$

When this eigenspace is one-dimensional over $\overline{\mathbf{Q}_p}$ we write $\mathbf{f}_x$ for the canonically normalized generator. For any $x \in \mathscr{C}(N)(\overline{\mathbf{Q}_p})$, the theory of pseudorepresentations yields a continuous two-dimensional $G_{\mathbf{Q}}$-representation V_x over the residue field E_x of x with tame Artin conductor dividing

N such that for all primes $\ell \nmid Np$ we have

$$\begin{aligned} \mathrm{tr}\mathrm{Frob}_\ell|V_x &= \phi_x(T_\ell), \\ \det \mathrm{Frob}_\ell|V_x &= \ell\chi_{\mathscr{W},w(x)}(\ell)\phi_x(\langle\ell\rangle) \end{aligned}$$

with $\mathrm{Frob}_\ell \in G_{\mathbf{Q}}$ a geometric Frobenius. The function $\alpha = \phi(U_p) \in \mathscr{O}(\mathscr{C}(N))$ is nonvanishing on $\mathscr{C}(N)$ and plays a distinguished role in the theory; in particular, the image of $\mathscr{C}(N)$ in $\mathscr{W} \times \mathbf{A}^1$ under the map $x \mapsto (w(x), 1/\alpha(x))$ is a hypersurface over which $\mathscr{C}(N)$ is finite.

Given an integer $M|N$, we say a point $x \in \mathscr{C}(N)(\overline{\mathbf{Q}_p})$ is *crystalline of conductor M* if there is a (necessarily unique) normalized cuspidal elliptic newform

$$f_x = \sum_{n\geq 1} a_n(f_x)q^n \in S_k(\Gamma_1(M))$$

of weight $k = k_x \geq 2$ and nebentype ε_{f_x} such that $\phi_x(T_\ell) = a_\ell(f_x)$ and $\phi_x(\langle\ell\rangle) = \varepsilon_{f_x}(\ell)$ for all $\ell \nmid Np$ and furthermore α_x is a root of the Hecke polynomial $X^2 - a_p(f_x) + p^{k_x-1}\varepsilon_{f_x}(p)$.[1] It's not hard to see that any newform of weight ≥ 2 and level dividing N occurs this way. Given any point $x \in \mathscr{C}(N)(\overline{\mathbf{Q}_p})$ with $w(x) = k, k \geq 2$, we say x is *noncritical* if the inclusion of generalized eigenspaces

$$M_k(\Gamma_1(N) \cap \Gamma_0(p))_{\ker\phi_x} \subset M_k^\dagger(\Gamma_1(N))_{\ker\phi_x}$$

is an equality, and *critical* otherwise.

Definition 1.1.1. *A point $x \in \mathscr{C}(N)(\overline{\mathbf{Q}_p})$ is* **noble** *if it is crystalline of conductor N, and furthermore x is noncritical and α_x is a simple root of the pth Hecke polynomial of f_x.*

Note that a crystalline point of conductor N is noble exactly when the generalized eigenspaces in the definition of noncriticality are honest eigenspaces of common dimension one, in which case the p-stabilized form

$$\mathbf{f}_x = f_x(q) - p^{k_x-1}\varepsilon_{f_x}(p)\alpha_x^{-1}f_x(q^p)$$

defines the canonical generator of both spaces. Our motivation for singling out these points is geometric: the eigencurve is smooth and étale over $\mathscr{W}$ locally at any noble point [Bel12], and certain sheaves of interest are locally free of minimal rank around such points. It is conjectured that the final condition in the definition of nobility always holds [CE98], and that a crystalline point of conductor N fails to be noble if and only if f_x is a CM form and $v_p(\alpha_x) = k_x - 1$; in particular, the map

$$\begin{aligned} \{\text{noble points}\} &\to \{\text{newforms of level } N \text{ and weight } \geq 2\} \\ x &\mapsto f_x \end{aligned}$$

is conjecturally surjective, one-to-one on the set of CM forms with p split in the CM field, and two-to-one on all other forms. These results are all known for points of weight 2, and in particular, any point such that f_x is associated with a non-CM (modular) elliptic curve over $\mathbf{Q}$ is noble.

Let $\mathscr{C}_0^{M-\mathrm{new}}(N)$ denote the Zariski closure in $\mathscr{C}(N)$ of the crystalline points of conductor M; this is a union of irreducible components of $\mathscr{C}(N)$, and the complement of $\cup_{M|N}\mathscr{C}_0^{M-\mathrm{new}}(N)$ in $\mathscr{C}(N)$ is an open dense subspace of the union of Eisenstein components of $\mathscr{C}(N)$.

[1] These are precisely the points for which the Galois representation V_x has tame Artin conductor M and is crystalline with distinct nonnegative Hodge-Tate weights at p.

Definition 1.1.2. *The* smooth cuspidal eigencurve of level N, *denoted by* $\mathscr{C}_N$ *or simply by* $\mathscr{C}$, *is the normalization of* $\mathscr{C}_0^{N-\mathrm{new}}(N)$.

The curve $\mathscr{C}$ is a disjoint union of connected smooth reduced rigid analytic curves, with a natural morphism $i : \mathscr{C} \to \mathscr{C}(N)$. We will slightly abusively write $w = w \circ i$, $\phi = i^*\phi$, $\alpha = i^*\alpha$ for the natural maps and functions inherited from those on $\mathscr{C}(N)$. Any point in the smooth locus of $\mathscr{C}_0^{N-\mathrm{new}}(N)$ determines a unique point in $\mathscr{C}$, and i is an isomorphism locally around any such point; in particular, this holds for any noble point. We find it very convenient to introduce the involution $x \mapsto x^c$ of $\mathscr{C}$ given on geometric points by

$$\begin{aligned} \phi_{x^c}(T_\ell) &= \phi_x(T_\ell \langle \ell \rangle^{-1}), \\ \phi_{x^c}(U_p) &= \phi_x(U_p \langle p \rangle^{-1}), \\ \phi_{x^c}(\langle d \rangle) &= \phi_x(\langle d^{-1} \rangle). \end{aligned}$$

Note that on crystalline points we have $f_{x^c} = f_x^c$ where $f^c = f \otimes \varepsilon_f^{-1}$ is the complex conjugate of f, and likewise for α_x.

As we construct it here, the curve $\mathscr{C}$ carries a rank two locally free sheaf $\mathscr{V}$ with a continuous $G_{\mathbf{Q}}$-action unramified outside Np, such that for every noble point $x \in \mathscr{C}(\overline{\mathbf{Q}_p})$ there is an isomorphism $\mathscr{V}_x \simeq V_{f_x}(k_x) \simeq V^*_{f_x^c}(1)$ which is moreover realized by a *canonical* isomorphism

$$\mathscr{V}_x \cong \left(H^1_{\text{ét}} \left(Y_1(Np)_{/\overline{\mathbf{Q}}}, (\mathrm{sym}^{k_x-2} T_p E)(2) \right) \otimes_{\mathbf{Z}_p} E_x \right) [\ker \phi_x]$$

of Galois modules. Here T_pE is the étale sheaf defined by the Tate module of the universal elliptic curve over $Y_1(Np)$. The sheaf $\mathscr{V}$ and its fibers provide canonical étale-cohomological realizations of the Galois representations associated with the overconvergent eigenforms parametrized by $\mathscr{C}$. Following ideas of Pottharst, we define coherent Galois cohomology sheaves $\mathscr{H}^1(\mathbf{Q}(\zeta_m), \mathscr{V}(-r))$ over $\mathscr{C}$ whose sections over an admissible affinoid open U are given by

$$\mathscr{H}^1(\mathbf{Q}(\zeta_m), \mathscr{V}(-r))(U) = H^1(G_{\mathbf{Q}(\zeta_m), Nmp\infty}, \mathscr{V}(U)(-r)).$$

We denote the global sections of this sheaf by $H^1(\mathbf{Q}(\zeta_m), \mathscr{V}(-r))$.

As a sample of our results, we state the following theorem.

Theorem 1.1.3. *Notation as above, let ν be any primitive Dirichlet character of even order and conductor A prime to Np, with coefficient field $\mathbf{Q}_p(\nu)$, and let r be an arbitrary integer.*

i. *There is a collection of global cohomology classes*

$$\mathfrak{z}(r, \nu) = \left(\mathfrak{z}_m(r, \nu) \in H^1(\mathbf{Q}(\zeta_m), \mathscr{V}(-r)) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\nu) \right)_{m \geq 1, (m,A)=1}$$

such that for any prime $\ell \nmid NA$, we have the corestriction relation

$$\mathrm{Cor}_{\mathbf{Q}(\zeta_{m\ell})/\mathbf{Q}(\zeta_m)} \left(\mathfrak{z}_{\ell m}(r, \nu) \right) = \begin{cases} \mathfrak{z}_m(r, \nu) & \text{if } \ell | mp \\ P_\ell(\ell^{-r} \sigma_\ell^{-1}) \cdot \mathfrak{z}_m(r, \nu) & \text{if} \ell \nmid mp, \end{cases}$$

where $\sigma_\ell \in \mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q})$ denotes an arithmetic Frobenius at ℓ and $P_\ell(X) \in \mathscr{O}(\mathscr{C})[X]$ is the polynomial

$$\begin{aligned} P_\ell(X) &= 1 - \phi(T_\ell \langle \ell \rangle^{-1}) X + \phi(\langle \ell \rangle^{-1}) \chi_{\mathscr{W}}(\ell) \ell X^2 \\ &= \det\left(1 - X \cdot \mathrm{Frob}_\ell\right) | \mathscr{V}^*(1). \end{aligned}$$

ii. *The class* $\mathfrak{z}_m(r,\nu)$ *is unramified outside the primes dividing* p.
iii. *If* x *is a noble point such that either* $k_x > 2$ *or* $k_x = 2$ *and* $L(1, f_x \otimes \nu) \neq 0$, *the specialization*

$$\mathfrak{z}(r,\nu)_x = \left(\mathfrak{z}_m(r,\nu)_x \in H^1(\mathbf{Q}(\zeta_m), \mathscr{V}_x(-r)) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\nu)\right)_{m\geq 1,\,(m,A)=1}$$

is nonzero.

The corestriction relation in part i. here is the famous Euler system relation.

We also construct classes in Iwasawa cohomology which interpolate the classes in Theorem 1.1.3 for varying r and m. More precisely, set $\Gamma_m = \mathrm{Gal}(\mathbf{Q}(\zeta_{mp^\infty})/\mathbf{Q}(\zeta_m))$ and let Λ_m be the sheaf of rings over $\mathscr{C}$ defined by

$$\Lambda_m(U) = \left(\mathscr{O}(U)^\circ \widehat{\otimes} \mathbf{Z}_p[[\Gamma_m]]\right)[\frac{1}{p}]$$

for U an affinoid open (we notate these and other allied objects in the case $m = 1$ by dropping m).[2] We construct a sheaf $\mathscr{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\zeta_m), \mathscr{V}(-r))$ of Λ_m-modules characterized by the equality

$$\mathscr{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\zeta_m), \mathscr{V}(-r))(U) = H^1(G_{\mathbf{Q}(\zeta_m),Nmp\infty}, \mathscr{V}(U)(-r) \otimes_{\mathscr{O}(U)} \Lambda^\iota_m(U)).$$

For any integer r and any integer $j \geq 0$, there is a natural morphism of sheaves

$$\theta_{r,j} : \mathscr{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\zeta_m), \mathscr{V}(-r)) \to \mathscr{H}^1(\mathbf{Q}(\zeta_{mp^j}), \mathscr{V}(-r))$$

induced by the surjections $\Lambda_m(U) \twoheadrightarrow \mathscr{O}(U) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p[\mathrm{Gal}(\mathbf{Q}(\zeta_{mp^j})/\mathbf{Q}(\zeta_m))]$ together with Shapiro's lemma. Let H^1_{Iw} denote the global sections of $\mathscr{H}^1_{\mathrm{Iw}}$.

Theorem 1.1.4. *There are naturally defined cohomology classes*

$$\mathfrak{Z}_m(r,\nu) \in H^1_{\mathrm{Iw}}(\mathbf{Q}(\zeta_m), \mathscr{V}(-r)) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\nu)$$

whose image under $\theta_{r,j}$ *equals* $\mathfrak{z}_{mp^j}(r,\nu)$ *for all* $r \in \mathbf{Z}$ *and all* $j \in \mathbf{Z}_{\geq 1}$, *and such that under the natural twisting isomorphisms*

$$\mathrm{Tw}_{r_1-r_2} : \mathscr{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\zeta_m), \mathscr{V}(-r_1)) \xrightarrow{\sim} \mathscr{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\zeta_m), \mathscr{V}(-r_2))$$

we have $\mathrm{Tw}_{r_1-r_2}(\mathfrak{Z}_m(r_1,\nu)) = \mathfrak{Z}_m(r_2,\nu)$.

Theorem 1.1.3, while aesthetically pleasing, is somewhat useless in applications, since the Euler system machinery deals in integral structures. In fact Theorems 1.1.3 and 1.1.4 follow from our main technical result, which takes integral structures into account and which we now describe. Given $k \geq 2$ and $M \geq 1$, set

$$V^\circ_k(M) = H^1_{\text{ét}}\left(Y_1(M)_{/\overline{\mathbf{Q}}}, \mathrm{sym}^{k-2} T_p E\right)(2-k).$$

This space (denoted $V_{k,\mathbf{Z}_p}(Y_1(M))$ in [Kat04]) admits commuting actions of $G_{\mathbf{Q}}$ and a suitable algebra of Hecke operators, and its eigenspace for the Hecke eigenvalues of a newform f of weight k and level M provides a canonical realization of the Galois representation associated with f. Generalizing ideas of Stevens and others, given a suitable "small" formal subscheme $\mathfrak{U} \subset \mathfrak{W}$ and a

[2] If A is a reduced affinoid algebra and R is an adic Noetherian $\mathbf{Z}_p$-algebra which is p-adically complete, then $A^\circ \widehat{\otimes} R$ denotes the $1 \otimes I$-adic completion of $A^\circ \otimes_{\mathbf{Z}_p} R$ where I is the largest ideal of definition of R (so in particular $p \in I$).

sufficiently large integer s, we define a module $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ of $\mathscr{O}(\mathfrak{U})$-valued locally analytic distributions on $\mathbf{Z}_p$ with a continuous action of $\Gamma_0(p)$. Here "small" means basically that $\mathfrak{U} = \mathrm{Spf}(R)$ where $R = \mathscr{O}(\mathfrak{U})$ is a $\mathbf{Z}_p$-flat and module-finite $\mathbf{Z}_p[[X_1, \ldots, X_d]]$-algebra - in particular, the Berthelot generic fiber $\mathfrak{U}^{\mathrm{rig}} \subset \mathscr{W}$ is typically not quasicompact. On the other hand, working with small opens like this allows us to construct a decreasing filtration on the module $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ by sub-$\mathscr{O}(\mathfrak{U})$-modules

$$\mathcal{D}_{\mathfrak{U}}^{s,\circ} = \mathrm{Fil}^0\mathcal{D}_{\mathfrak{U}}^{s,\circ} \supset \mathrm{Fil}^1\mathcal{D}_{\mathfrak{U}}^{s,\circ} \supset \cdots \supset \mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ} \supset \cdots$$

with several extremely felicitous properties: this filtration is $\Gamma_0(p)$-stable, $\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ is a finite abelian group of exponent p^i, $\Gamma(p^{s+i})$ acts trivially on $\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}$, and $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ is separated and complete for the topology defined by $\mathrm{Fil}^\bullet\mathcal{D}_{\mathfrak{U}}^{s,\circ}$. With these facts in hand we are able to show that the module

$$\mathbf{V}_{\mathfrak{U}}^{s,\circ}(N) := \lim_{\infty \leftarrow i} H^1_{\text{ét}}\left(Y_1(pN)_{/\overline{\mathbf{Q}}}, \mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}\right) \; (2)$$

is canonically isomorphic as a Hecke module to the cohomology of the local system induced by $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ on the analytic space $Y_1(Np)(\mathbf{C})$. The modules $\mathbf{V}_{\mathfrak{U}}^{s,\circ}(N)$ and variants thereof are the main technical objects in this article. By construction $\mathbf{V}_{\mathfrak{U}}^{s,\circ}(N)$ admits commuting Hecke and Galois actions, and we will see that in addition the Galois action is p-adically continuous and unramified outside the primes dividing Np. For any $k \in \mathfrak{U} \cap \mathbf{Z}_{\geq 2}$, there is a natural Hecke- and Galois-equivariant "integration" map

$$i_k : \mathbf{V}_{\mathfrak{U}}^{s,\circ}(N) \to V_k^\circ(Np)(k).$$

Theorem 1.1.5. (cf. Proposition 3.2.1) *For any integer r, any integer $A \geq 1$ prime to p, any residue class $a(A)$, any integer $m \geq 1$, and any integers c, d with $(cd, 6Apm) = (d, Np) = 1$, there is a canonically defined cohomology class*

$${}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U}, r, a(A)) \in H^1(G_{\mathbf{Q}(\zeta_m),Nmp\infty}, \mathbf{V}_{\mathfrak{U}}^{s,\circ}(N)(-r))$$

whose specialization under the integration map i_k for any $k \in \mathbf{Z}_{\geq 2} \cap \mathfrak{U}$ satisfies

$$i_k({}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U}, r, a(A))) = {}_{c,d}\mathbf{z}^{(p)}_{1,Np,m}(k, r, k-1, a(A), \mathrm{prime}(mpA)) \in H^1(G_{\mathbf{Q}(\zeta_m),Nmp\infty}, V_k^\circ(Np)(k-r)),$$

where ${}_{c,d}\mathbf{z}^{(p)}_{1,Np,m}(k, r, r', a(A), S)$ is the p-adic zeta element defined in §8.9 of [Kat04].

These classes satisfy the Euler system relations as m varies and are compatible under changing $\mathfrak{U}$ and s, among other properties. With this result in hand, we prove Theorems 1.1.3 and 1.1.4 by building the eigencurve and all associated structures from the finite-slope direct summands of the modules $\mathbf{V}_{\mathfrak{U}}^s(N) = \mathbf{V}_{\mathfrak{U}}^{s,\circ}(N)[\frac{1}{p}]$. Precisely, given a pair $(\mathfrak{U}, h)$ with $\mathfrak{U} \subset \mathfrak{W}$ as before and $h \in \mathbf{Q}_{\geq 0}$, we say this pair is a *slope datum* if the module $\mathbf{V}_{\mathfrak{U}}^s(N)$ admits a slope-$\leq h$ direct summand $\mathbf{V}_{\mathfrak{U},h}^s(N)$ for the action of the U_p-operator. It turns out that any such direct summand is module-finite over $\mathscr{O}(\mathfrak{U})[\frac{1}{p}]$ and preserved by the Hecke and Galois actions. Furthermore, the natural map $\mathbf{V}_{\mathfrak{U}}^{s+1,\circ}(N) \to \mathbf{V}_{\mathfrak{U}}^{s,\circ}(N)$ induces Hecke- and Galois-equivariant *isomorphisms* $\mathbf{V}_{\mathfrak{U},h}^{s+1}(N) \cong \mathbf{V}_{\mathfrak{U},h}^s(N)$; we set $\mathscr{V}_{\mathfrak{U},h}(N) = \lim_{\leftarrow s} \mathbf{V}_{\mathfrak{U},h}^s(N)$. We construct the full eigencurve $\mathscr{C}(N)$ by gluing the quasi-Stein rigid spaces $\mathscr{C}_{\mathfrak{U}^{\mathrm{rig}},h}(N)$ associated with the finite $\mathscr{O}(\mathfrak{U})[\frac{1}{p}]$-algebras

$$\mathbf{T}_{\mathfrak{U},h}(N) = \text{image of } \mathbf{T} \otimes_{\mathbf{Z}} \mathscr{O}(\mathfrak{U})[\tfrac{1}{p}] \text{ in } \mathrm{End}_{\mathscr{O}(\mathfrak{U})[\frac{1}{p}]}\left(\mathscr{V}_{\mathfrak{U},h}(N)\right)$$

(or rather, associated with their "generic fibers": we have $\mathbf{T}_{\mathfrak{U},h}(N) \cong \mathscr{O}(\mathscr{C}_{\mathfrak{U}^{\mathrm{rig}},h}(N))^b$). The $\mathbf{T}_{\mathfrak{U},h}(N)$-modules $\mathscr{V}_{\mathfrak{U},h}(N)$ and $H^1(G_{\mathbf{Q}(\zeta_m),Nmp\infty}, \mathscr{V}_{\mathfrak{U},h}(N)(-r))$ glue together over this covering into coherent sheaves $\mathscr{V}(N)$ and $\mathscr{H}^1(\mathbf{Q}(\zeta_m), \mathscr{V}(N)(-r))$, respectively, and the classes described in Theorem 1.1.5 glue into global sections ${}_{c,d}\mathfrak{z}_{N,m}(r, a(A))$ of the latter sheaf. Choosing ν as above and setting

$$\mu_{c,d}(r,\nu) = (c^2 - c^r\chi_{\mathscr{W}}(c)^{-1}\nu(c)^{-1}\sigma_c)(d^2 - d^r\nu(d)\sigma_d) \in \left(\mathscr{O}(\mathscr{W}) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\nu)\right)[\mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q}],$$

we are able to show that for a judicious choice of c and d, the quantity $\mu_{c,d}$ is a unit in

$$\left(\mathscr{O}(\mathscr{W}) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\nu)\right)[\mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q}],$$

and

$$\mathfrak{z}_{N,m}(r,\nu) = \mu_{c,d}(r,\nu)^{-1} \sum_{a\in(\mathbf{Z}/A\mathbf{Z})^\times} \nu(a)_{c,d}\mathfrak{z}_{N,m}(r, a(A))$$

is well-defined independently of choosing c, d. Finally, we pull everything back under $i : \mathscr{C} \to \mathscr{C}(N)$. The proof of Theorem 1.1.4 is only slightly different, although it requires an additional intermediate result which seems interesting in its own right: we show that the module

$$\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N) = \text{image of } \mathbf{V}^{s,\circ}_{\mathfrak{U}}(N) \text{ in } \mathbf{V}^{s}_{\mathfrak{U},h}(N)$$

is a *finitely presented* $\mathscr{O}(\mathfrak{U})$-module. Since $\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)[\frac{1}{p}] \cong \mathbf{V}^{s}_{\mathfrak{U},h}(N)$, this yields natural integral structures on Coleman families. We refer the reader to §2 and §3 for the details of all these constructions.

1.2 Critical slope p-adic L-functions

We now explain an application of these results to p-adic L-functions. To do this, we need a little more notation. For the remainder of the paper we set

$$\Gamma = \Gamma_1 = \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p) \cong \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}) \cong \mathbf{Z}_p^\times.$$

Let $\mathscr{X}$ be the rigid generic fiber of the formal scheme $\mathrm{Spf}(\mathbf{Z}_p[[\Gamma]])$, so there is a natural bijection of sets (or even of groups)

$$\mathscr{X}(\overline{\mathbf{Q}_p}) = \mathrm{Hom}_{\mathrm{cts}}(\Gamma, \overline{\mathbf{Q}_p}^\times).$$

Given a point $y \in \mathscr{X}(\overline{\mathbf{Q}_p})$ we write χ_y for the corresponding character, and likewise given a character χ we write y_χ for the corresponding point. Let $\mathscr{G}$ be the sheaf of rings over $\mathscr{C}$ defined by $\mathscr{G}(U) = \mathscr{O}(U \times \mathscr{X})$, so $\mathscr{G}$ is a sheaf of Λ-algebras. We regard an element $g \in \mathscr{G}(U)$ as an analytic function of the two variables $x \in U(\overline{\mathbf{Q}_p})$ and $\chi \in \mathrm{Hom}(\Gamma, \overline{\mathbf{Q}_p}^\times)$, writing g_x for its specialization to an element of $\mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} E_x$ and $g_x(\chi)$ or $g(x,\chi) \in \overline{\mathbf{Q}_p}$ for its value at the point (x, y_χ).

Let $f = \sum_{n\geq 1} a_n(f)q^n \in S_k(\Gamma_1(N))$ be a cuspidal newform with character ε and coefficient field $\mathbf{Q}(f) \subset \mathbf{C}$. Given any element $\gamma \in H^1(Y_1(N)(\mathbf{C}), \mathrm{sym}^{k-2}(\mathbf{Q}(f)^2))$ in the f-eigenspace whose projections $\gamma^\pm$ to the ± 1-eigenspaces for complex conjugation are nonzero, we define the periods of f by

$$\omega_f = \Omega^+_{f,\gamma}\gamma^+ + \Omega^-_{f,\gamma}\gamma^- \in H^1_{\mathrm{dR}}(Y_1(N), \mathrm{sym}^{k-2}(\mathbf{C}^2)).$$

By a fundamental theorem of Eichler and Shimura, for any integer $0 \leq j \leq k-2$ and any Dirichlet character η, the ratio

$$L^{\mathrm{alg}}(j+1, f\otimes\eta) = \frac{j!L(j+1, f\otimes\eta)}{(2\pi i)^{j+1}\tau(\eta)\Omega^\pm_{f,\gamma}}$$

is *algebraic,* and in fact lies in the extension $\mathbf{Q}(f,\eta)$; here the sign here is determined by $(-1)^j = \pm\eta(-1)$.

A p-adic L-function associated with f interpolates the special values $L^{\mathrm{alg}}(j+1, f \otimes \eta)$ as η varies over Dirichlet characters of p-power conductor. More precisely, let $\mathscr{X}^{(k)} \subset \mathscr{X}(\overline{\mathbf{Q}_p})$ denote the set of characters of the form $t \mapsto t^j \eta(t)$ with $0 \leq j \leq k-2$ and η nontrivial of finite order. Let $\mathbf{f}$ be either p-stabilization of f, with $U_p \mathbf{f} = \alpha_{\mathbf{f}} \mathbf{f}$, and let $\mathbf{Q}_p(\mathbf{f})$ be the extension of $\mathbf{Q}_p$ generated by $\mathbf{Q}(f)$ and $\alpha_{\mathbf{f}}$. There are then two natural p-adic L-functions

$$L_{p,\bullet}(\mathbf{f}) \in \mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\mathbf{f}),\ \bullet \in \{\mathrm{an}, \mathrm{alg}\}$$

associated with $\mathbf{f}$, which we call the *analytic* and *algebraic* p-adic L-functions. Both L-functions satisfy the interpolation formula

$$L_{p,\bullet}(\mathbf{f})(x^j \eta(x)) = \alpha_{\mathbf{f}}^{-n} p^{(j+1)n} L^{\mathrm{alg}}(j+1, f \otimes \eta^{-1})$$

for all $t^j \eta(t) \in \mathscr{X}^{(k)}$ (here p^n is the conductor of η) and they both have growth "of order at most $v_p(\alpha_{\mathbf{f}})$" in a certain technical sense. After the stimulating initial work of Mazur—Swinnerton-Dyer [MSD74], the analytic p-adic L-function was constructed by many people using the theory of modular symbols in various guises; the cleanest approach is via Stevens's theory of overconvergent modular symbols, as developed by Stevens, Pollack-Stevens, and Bellaïche [Ste94, PS11, PS13, Bel12]. The algebraic p-adic L-function was constructed by Kato as the image of a certain globally defined zeta element in Iwasawa cohomology under the Perrin-Riou regulator map.

According to a theorem of Višik [Viš76], any element of $\mathscr{O}(\mathscr{X})$ with growth of order less than $k-1$ is determined uniquely by its values at the points in $\mathscr{X}^{(k)}$. When $v_p(\alpha_{\mathbf{f}}) < k-1$, this immediately implies the equality $L_{p,\mathrm{an}}(\mathbf{f}) = L_{p,\mathrm{alg}}(\mathbf{f})$. On the other hand, suppose we are in the *critical slope case* where $v_p(\alpha_{\mathbf{f}}) = k-1$. Even though the two p-adic L-functions agree at all points in $\mathscr{X}^{(k)}$, Višik's theorem does not apply, and Pollack and Stevens have pointed out (Remark 9.7 of [PS13]) that comparing $L_{p,\mathrm{an}}$ and $L_{p,\mathrm{alg}}$ is a genuine question in this situation.

Theorem 1.2.1. *If $v_p(\alpha_{\mathbf{f}}) = k-1$ and $V_f|G_{\mathbf{Q}_p}$ is indecomposable, then $L_{p,\mathrm{an}}(\mathbf{f}) = L_{p,\mathrm{alg}}(\mathbf{f})$.*

To the best of our knowledge, this theorem is the first comparison of two p-adic L-functions which isn't an immediate consequence of Weierstrass preparation, Višik's theorem, or some other "soft" result in p-adic analysis.

It's natural to ask whether our proof of this theorem extends to the case where $V_f|G_{\mathbf{Q}_p}$ is split, or equivalently [BE10, Ber14] where $\mathbf{f}$ is θ-critical. If f satisfies one of these equivalent conditions, a notorious conjecture of Greenberg asserts that f is a CM form; assuming the truth of Greenberg's conjecture, Lei-Loeffler-Zerbes proved the equality $L_{p,\mathrm{an}}(\mathbf{f}) = L_{p,\mathrm{alg}}(\mathbf{f})$ in the θ-critical case [LLZ13]. Their argument relies on a comparison of Kato's Euler system with the Euler system of elliptic units, together with a formula of Bellaïche expressing $L_{p,\mathrm{an}}(\mathbf{f})$ for $\mathbf{f}$ a critical-slope CM form in terms of a Katz p-adic L-function. Unfortunately, without assuming the truth of Greenberg's conjecture, it's an open problem to show that either of the two functions $L_{p,\bullet}(\mathbf{f})$ is not *identically zero* when $\mathbf{f}$ is θ-critical! If these functions were both known to be nonzero, our arguments would extend to the θ-critical case as well; however, proving this nonvanishing seems to be very hard.

Ideally, the proof of this theorem would proceed as follows. Let U be a suitably small affinoid neighborhood of the point $x_{\mathbf{f}}$ corresponding to $\mathbf{f}$. The indecomposability of $V_f|G_{\mathbf{Q}_p}$ implies $x_{\mathbf{f}}$ is a noble point [Bel12]. By the work of many authors (cf. ibid.), any choice of a section $\gamma \in \mathscr{V}(U)$ as above gives rise to a canonically defined two-variable interpolation of $L_{p,\mathrm{an}}$, i.e. a function $\mathbf{L}_{p,\mathrm{an}} \in \mathscr{O}(U \times \mathscr{X})$ with the following properties:

- $\mathbf{L}_{p,\mathrm{an},x} = L_{p,\mathrm{an}}(\mathbf{f}_x)$ for all noble points $x \in U(\overline{\mathbf{Q}_p})$, and

- $\mathbf{L}_{p,\mathrm{an},x}$ has growth of order at most $v_p(\alpha_x)$ for any $x \in U(\overline{\mathbf{Q}_p})$.

Suppose we could construct an analogous function $\mathbf{L}_{p,\mathrm{alg}}$ interpolating $L_{p,\mathrm{alg}}$ and with exactly the same growth properties as $\mathbf{L}_{p,\mathrm{an}}$. By the remarks immediately preceding Theorem 1.2.1, for any noble point x of noncritical slope the difference

$$\mathbf{L}_{p,\mathrm{an},x} - \mathbf{L}_{p,\mathrm{alg},x} = L_{p,\mathrm{an}}(\mathbf{f}_x) - L_{p,\mathrm{alg}}(\mathbf{f}_x)$$

is identically zero in $\mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} E_x$, so the Zariski-density of such points in U implies the equality $\mathbf{L}_{p,\mathrm{an}} = \mathbf{L}_{p,\mathrm{alg}}$. Specializing this equality at our original point $x_{\mathbf{f}}$ implies Theorem 1.2.1. Of course, the existence of $\mathbf{L}_{p,\mathrm{alg}}$ is the really nontrivial ingredient in this argument. We don't actually construct $\mathbf{L}_{p,\mathrm{alg}}$ on the nose in this paper, but we do construct a function close enough in behavior that a slightly modified version of the above argument goes through.

To give this construction, we introduce a regulator map which interpolates the Perrin-Riou regulator map over $\mathscr{C}$. Let $\mathscr{R}$ be the sheaf of relative Robba rings over $\mathscr{C}$, so $\mathscr{R}(U) \cong \mathscr{R}_{\mathbf{Q}_p} \widehat{\otimes} \mathscr{O}(U)$ for any admissible open affinoid U. The sheaf $\mathscr{R}$ carries commuting $\mathscr{O}_{\mathscr{C}}$-linear actions of Γ and an operator φ. Set $\mathscr{V}_0 = \mathscr{V}(-1)$ and $\mathscr{V}_0^* = \mathscr{H}\mathrm{om}_{\mathscr{O}_{\mathscr{C}}}(\mathscr{V}_0, \mathscr{O}_{\mathscr{C}})$, so $\mathscr{V}_{0,x}^* \simeq V_{f_x^c}$ at any point. Let $\mathscr{D}^* = \mathbf{D}_{\mathrm{rig}}^{\dagger}(\mathscr{V}_0^* | G_{\mathbf{Q}_p})$ be the relative (φ, Γ)-module sheaf of $\mathscr{V}_0^*$ over $\mathscr{C}$, so $\mathscr{D}^*$ is a locally free $\mathscr{R}$-module of rank two with $\mathscr{R}$-semilinear actions of Γ and φ. Inside $\mathscr{D}^*$ we have the sheaf of crystalline periods defined by $\mathscr{D}_{\mathrm{crys}}^* = (\mathscr{D}^*)^{\Gamma=1, \varphi=\alpha^c}$. According to a beautiful result of Kedlaya-Pottharst-Xiao and Liu [KPX14, Liu14], $\mathscr{D}_{\mathrm{crys}}^*$ is a line bundle on $\mathscr{C}$, and for each noble point x there is a canonical identification

$$\mathscr{D}_{\mathrm{crys},x}^* = \mathbf{D}_{\mathrm{crys}}(\mathscr{V}_{0,x}^*)^{\varphi=\alpha_x^c}.$$

Theorem 1.2.2. *There is a canonical morphism of sheaves of Λ-modules*

$$\mathrm{Log} : \mathscr{H}_{\mathrm{Iw}}^1(\mathbf{Q}_p, \mathscr{V}_0) \to \mathscr{H}\mathrm{om}_{\mathscr{O}_{\mathscr{C}}}(\mathscr{D}_{\mathrm{crys}}^*, \mathscr{G})$$

which interpolates the Perrin-Riou regulator map on all noble points: given any affinoid $U \subset \mathscr{C}$, any $\mathbf{z} \in \mathscr{H}_{\mathrm{Iw}}^1(\mathbf{Q}_p, \mathscr{V}_0)(U)$, and any $v \in \mathscr{D}_{\mathrm{crys}}^*(U)$ we have

$$\mathrm{Log}(\mathbf{z})(v)_x = v_x^* \log_x(\mathbf{z}_x) \in \mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} E_x$$

for any noble point $x \in U(\overline{\mathbf{Q}_p})$, where

$$\log_x : H_{\mathrm{Iw}}^1(\mathbf{Q}_p, \mathscr{V}_{0,x}) \to \mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{crys}}(\mathscr{V}_{0,x})$$

is the Perrin-Riou regulator map and v_x^* is the linear functional induced by v_x via the natural E_x-bilinear pairing $\mathbf{D}_{\mathrm{crys}}(\mathscr{V}_{0,x}) \times \mathscr{D}_{\mathrm{crys},x}^* \to E_x$. Furthermore, $\mathrm{Log}(\mathbf{z})(v)_x$ has growth of order at most $v_p(\alpha_x)$ for any $x \in U(\overline{\mathbf{Q}_p})$.

We note that in the setting of Coleman families, Nuccio and Ochiai have also constructed this map by an argument which avoids (φ, Γ)-modules [NO16].

Now choose a pair of characters ν^+, ν^- as in §1.1, with conductors $A^{\pm}$ and with $\nu^{\pm}(-1) = \pm 1$, and set

$$\begin{aligned} \mathbf{L}_{\nu^+,\nu^-} &= e^+ \prod_{\ell | \frac{A^-}{(A^+,A^-)}} P_\ell(\ell^{-1}\sigma_\ell^{-1}) \cdot \mathrm{Log}(\mathrm{res}_p \mathfrak{z}_1(1, \nu^+)) \\ &\quad + e^- \prod_{\ell | \frac{A^+}{(A^+,A^-)}} P_\ell(\ell^{-1}\sigma_\ell^{-1}) \cdot \mathrm{Log}(\mathrm{res}_p \mathfrak{z}_1(1, \nu^-)) \end{aligned}$$

with P_ℓ as in Theorem 1.1. (Here $e^{\pm} = \frac{1 \pm \chi_{\mathscr{W}}(-1) \cdot \sigma_{-1}}{2} \in \Lambda$.)

Theorem 1.2.3. *For any noble point $x \in U(\overline{\mathbf{Q}_p}) \subset \mathscr{C}(\overline{\mathbf{Q}_p})$ and any section v of $\mathscr{D}^*_{\mathrm{crys}}(U)$, nonvanishing at x, we have*

$$\mathbf{L}_{\nu^+,\nu^-}(v)_x = C_{\nu^+,\nu^-}(v_x) \prod_{\ell | A^+A^-} P_\ell(\ell^{-1}\sigma_\ell^{-1})_x \cdot L_{p,\mathrm{alg}}(\mathbf{f}_x^c) \in \mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} E_x$$

where $C_{\nu^+,\nu^-}(v_x) = e^+C_{\nu^+}(v_x) + e^-C_{\nu^-}(v_x)$ with $C_{\nu^\pm}(v_x)$ a nonzero multiple of $L_{\mathrm{alg}}(k-1, f^c \otimes \nu^{-1})$.

The finite product here has the effect of "deleting the Euler factors" of $L_{p,\mathrm{alg}}(\mathbf{f}_x)$ at primes dividing A^+A^-.

1.3 A conjecture

With an eye towards an optimal version of Theorem 1.2.3, we offer the following conjecture.

Conjecture 1.3.1. *There is a canonical morphism of abelian sheaves*

$$\mathbf{z}_{\mathscr{C}} : \mathscr{V} \to \mathscr{H}^1_{\mathrm{Iw}}(\mathbf{Q}, \mathscr{V})$$

such that for any noble point $x \in \mathscr{C}(\overline{\mathbf{Q}_p})$ and any $\gamma \in U \subset \mathscr{C}$ we have

$$\mathbf{z}_{\mathscr{C}}(\gamma)_x = \mathbf{z}^{(p)}_{\gamma_x, \mathbf{f}_x}(k_x) \in H^1_{\mathrm{Iw}}(\mathbf{Q}, \mathscr{V}_x),$$

where $\mathbf{z}^{(p)}_{\bullet,\mathbf{f}}$ denotes Kato's "optimal zeta element"

$$\begin{array}{ccc} V_{\mathbf{f}} & \to & H^1_{\mathrm{Iw}}(\mathbf{Q}, V_{\mathbf{f}}) \\ \gamma & \mapsto & \mathbf{z}^{(p)}_{\gamma,\mathbf{f}} \end{array}$$

associated with a nobly refined form $\mathbf{f}$.

It seems likely that $\mathbf{z}_{\mathscr{C}}$ can be obtained by patching suitable linear combinations of the classes ${}_{c,d}\mathfrak{Z}_{N,1}(\mathfrak{U}, 0, a(A))_h$ defined in §3.2 below. We shall give some partial results towards this conjecture in [Han16].

Notation

We write $K_0(N)$, $K_1(N)$ and $K(N)$ for the usual adelic congruence subgroups of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$, and $\Gamma_0(N), \Gamma_1(N), \Gamma(N)$ for their intersections with $\mathrm{SL}_2(\mathbf{Z})$. If $N = p^n$ is a prime power we sometimes conflate the former groups with their projections under the natural map

$$\mathrm{pr}_p : \mathrm{GL}_2(\widehat{\mathbf{Z}}) \to \mathrm{GL}_2(\mathbf{Z}_p),$$

since $K = \mathrm{pr}_p^{-1}(\mathrm{pr}_p(K))$ in this case. Let $I = K_0(p) \subset \mathrm{GL}_2(\mathbf{Z}_p)$ be the subgroup consisting of matrices which are upper-triangular modulo p, and let $\Delta \subset \mathrm{GL}_2(\mathbf{Q}_p)$ be the multiplicative monoid given by

$$\Delta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p) \mid \det \neq 0,\ c \in p\mathbf{Z}_p \text{ and } a \in \mathbf{Z}_p^\times \right\}.$$

Note that Δ is generated by I together with the matrix $\mathrm{diag}(1, p)$.

Given $K \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ open of finite index, we define the associated open modular curve first as a complex analytic space by

$$Y(K)(\mathbf{C}) = \mathrm{GL}_2^+(\mathbf{Q})\backslash(\mathfrak{h} \times \mathrm{GL}_2(\mathbf{A}_f))/K.$$

This is a possibly disconnected Riemannian orbifold, and the determinant map

$$\begin{array}{rcl} Y(K)(\mathbf{C}) & \to & \mathbf{Q}_{>0}^\times\backslash\mathbf{A}_f^\times / \det(K)^\times \simeq \widehat{\mathbf{Z}}^\times / \det(K)^\times \\ \mathrm{GL}_2^+(\mathbf{Q})(z, g_f)K & \to & \mathbf{Q}_{>0}^\times \det g_f \det(K)^\times \end{array}$$

induces a bijection between the set of connected components of $Y(K)(\mathbf{C})$ and the finite group $\widehat{\mathbf{Z}}^\times / \det(K)^\times$. There is a canonical affine curve $Y(K)$ defined over $\mathbf{Q}$ such that $Y(K)(\mathbf{C})$ is the analytic space associated with $Y(K) \times_{\mathrm{Spec}\mathbf{Q}} \mathrm{Spec}\mathbf{C}$.

We write L, M for a pair of positive integers with $L + M \geq 5$. Let $Y(L, M)$ be the modular curve over $\mathbf{Q}$ representing the functor which sends a $\mathbf{Q}$-scheme S to the set of isomorphism classes of triples (E, e_1, e_2) where E/S is an elliptic curve and $e_1, e_2 \in E(S)$ are two sections of $E \to S$ such that $Le_1 = Me_2 = 0$ and the map

$$\begin{array}{rcl} (\mathbf{Z}/L\mathbf{Z}) \times (\mathbf{Z}/M\mathbf{Z}) & \to & E(S) \\ (a, b) & \mapsto & ae_1 + be_2 \end{array}$$

is an injective group homomorphism. If $L|L'$ and $M|M'$ there is a natural covering map

$$\begin{array}{rcl} Y(L', M') & \to & Y(L, M) \\ (E, e_1, e_2) & \mapsto & (E, \frac{L'}{L}e_1, \frac{M'}{M}e_2). \end{array}$$

For $(a, b) \in (\mathbf{Z}/L\mathbf{Z})^\times \times (\mathbf{Z}/M\mathbf{Z})^\times$, let $\langle a \mid b\rangle$ denote the automorphism of $Y(L, M)$ defined by $(E, e_1, e_2) \mapsto (E, ae_1, abe)$, and let $\langle a \mid b\rangle^*$ denote the induced pullback map on functions. Let T_n and T_n' denote the Hecke operators and dual Hecke operators as defined by Kato, so $T_\ell = T_\ell' \langle 1/\ell \mid \ell\rangle$ if $\ell \nmid M$. We set $Y_1(M) = Y(1, M)$ as usual, and write $\langle 1 \mid b\rangle = \langle b\rangle$ in this case (but not in general). Following Kato we write $Y_1(N) \otimes \mathbf{Q}(\zeta_m)$ for $Y_1(N) \times_{\mathrm{Spec}\mathbf{Q}} \mu_m^\circ$, where μ_m° denotes the scheme of primitive mth roots of unity.

We often denote "$\mathbf{Z}_p$-integral modules" with a superscripted $\circ$, and denote the outcome of applying $(-) \otimes_{\mathbf{Z}} \mathbf{Q}$ by removing the $\circ$. Sometimes we apply this convention in reverse: in particular, if M is a $\mathbf{Q}_p$-Banach module, we denote its unit ball by M°. If X is a reduced rigid analytic space, we write $\mathscr{O}(X)$ for the ring of global functions on X, $\mathscr{O}(X)^b$ for the ring of bounded global functions, and $\mathscr{O}(X)^\circ$ for the ring of power-bounded functions.

All Galois cohomology groups are taken to be continuous cohomology.

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2 Background on overconvergent cohomology

In this section we prove some foundational results on "étale overconvergent cohomology." As noted in the introduction, we work with small formal opens in the weight space, as opposed to the more familiar affinoid opens. However, there is a serious payoff for this slight complication: the filtrations which this point of view allows are so well-behaved that they completely obliterate any possible difficulties involving higher derived functors of inverse limits. The basic constructions and finiteness results in this section generalize with very little effort to yield a workable definition of étale overconvergent cohomology for any Shimura variety: this should be particularly attractive in the context of quaternionic Shimura curves and Kottwitz's simple Shimura varieties, where the middle-dimensional cohomology gives the "right" Galois representation.

2.1 Locally analytic distributions

Let s be a nonnegative integer. Consider the ring of functions

$$\mathbf{A}^s = \{f : \mathbf{Z}_p \to \mathbf{Q}_p \mid f \,\mathrm{analytic\,on\,each}\, p^s\mathbf{Z}_p - \mathrm{coset}\}\,.$$

Recall that by a fundamental result of Amice, the functions $e_j^s(x) = \lfloor p^{-s}j\rfloor! \begin{pmatrix} x \\ j \end{pmatrix}$ define an orthonormal basis of $\mathbf{A}^s$. The ring $\mathbf{A}^s$ is affinoid, and we set $\mathbf{B}_s = \mathrm{Sp}(\mathbf{A}^s)$, so e.g.

$$\mathbf{B}_s(\mathbf{C}_p) = \left\{ x \in \mathbf{C}_p \mid \inf_{a \in \mathbf{Z}_p} |x-a| \leq p^{-s} \right\}.$$

Let $\mathscr{W}$ be the Berthelot generic fiber (cf. [dJ95], §7) of the formal scheme $\mathfrak{W} = \mathrm{Spf}(\mathbf{Z}_p[[\mathbf{Z}_p^\times]])$, and let $\chi_{\mathscr{W}} : \mathbf{Z}_p^\times \to \mathscr{O}(\mathscr{W})^\times$ be the universal character induced by the inclusion of grouplike elements $\mathbf{Z}_p^\times \subset \mathbf{Z}_p[[\mathbf{Z}_p^\times]]^\times \subset \mathscr{O}(\mathscr{W})^\times$. The canonical splitting

$$\begin{array}{rcl} \mathbf{Z}_p^\times & = & \mu_{p-1} \times (1+p\mathbf{Z}_p)^\times \\ z & \mapsto & (\omega(z), \langle z \rangle) \end{array}$$

induces natural identifications $\mathscr{O}(\mathfrak{W}) = \mathbf{Z}_p[\mu_{p-1}] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[T]]$ and $\mathscr{O}(\mathscr{W}) = \mathbf{Z}_p[\mu_{p-1}] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \langle\langle T \rangle\rangle$, where $\mathbf{Q}_p \langle\langle T \rangle\rangle \subset \mathbf{Q}_p[[T]]$ is the ring of power series convergent on the open unit disk, by mapping the grouplike element $[1+p] \in \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ to $1+T$. The universal character is given by the convergent series

$$\chi_{\mathscr{W}}(z) = [\omega(z)] \otimes \sum_{n=0}^{\infty} T^n \begin{pmatrix} \frac{\log_p \langle z \rangle}{\log_p (1+p)} \\ n \end{pmatrix} \in \mathscr{O}(\mathscr{W}),\, z \in \mathbf{Z}_p^\times.$$

The pair $(\mathscr{W}, \chi_{\mathscr{W}})$ is universal for pairs (Ω, χ_Ω), where $\Omega = \mathrm{Sp}A$ is an affinoid space and $\chi_\Omega : \mathbf{Z}_p^\times \to \mathscr{O}(\Omega)^\times = A^\times$ is a continuous character; given any such pair, there is a unique morphism $f : \Omega \to \mathscr{W}$ such that $\chi_\Omega = f^*\chi_{\mathscr{W}}$, with f characterized by $f^*[\zeta_{p-1}] = \chi_\Omega(\zeta_{p-1})$ and $f^*T = \chi_\Omega(1+p) - 1$.

Clearly $\chi_{\mathscr{W}}$ factors through the universal character $\chi_{\mathfrak{W}} : \mathbf{Z}_p^\times \to \mathscr{O}(\mathfrak{W})^\times$, and $(\mathfrak{W}, \chi_{\mathfrak{W}})$ satisfies an analogous universal property for formal schemes. If $\lambda \in \mathscr{W}(\overline{\mathbf{Q}_p})$ is a point, we shall always denote the associated character by λ as well. We embed $\mathbf{Z}$ into $\mathscr{W}(\mathbf{Q}_p)$ (and into $\mathfrak{W}(\mathbf{Z}_p)$) by mapping k to the character $\lambda_k(z) = z^{k-2}$.

Now let $\mathfrak{U} \subset \mathfrak{W}$ be a formal subscheme of the form $\mathfrak{U} = \mathrm{Spf}(R_{\mathfrak{U}})$ with $R_{\mathfrak{U}}$ a $\mathbf{Z}_p$-flat, normal and module-finite $\mathbf{Z}_p[[X_1, \ldots, X_d]]$-algebra, such that the induced map $\mathfrak{U}^{\mathrm{rig}} \to \mathscr{W}$ on Berthelot generic fibers is an open immersion with image contained in an admissible affinoid open subset of $\mathscr{W}$. We write $\mathscr{O}(\mathfrak{U})$ interchangeably for $R_{\mathfrak{U}}$. Note that $\mathscr{O}(\mathfrak{U})$ is reduced, and that we have identifications $\mathscr{O}(\mathfrak{U}^{\mathrm{rig}})^b = \mathscr{O}(\mathfrak{U})[\frac{1}{p}]$ and $\mathscr{O}(\mathfrak{U}^{\mathrm{rig}})^\circ = \mathscr{O}(\mathfrak{U})$. We may choose an ideal $\mathfrak{a} \subset R_{\mathfrak{U}}$ containing p such that $R_{\mathfrak{U}}$ is $\mathfrak{a}$-adically separated and complete and each $R_{\mathfrak{U}}/\mathfrak{a}^n$ is a finite abelian group. Let $s_{\mathfrak{U}}$ be the least nonnegative integer such that $\chi_\Omega(1 + p^{s_{\mathfrak{U}}+1}x) \in \mathscr{O}(\Omega)\langle x \rangle$ for some admissible affinoid $\mathscr{W} \supset \Omega \Supset \mathfrak{U}^{\mathrm{rig}}$. Define

$$\begin{aligned} \mathscr{A}_{\mathfrak{U}}^{s,\circ} &= \mathscr{O}(\mathbf{B}_s \times \mathfrak{U}^{\mathrm{rig}})^\circ \\ &= \mathbf{A}^{s,\circ} \widehat{\otimes} R_{\mathfrak{U}} \end{aligned}$$

where the indicated completion is the $1 \otimes \mathfrak{a}$-adic completion. In the Amice basis we have

$$\mathscr{A}_{\mathfrak{U}}^{s,\circ} = \left\{ \sum_{j \geq 0} r_j e_j^s(x) \mid r_j \in R_{\mathfrak{U}} \,\mathrm{with}\, r_j \to 0 \, \mathfrak{a} - \mathrm{adically} \right\}.$$

The formula

$$(\gamma \cdot_{\mathfrak{U}} f)(x) = \chi_{\mathfrak{U}}(a + cx) f\left(\frac{b + dx}{a + cx}\right)$$

defines a continuous left action of Δ on $\mathscr{A}_{\mathfrak{U}}^{s,\circ}$ for any $s \geq s_{\mathfrak{U}}$. Set $\mathscr{D}_{\mathfrak{U}}^{s,\circ} = \mathrm{Hom}_{\mathscr{O}(\mathfrak{U})}(\mathscr{A}_{\mathfrak{U}}^{s,\circ}, \mathscr{O}(\mathfrak{U}))$, with the dual right action, and set

$$\begin{aligned} \mathscr{A}_{\mathfrak{U}}^{s} &= \mathscr{A}_{\mathfrak{U}}^{s,\circ} \otimes_{\mathbf{Z}} \mathbf{Q} \\ &\cong \mathscr{O}(\mathbf{B}_s \times \mathfrak{U}^{\mathrm{rig}})^b \end{aligned}$$

and

$$\begin{aligned} \mathscr{D}_{\mathfrak{U}}^{s} &= \mathscr{D}_{\mathfrak{U}}^{s,\circ} \otimes_{\mathbf{Z}} \mathbf{Q} \\ &\cong \mathcal{L}_{\mathscr{O}(\mathfrak{U}^{\mathrm{rig}})^b}\left(\mathscr{A}_{\mathfrak{U}}^{s}, \mathscr{O}(\mathfrak{U}^{\mathrm{rig}})^b\right). \end{aligned}$$

The following lemma quantifies the continuity of the Δ-action on these modules.

Lemma 2.1.1. *For any $s \geq s_{\mathfrak{U}}$ and $n \geq 1$, the principal congruence subgroup $K(p^{s+n}) \subset \mathrm{GL}_2(\mathbf{Z}_p)$ acts trivially on $\mathscr{A}_{\mathfrak{U}}^{s,\circ}/p^n$ and $\mathscr{D}_{\mathfrak{U}}^{s,\circ}/p^n$.*

Proof. Let $\mathcal{C}^s$ denote the Banach space of functions $F : I \to \mathbf{Q}_p$ which are analytic on each coset of $K(p^s)$, in the sense that

$$F\left(\gamma \begin{pmatrix} 1 + p^s X_1 & p^s X_2 \\ p^s X_3 & 1 + p^s X_4 \end{pmatrix}\right) \in \mathbf{Q}_p \langle X_1, X_2, X_3, X_4 \rangle$$

for any fixed $\gamma \in I$. Regard $\mathcal{C}^s$ as a left I-module via right translation. The key observation is that $K(p^{n+s})$ acts trivially on $\mathcal{C}^{s,\circ}/p^n$ and thus on $(\mathcal{C}^{s,\circ} \widehat{\otimes_{\mathbf{Z}_p}} \mathscr{O}(\mathfrak{U}))/p^n$. To see this, fix $\gamma \in I$ and $F \in$

$\mathcal{C}^{s,\circ}$, and for $g \in K(p^s)$ write $g = \begin{pmatrix} 1+p^sX_1(g) & p^sX_2(g) \\ p^sX_3(g) & 1+p^sX_4(g) \end{pmatrix}$. Define $\varphi_{F,\gamma} \in \mathbf{Z}_p\langle X_1, \ldots, X_4\rangle$ by

$$\varphi_{F,\gamma}(X_1(g), \ldots, X_4(g)) = F(\gamma g).$$

Then for any $h \in K(p^{s+n})$ we have

$$\sup_i |X_i(g) - X_i(gh)| \leq p^{-n},$$

so

$$\begin{aligned} |F(\gamma g) - F(\gamma gh)| &= |\varphi_{F,\gamma}(X_1(g), \ldots, X_4(g)) - \varphi_{F,\gamma}(X_1(gh), \ldots, X_4(gh))| \\ &\leq p^{-n} \end{aligned}$$

by Proposition 7.2.1/1 of [BGR84].

Let B^- denote the lower-triangular matrices in I. A simple calculation shows that the map $F \mapsto F\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ gives an I-equivariant isomorphism onto $\mathcal{A}_{\mathfrak{U}}^{s,\circ}$ from the subspace of functions $F \in \mathcal{C}^{s,\circ}\widehat{\otimes_{\mathbf{Z}_p}}\mathscr{O}(\mathfrak{U})$ satisfying

$$F\left(\begin{pmatrix} a & \\ c & d \end{pmatrix} g\right) = \chi_{\mathfrak{U}}(a)F(g)\,\forall \begin{pmatrix} a & \\ c & d \end{pmatrix} \in B^-.$$

This is a closed subspace of $\mathcal{C}^{s,\circ}\widehat{\otimes_{\mathbf{Z}_p}}\mathscr{O}(\mathfrak{U})$, and so $\mathcal{A}_{\mathfrak{U}}^{s,\circ}/p^n$ injects I-equivariantly into $(\mathcal{C}^{s,\circ}\widehat{\otimes_{\mathbf{Z}_p}}\mathscr{O}(\mathfrak{U}))/p^n$. $\square$

Next we construct the filtration on $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ described in the introduction. For any $s > s_{\mathfrak{U}}$, define $Q_{\mathfrak{U}}^{s,i}$ as the image of $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ in $\mathcal{D}_{\mathfrak{U}}^{s-1,\circ}/p^i$. Then $Q_{\mathfrak{U}}^{s,i}$ is a *finitely generated* $R_{\mathfrak{U}}/p^i$-submodule of $\mathcal{D}_{\mathfrak{U}}^{s-1,\circ}/p^i$, stable under Δ, and the natural map

$$\mathcal{D}_{\mathfrak{U}}^{s,\circ} \to \varprojlim_i Q_{\mathfrak{U}}^{s,i}$$

is an isomorphism. By Lemma 2.1.1, $K(p^{s+i})$ acts trivially on $Q_{\mathfrak{U}}^{s,i}$.

Definition 2.1.2. *We define*

$$\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}$$

as the kernel of the composite map

$$\mathcal{D}_{\mathfrak{U}}^{s,\circ} \to Q_{\mathfrak{U}}^{s,i} \to Q_{\mathfrak{U}}^{s,i} \otimes_{R_{\mathfrak{U}}/p^i} R_{\mathfrak{U}}/\mathfrak{a}^i.$$

To unwind this, note the canonical identification

$$\begin{aligned} \iota_{\mathfrak{U}}^s : \mathcal{D}_{\mathfrak{U}}^{s,\circ} &\cong \prod_{j\geq 0} R_{\mathfrak{U}} \\ \mu &\mapsto (\mu(e_j^s))_{j\geq 0}. \end{aligned}$$

Setting $c_j^s = \frac{\lfloor p^{1-s}j\rfloor!}{\lfloor p^{-s}j\rfloor!}$, we have $e_j^{s-1} = c_j^s e_j^s$, and the coefficient c_j^s is a p-adic integer tending uniformly to zero as $j \to \infty$, in fact with $v_p(c_j^s) = \lfloor p^{-s}j\rfloor$. Therefore we obtain

$$Q_{\mathfrak{U}}^{s,i} = \bigoplus_{j \text{ with } v_p(c_j^s)<i} R_{\mathfrak{U}}/p^{i-v_p(c_j^s)}$$

with only finitely many j's appearing, and thus

$$\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ} = \bigoplus_{j \text{ with } v_p(c_j^s)<i} R_{\mathfrak{U}}/(\mathfrak{a}^i, p^{i-v_p(c_j^s)}).$$

Taking the inverse limit over i, we easily see that $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ is separated and complete for the filtration defined by the Fil^i's, and that this filtration possesses all the other claimed properties.

Now suppose $k \in \mathbf{Z}_{\geq 2}$ and $\mathfrak{U}$ is such that $k \in \mathfrak{U}(\mathbf{Q}_p)$; let $\mathfrak{p}_k \subset \mathscr{O}(\mathfrak{U})$ be the prime cutting out k, so $R_{\mathfrak{U}}/\mathfrak{p}_k \cong \mathbf{Z}_p$. Let $\mathcal{A}_k^s$ be $\mathbf{A}^s$ with the left action

$$(\gamma \cdot_k f)(x) = (a+cx)^{k-2} f\left(\frac{b+dx}{a+cx}\right),$$

and let $\mathcal{D}_k^s$ be the continuous $\mathbf{Q}_p$-linear dual of $\mathcal{A}_k^s$, with unit ball $\mathcal{D}_k^{s,\circ}$. Analogously with $\mathcal{D}_{\mathfrak{U}}^s$, there is a natural isomorphism

$$\iota_k^s : \mathcal{D}_k^{s,\circ} \cong \prod_{j\geq 0} \mathbf{Z}_p.$$

The compatibility of $\iota_{\mathfrak{U}}^s$ and ι_k^s induces a natural Δ-equivariant surjection

$$\sigma_k : \mathcal{D}_{\mathfrak{U}}^{s,\circ} \twoheadrightarrow \mathcal{D}_k^{s,\circ}$$

factoring over the map $\mathcal{D}_{\mathfrak{U}}^{s,\circ} \to \mathcal{D}_{\mathfrak{U}}^{s,\circ} \otimes_{R_{\mathfrak{U}}} R_{\mathfrak{U}}/\mathfrak{p}_k$.

Let $\mathscr{L}_k(A)$ denote the module of polynomials $A[X]^{\deg \leq k-2}$ endowed with the right $\mathrm{GL}_2(A)$-action

$$(p \cdot_k \gamma)(X) = (d+cX)^{k-2} p\left(\frac{b+aX}{d+cX}\right),$$

and set in particular $\mathscr{L}_k = \mathscr{L}_k(\mathbf{Q}_p)$ and $\mathscr{L}_k^\circ = \mathscr{L}_k(\mathbf{Z}_p)$. By a simple calculation, the map

$$\begin{array}{rcl} \rho_k : \mathcal{D}_k^{s,\circ} & \to & \mathscr{L}_k^\circ \\ \mu & \mapsto & \displaystyle\int (X+x)^{k-2}\mu(x) \\ & & = \displaystyle\sum_{j=0}^{k-2} \binom{k-2}{j} \mu(x^j) X^{k-2-j} \end{array}$$

is Δ-equivariant.

Definition 2.1.3. *The* integration map in weight k *is the* Δ*-equivariant map* $i_k : \mathcal{D}_{\mathfrak{U}}^{s,\circ} \to \mathscr{L}_k^\circ$ *defined by* $i_k = \rho_k \circ \sigma_k$.

Lemma 2.1.4. *The map* i_k *fits into a* Δ*-equivariant commutative diagram*

$$\begin{array}{ccc} \mathcal{D}_{\mathfrak{U}}^{s,\circ} & \xrightarrow{i_k} & \mathscr{L}_k^\circ \\ \downarrow & & \downarrow \\ \mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i & \longrightarrow & \mathscr{L}_k^\circ/p^i \end{array}$$

compatibly with varying i, s *and* $\mathfrak{U}$ *in the evident manner.*

Proof. We observe that i_k can be realized as the composite map

$$\mathcal{D}_{\mathfrak{U}}^{s,\circ} \to \mathcal{D}_k^{s,\circ} \to \mathcal{D}_k^{s-1,\circ} \xrightarrow{\rho_k} \mathscr{L}_k^{\circ},$$

and the composition of the first two of these maps carries $\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ into $p^i\mathcal{D}_k^{s-1,\circ} \subset \mathcal{D}_k^{s-1,\circ}$. To see the latter, note that

$$\begin{aligned} \mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i \otimes_{R_{\mathfrak{U}}} R_{\mathfrak{U}}/\mathfrak{p}_k &= Q_{\mathfrak{U}}^{s,i} \otimes_{R_{\mathfrak{U}}} R_{\mathfrak{U}}/\mathfrak{p}_k \\ &\subset (\mathcal{D}_{\mathfrak{U}}^{s-1,\circ}/p^i) \otimes_{R_{\mathfrak{U}}} R_{\mathfrak{U}}/\mathfrak{p}_k \\ &= \mathcal{D}_k^{s-1,\circ}/p^i, \end{aligned}$$

where the first line follows from the fact that $\bar{\mathfrak{a}} \subset (p)$ in $R_{\mathfrak{U}}/\mathfrak{p}_k = \mathbf{Z}_p$. $\square$

The Dirac distribution

$$\mu_{\mathrm{Dir}} \in \mathcal{D}_{\mathfrak{U}}^{s,\circ}$$

defined by

$$\mu_{\mathrm{Dir}}(f) = f(0)$$

plays a key role in our constructions. Note that the various μ_{Dir}'s are carried to each other under the "change of s" or "change of $\mathfrak{U}$" maps, so we are justified in omitting $\mathfrak{U}$ and s from the notation. We also write μ_{Dir} for the image of μ_{Dir} in $\mathcal{D}_{\mathfrak{U}}^{s}$ or any quotient of $\mathcal{D}_{\mathfrak{U}}^{s,\circ}$.

Lemma 2.1.5. We have $i_k(\mu_{\mathrm{Dir}}) = X^{k-2}$.

2.2 Étale overconvergent cohomology

Let $Y = Y(K)$ be any modular curve with the p-part of K contained in I, so any of the modules introduced in §2.1 defines a local system on the analytic space $Y(\mathbf{C})$ associated with Y.

Lemma 2.2.1. *The natural map*

$$H^1(Y(\mathbf{C}), \mathcal{D}_{\mathfrak{U}}^{s,\circ}) \to \varprojlim_i H^1(Y(\mathbf{C}), \mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ})$$

is an isomorphism.

Proof. This map is surjective with kernel isomorphic to $\lim^1\left(H^0(Y(\mathbf{C}), \mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ})\right)$. By the basic finiteness properties of $\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}$, this is a projective system of finite abelian groups, and so its $\lim^1$ vanishes. $\square$

The modules $\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}$ define locally constant sheaves of finite abelian groups on the étale sites of $Y_{/\overline{\mathbf{Q}}}$ and Y. With the previous lemma in mind, we define

$$H^1_{\text{ét}}(Y_{/\overline{\mathbf{Q}}}, \mathcal{D}_{\mathfrak{U}}^{s,\circ}) = \varprojlim_i H^1_{\text{ét}}(Y_{/\overline{\mathbf{Q}}}, \mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^i\mathcal{D}_{\mathfrak{U}}^{s,\circ}).$$

This is isomorphic to $H^1(Y(\mathbf{C}), \mathcal{D}_{\mathfrak{U}}^{s,\circ})$ as an $R_{\mathfrak{U}}$-module and Hecke module, and is equipped additionally with a p-adically continuous[3] $R_{\mathfrak{U}}$-linear action of $G_{\mathbf{Q}}$ commuting with the Hecke action. If L, M is a pair of integers as earlier with $p|M$, or N is an integer prime to p, we define in particular

$$\begin{aligned} \mathbf{V}_{\mathfrak{U}}^{s,\circ}(L,M) &= H^1_{\text{ét}}(Y(L,M)_{/\overline{\mathbf{Q}}}, \mathcal{D}_{\mathfrak{U}}^{s,\circ})(2), \\ \mathbf{V}_{\mathfrak{U}}^{s,\circ}(N) &= \mathbf{V}_{\mathfrak{U}}^{s,\circ}(1, Np). \end{aligned}$$

[3] In the weakest possible sense: the $G_{\mathbf{Q}}$-action on $H^1_{\text{ét}}(Y_{/\overline{\mathbf{Q}}}, \mathcal{D}_{\mathfrak{U}}^{s,\circ}) \otimes_{\mathbf{Z}} \mathbf{Z}/p^n\mathbf{Z}$ factors through a finite quotient of $G_{\mathbf{Q}}$ for any n, but please note we are *not* claiming this module is separated or complete for the p-adic topology!

We also define

$$V_k^\circ(L,M) = H^1_{\text{ét}}(Y(L,M)_{/\overline{\mathbf{Q}}}, \mathscr{L}_k^\circ)(2-k).$$

Note that $\mathscr{L}_k^\circ/p^n$ is naturally isomorphic to the etale sheaf $\mathrm{sym}^{k-2}T_pE/p^n$. By Lemma 2.1.4, the map $\mathcal{D}_{\mathfrak{U}}^{s,\circ} \to \mathscr{L}_k^\circ/p^j$ factors through a map $\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^j\mathcal{D}_{\mathfrak{U}}^{s,\circ} \to \mathscr{L}_k^\circ/p^j$, which in the inverse limit induces maps

$$\mathbf{V}_{\mathfrak{U}}^{s,\circ}(L,M) \to V_k^\circ(L,M)(k) = H^1_{\text{ét}}(Y(L,M)_{/\overline{\mathbf{Q}}}, \mathscr{L}_k^\circ)(2)$$

and

$$\mathbf{V}_{\mathfrak{U}}^{s,\circ}(N) \to V_k^\circ(Np)(k).$$

Before continuing, we need some results about slope decompositions. Recall the notion of an *augmented Borel-Serre complex* of level K from [Han14]: this is a functor $C^\bullet(-) = C^\bullet(K,-)$ from right K-modules to complexes of abelian groups with various good properties, such that $H^*(C^\bullet(M)) \cong H^*(Y(K)(\mathbf{C}), M)$. In particular, $C^i(M) = M^{\oplus r(i)}$ with $r(i)$ independent of M and zero for $i \notin [0,2]$, and $C^\bullet(M)$ inherits any additional structures carried by M which are compatible with the K-action; furthermore, if the K-action on M extends to a right action of some monoid S with $K \subset \mathrm{S} \subset \mathrm{GL}_2(\mathbf{A}_f)$, any Hecke operator $T \in \mathscr{C}_c^\infty(K\backslash\backslash \mathrm{S}, \mathbf{Z})$ lifts to an endomorphism $\tilde{T} \in \mathrm{End}_{\mathrm{Ch}}(C^\bullet(M))$.

Now take $K = K_1(Np)$, and suppose $(\mathfrak{U}, h)$ is a *slope datum* in the sense of [Han14], i.e. we may choose an augmented Borel-Serre complex such that $C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)$ admits a slope-$\leq h$ decomposition

$$C^\bullet(\mathcal{D}_{\mathfrak{U}}^s) \cong C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)_h \oplus C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)^h$$

with respect to the action of $\tilde{U}_p$. Note that even though $\mathcal{D}_{\mathfrak{U}}^s$ is not quite orthonormalizable, all the Riesz theory arguments of [Buz07, Col97] still go through (cf. [AIS13] and [CHJ16]): the determinant $\det(1-\tilde{U}_pX | C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)$ is a well-defined Fredholm series $F_{\mathfrak{U}}(X) \in \mathscr{O}(\mathfrak{U})\{\{X\}\}$ defined independently of s, and $C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)$ admits a slope-$\leq h$ decomposition if and only if $F_{\mathfrak{U}}(X)$ admits a slope-$\leq h$ factorization $F_{\mathfrak{U}}(X) = Q_{\mathfrak{U},h}(X)R(X)$, in which case $C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)_h = \ker Q^*_{\mathfrak{U},h}(\tilde{U}_p)$.[4] Taking the cohomology of this decomposition yields a Hecke- and Galois-stable direct sum decomposition $\mathbf{V}_{\mathfrak{U}}^s(N) = \mathbf{V}_{\mathfrak{U},h}^s(N) \oplus \mathbf{V}_{\mathfrak{U}}^s(N)^h$, and $\mathbf{V}_{\mathfrak{U},h}^s(N)$ is a finite projective $\mathscr{O}(\mathfrak{U})[\frac{1}{p}]$-module. We define $\mathbf{V}_{\mathfrak{U},h}^{s,\circ}(N)$ as the image of $\mathbf{V}_{\mathfrak{U}}^{s,\circ}(N)$ in $\mathbf{V}_{\mathfrak{U},h}^s(N)$, so $\mathbf{V}_{\mathfrak{U},h}^s(N) = \mathbf{V}_{\mathfrak{U},h}^{s,\circ}(N) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. The following proposition is a key finiteness result.

Proposition 2.2.2. *The $\mathscr{O}(\mathfrak{U})$-module $\mathbf{V}_{\mathfrak{U},h}^{s,\circ}(N)$ is finitely presented, and in particular is separated and complete for the $\mathfrak{a}$-adic topology, and thus for the p-adic topology as well.*

Proof. Fix the Borel-Serre complex $C^\bullet(-) = C^\bullet(Y_1(Np), -)$ for which $(\mathfrak{U}, h)$ is a slope datum, and let $Z^1(-)$ denote the cocycles in degree 1. The direct sum decomposition $C^\bullet(\mathcal{D}_{\mathfrak{U}}^s) = C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)_h \oplus C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)^h$ induces an analogous decomposition of the cocycles. We have the commutative diagram

$$\begin{tikzcd}
Z^1(\mathcal{D}_{\mathfrak{U}}^s)_h \arrow[r, two heads] \arrow[d, "i_h"', shift right] & H^1(\mathcal{D}_{\mathfrak{U}}^s)_h = \mathbf{V}_{\mathfrak{U},h}^s(N) \arrow[d, "i_h"', shift right] \\
Z^1(\mathcal{D}_{\mathfrak{U}}^s) \arrow[r, two heads] \arrow[u, bend right, "\mathrm{pr}_h"'] & H^1(\mathcal{D}_{\mathfrak{U}}^s) = \mathbf{V}_{\mathfrak{U}}^s(N) \arrow[u, bend right, "\mathrm{pr}_h"'] \\
Z^1(\mathcal{D}_{\mathfrak{U}}^{s,\circ}) \arrow[r, two heads] \arrow[u] & H^1(\mathcal{D}_{\mathfrak{U}}^{s,\circ}) = \mathbf{V}_{\mathfrak{U}}^{s,\circ}(N) \arrow[u]
\end{tikzcd}$$

[4] Here $Q_{\mathfrak{U},h}(X) \in \mathscr{O}(\mathfrak{U})[X]$ has leading coefficient a unit in $\mathscr{O}(\mathfrak{U})[\frac{1}{p}]$, and $Q^*(X) = X^{\deg Q}Q(1/X)$.

where i_h (resp. pr_h) denotes the natural inclusion (resp. projection). Choose elements $m_1, \dots, m_g \in \mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)$ which generate $\mathbf{V}^{s}_{\mathfrak{U},h}(N)$ as an $\mathscr{O}(\mathfrak{U})[\frac{1}{p}]$-module. Let M be the $\mathscr{O}(\mathfrak{U})$-submodule of $\mathbf{V}^{s}_{\mathfrak{U},h}(N)$ generated by $m_1, \dots, m_g$, and set $Q = \mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)/M$. Clearly any element of Q is killed by a finite power of p, and it's enough to show that Q has finite exponent $p^e < \infty$, since then $\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N) \subseteq p^{-e}M$ as $\mathscr{O}(\mathfrak{U})$-modules and $p^{-e}M$ is a finite $\mathscr{O}(\mathfrak{U})$-module by construction.

Suppose Q does not have finite exponent, so we may choose cohomology classes $c_1, c_2, \dots, c_n, \dots \in \mathbf{V}^{s,\circ}_{\mathfrak{U}}(N)$ such that the image of $\mathrm{pr}_h(c_n)$ in Q has exponent p^n. The module $\mathbf{V}^{s}_{\mathfrak{U},h}(N)$ is naturally a p-adic Banach space, and the sequence $\mathrm{pr}_h(c_n)$ is unbounded. Now choose any cocycles $z_n \in Z^1(\mathcal{D}^{s,\circ}_{\mathfrak{U}})$ lifting the c_n's. The sequence of z_n's is clearly bounded in the Banach topology on $Z^1(\mathcal{D}^{s}_{\mathfrak{U}})$, and $\mathrm{pr}_h : Z^1(\mathcal{D}^{s}_{\mathfrak{U}}) \to Z^1(\mathcal{D}^{s}_{\mathfrak{U}})_h$ is a continuous map of p-adic Banach spaces, so the sequence of cocycles $\mathrm{pr}_h(z_n)$ is bounded as well. But the topmost horizontal arrow in the diagram is a continuous map of p-adic Banach spaces carrying $\mathrm{pr}_h(z_n)$ to $\mathrm{pr}_h(c_n)$, so we have a contradiction. □

The map $\mathbf{V}^{s+1}_{\mathfrak{U}}(N) \to \mathbf{V}^{s}_{\mathfrak{U}}(N)$ induces canonical *isomorphisms* $\mathbf{V}^{s+1}_{\mathfrak{U},h}(N) \cong \mathbf{V}^{s}_{\mathfrak{U},h}(N)$, and we write $\mathscr{V}_{\mathfrak{U},h}(N)$ for the inverse limits along these isomorphisms.

Proposition 2.2.3. *The modules $\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)$ and $\mathbf{V}^{s}_{\mathfrak{U},h}(N)$ are p-torsion-free, and their natural $G_{\mathbf{Q}}$-actions are p-adically continuous.*

Proof. Immediate. □

2.3 Reconstructing the eigencurve

Let $\mathbf{B}[r] = \mathrm{Sp}\mathbf{Q}_p \langle p^r X \rangle$ be the rigid disk of radius p^r, with $\mathbf{A}^1 = \cup_r \mathbf{B}[r]$. Now let $F \in \mathscr{O}(\mathfrak{W})\{\{X\}\}$ be a Fredholm series, with $\mathscr{Z} \subset \mathscr{W} \times \mathbf{A}^1$ the associated Fredholm hypersurface. For any admissible open $U \subset \mathscr{W}$ and any $h \in \mathbf{Q}$, $U \times \mathbf{B}[h]$ is admissible open in $\mathscr{W} \times \mathbf{A}^1$, and we define an admissible open subset $\mathscr{Z}_{U,h} \subset \mathscr{Z}$ by $\mathscr{Z}_{U,h} = \mathscr{Z} \cap (U \times \mathbf{B}[h])$. If U is affinoid then $\mathscr{Z}_{U,h}$ is affinoid as well (and in fact $\mathscr{Z}_{U,h} = \left(\mathrm{Sp}\mathscr{O}(U) \left\langle p^h X \right\rangle / (F(X))\right)$). We say $\mathscr{Z}_{U,h}$ is *slope-adapted* if the natural map $\mathscr{Z}_{U,h} \to U$ is finite and flat.

Lemma 2.3.1 (Buzzard)**.** *There is an admissible covering of $\mathscr{Z}$ by slope-adapted affinoids $\mathscr{Z}_{U,h}$.*

Proof. This is Theorem 4.6 of [Buz07]. □

Lemma 2.3.2 (Buzzard)**.** *If $U \subset \mathscr{W}$ is an affinoid subdomain and $h \in \mathbf{Q}$ with $\mathscr{Z}_{U,h}$ slope-adapted, we can choose an affinoid subdomain $V \subset \mathscr{W}$ and an $h' \geq h$ such that $U \Subset V$, $\mathscr{Z}_{V,h'}$ is slope-adapted, and $\mathscr{Z}_{V,h'} \cap (U \times \mathbf{A}^1) = \mathscr{Z}_{U,h}$.*

Proof. Immediate from Lemma 4.5 of [Buz07]. Note that $\mathscr{Z}_{U,h} = \mathscr{Z}_{U,h'}$. □

Lemma 2.3.3. *For some index set I, we may choose pairs of affinoid subdomains $U'_i \Subset U_i$ and rationals h_i such that $\mathscr{Z}_{U_i,h_i} \cap (U'_i \times \mathbf{A}^1) = \mathscr{Z}_{U'_i,h_i}$ and both $(\mathscr{Z}_{U_i,h_i})_{i \in I}$ and $(\mathscr{Z}_{U'_i,h_i})_{i \in I}$ are admissible coverings of $\mathscr{Z}$ by slope-adapted affinoids.*

Proof. Immediate from the previous two lemmas. □

Lemma 2.3.4. *Notation as in the previous lemma, we may choose formal opens $\mathfrak{U}_i \subset \mathfrak{W}$ such that $U'_i \subset \mathfrak{U}_i^{\mathrm{rig}} \subset U_i$ and such that $(\mathscr{Z}_{\mathfrak{U}_i^{\mathrm{rig}},h_i})_{i \in I}$ is an admissible covering of $\mathscr{Z}$ by slope-adapted admissible opens.*

Proof. For each i, choose a finite map $f_i^* : \mathbf{Q}_p \langle X_1, \dots, X_d \rangle \to \mathscr{O}(U_i)$ for some d such that the associated map $f_i : U_i \to \mathrm{Sp}\mathbf{Q}_p \langle X_1, \dots, X_d \rangle$ carries U_i' into $\mathrm{Sp}\mathbf{Q}_p \langle p^{-r}X_1, \dots, p^{-r}X_d \rangle$ for some $r \in \mathbf{Q}_{>0}$. The existence of such a map is the definition of the relative compacity $U_i' \Subset U_i$. By Corollary 6.4.1/6 of [BGR84], $\mathscr{O}(U_i)^\circ$ is module-finite over $\mathbf{Z}_p \langle X_1, \dots, X_d \rangle$. Now let

$$R_i' = \left(\mathbf{Z}_p[[X_1, \dots, X_d]] \otimes_{\mathbf{Z}_p \langle X_1, \dots, X_d \rangle, f_i^*} \mathscr{O}(U_i)^\circ\right) / p - \text{power torsion}$$

and let R_i be the normalization of R_i'; setting $\mathfrak{U}_i = \mathrm{Spf}(R_i)$ gives a formal scheme of the desired type with $\mathfrak{U}_i^{\mathrm{rig}} \subset U_i$, and the map $\mathscr{O}(U_i) \to \mathscr{O}(U_i')$ clearly factors over a map $R_i[\frac{1}{p}] \to \mathscr{O}(U_i')$ (since $R_i'[\frac{1}{p}] \cong R_i[\frac{1}{p}]$ and the X_i's are topologically nilpotent in $\mathscr{O}(U_i')$ by design), so $U_i' \subset \mathfrak{U}_i^{\mathrm{rig}}$ as desired. $\square$

Notation as in the previous subsection, fix a choice of augmented Borel-Serre complex, and let $F \in \mathscr{O}(\mathfrak{W})\{\{X\}\}$ be the Fredholm series such that $F|_{\mathfrak{U}} = \det(1 - \tilde{U}_p X)|C^\bullet(\mathcal{D}_{\mathfrak{U}}^s)$ for all $\mathfrak{U} \subset \mathfrak{W}$. Let $\mathscr{Z} \subset \mathscr{W} \times \mathbf{A}^1$ be the associated Fredholm hypersurface. Given any slope-adapted admissible open $\mathscr{Z}_{\mathfrak{U}^{\mathrm{rig}},h} \subset \mathscr{Z}$, we have a natural identification $\mathscr{O}(\mathscr{Z}_{\mathfrak{U}^{\mathrm{rig}},h}) = \mathscr{O}(\mathfrak{U}^{\mathrm{rig}})[X]/Q_{\mathfrak{U},h}(X)$, and the $\mathscr{O}(\mathfrak{U}^{\mathrm{rig}})$-module

$$\mathscr{V}_{\mathfrak{U}^{\mathrm{rig}},h}(N) := \mathscr{V}_{\mathfrak{U},h}(N) \otimes_{\mathscr{O}(\mathfrak{U})[\frac{1}{p}]} \mathscr{O}(\mathfrak{U}^{\mathrm{rig}})$$

has a natural structure as an $\mathscr{O}(\mathscr{Z}_{\mathfrak{U}^{\mathrm{rig}},h})$-module, by letting X act as U_p^{-1}. These actions are compatible as $\mathfrak{U}$ and h vary, and in particular we may glue the $\mathscr{V}_{\mathfrak{U}^{\mathrm{rig}},h}(N)$'s over a chosen admissible covering of the type provided by Lemma 2.3.4 into a coherent sheaf $\mathscr{V}(N)$ over $\mathscr{Z}$ such that $\mathscr{V}(N)(\mathscr{Z}_{\mathfrak{U}^{\mathrm{rig}},h}) \cong \mathscr{V}_{\mathfrak{U}^{\mathrm{rig}},h}(N)$. For any constituent $\mathscr{Z}_{\mathfrak{U}^{\mathrm{rig}},h}$ of our chosen covering, let $\mathbf{T}_{\mathfrak{U},h}(N)$ denote the image of $\mathscr{O}(\mathfrak{U})[\frac{1}{p}] \otimes_{\mathbf{Z}} \mathbf{T}$ in $\mathrm{End}_{\mathscr{O}(\mathfrak{U})[\frac{1}{p}]}(\mathscr{V}_{\mathfrak{U},h}(N))$. This is clearly a module-finite $\mathscr{O}(\mathfrak{U})[\frac{1}{p}]$-algebra, and we have natural quasi-Stein rigid spaces $\mathscr{C}_{\mathfrak{U}^{\mathrm{rig}},h}(N)$, finite over $\mathscr{Z}_{\mathfrak{U}^{\mathrm{rig}},h}$, such that

$$\mathscr{O}(\mathscr{C}_{\mathfrak{U}^{\mathrm{rig}},h}(N)) \cong \mathbf{T}_{\mathfrak{U},h}(N) \otimes_{\mathscr{O}(\mathfrak{U})[\frac{1}{p}]} \mathscr{O}(\mathfrak{U}^{\mathrm{rig}}).$$

We finally obtain the eigencurve $\mathscr{C}(N)$ by gluing these rigid spaces over our chosen covering of $\mathscr{Z}$. Since the $\mathscr{V}_{\mathfrak{U}^{\mathrm{rig}},h}(N)$'s are naturally finite $\mathscr{O}(\mathscr{C}_{\mathfrak{U}^{\mathrm{rig}},h}(N))$-modules, we may glue them into a coherent sheaf $\mathscr{V}(N)$ over $\mathscr{C}(N)$ (under the spectral projection $\pi : \mathscr{C}(N) \to \mathscr{Z}$ we have $\pi_*\mathscr{V}(N) = \mathscr{V}(N)$, with a slight abuse of notation). The Galois action on each $\mathscr{V}_{\mathfrak{U}^{\mathrm{rig}},h}(N)$ glues into an action on $\mathscr{V}(N)$. Finally, for any number field K and any finite set of places S of K containing all places dividing $Np\infty$, the Galois cohomology modules $H^i(G_{K,S}, \mathscr{V}_{\mathfrak{U}^{\mathrm{rig}},h}(N))$ glue (cf. [Pot13]) into a coherent sheaf $\mathscr{H}^i(G_{K,S}, \mathscr{V}(N))$ over $\mathscr{C}(N)$.

Proposition 2.3.5. *The coherent sheaf $\mathscr{V}(N)$ over $\mathscr{C}(N)$ is torsion-free, and $\mathscr{V}(N)|_{\mathscr{C}_0^{M-\mathrm{new}}(N)}$ has generic rank $2\tau(N/M)$, where $\tau(A)$ denotes the number of divisors of A. In particular, the pullback of $\mathscr{V}(N)$ to the normalization $\mathscr{C}$ of $\mathscr{C}_0^{N-\mathrm{new}}(N)$ is locally free of rank two.*

Proof (sketch). Torsion-freeness follows from the torsion-freeness of $H^1(Y_1(Np), \mathcal{D}_{\mathfrak{U}}^s)$ over $\mathscr{O}(\mathfrak{U})[\frac{1}{p}]$, which in turn follows from the vanishing of $H^0(Y_1(Np)(\mathbf{C}), \mathcal{D}_\lambda^s)$ (here $\lambda \in \mathscr{W}$ is an arbitrary weight) by examining the long exact sequence in cohomology for

$$0 \to \mathcal{D}_{\mathfrak{U}}^s \xrightarrow{\cdot a} \mathcal{D}_{\mathfrak{U}}^s \to \mathcal{D}_\lambda^s \to 0,$$

where e.g. $a \in \mathscr{O}(\mathfrak{U})[\frac{1}{p}]$ is a prime element with zero locus λ. The statements about ranks follow from an straightforward analysis of the specializations of $\mathscr{V}(N)$ at crystalline points of conductor M and non-critical slope, by combining Stevens's control theorem, the Eichler-Shimura isomorphism, and basic newform theory. $\square$

Let $\mathbf{T}^{s,\circ}_{\mathfrak{U},h}(N)$ denote the image of $\mathbf{T} \otimes_{\mathbf{Z}} \mathscr{O}(\mathfrak{U})$ in $\mathrm{End}_{\mathscr{O}(\mathfrak{U})}\left(\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)\right)$. The ring $\mathbf{T}^{s,\circ}_{\mathfrak{U},h}(N)$ is a $\mathbf{Z}_p$-flat complete semilocal Noetherian ring of dimension two. Set $\mathfrak{C}^s_{\mathfrak{U},h}(N) = \mathrm{Spf}\mathbf{T}^{s,\circ}_{\mathfrak{U},h}(N)$. Since $\mathbf{T}^{s,\circ}_{\mathfrak{U},h}(N)[\frac{1}{p}] \cong \mathbf{T}_{\mathfrak{U},h}(N)$, the generic fiber of $\mathfrak{C}^s_{\mathfrak{U},h}(N)$ recovers $\mathscr{C}_{\mathfrak{U}^{\mathrm{rig}},h}(N)$. Is it possible to glue the formal schemes $\mathfrak{C}^s_{\mathfrak{U},h}(N)$, and the Galois representations $\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)$ over them, in any meaningful way?

3 Big zeta elements

Throughout this section, we let L, M denote a pair of positive integers with $p|M$, and let N denote a positive integer with $p \nmid N$. The main reference for this section is §8 of [Kat04]. We largely omit proofs in this section, but this seems a reasonable decision to us: these proofs are (unsurprisingly) given by cross-pollinating the proofs in [Kat04] with Theorem 3.1.1 below.

3.1 Chern class maps and zeta elements

Theorem 3.1.1. *There is a canonical family of maps*

$$\mathrm{Ch}^s_{L,M}(\mathfrak{U}, r) : \varprojlim_j K_2(Y(Lp^j, Mp^j)) \to H^1\left(G_{\mathbf{Q}}, \mathbf{V}^{s,\circ}_{\mathfrak{U}}(L,M)(-r)\right)$$

with the following properties.

i. *The diagram*

$$\begin{tikzcd} \lim K_2(Y(Lp^j, Mp^j)) \arrow[r, "\mathrm{Ch}^{s+1}_{L,M}(\mathfrak{U},r)"] \arrow[dr, "\mathrm{Ch}^s_{L,M}(\mathfrak{U}',r)"'] & H^1\left(G_{\mathbf{Q}}, \mathbf{V}^{s+1,\circ}_{\mathfrak{U}}(L,M)(-r)\right) \arrow[d] \\ & H^1\left(G_{\mathbf{Q}}, \mathbf{V}^{s,\circ}_{\mathfrak{U}'}(L,M)(-r)\right) \end{tikzcd}$$

commutes for any $\mathfrak{U}' \subseteq \mathfrak{U}$ and $s \geq s_{\mathfrak{U}}$, where the vertical arrow is induced by the map $\mathcal{D}^{s+1,\circ}_{\mathfrak{U}} \to \mathcal{D}^{s,\circ}_{\mathfrak{U}} \to \mathcal{D}^{s,\circ}_{\mathfrak{U}'}$.

ii. *For any $k \in \mathbf{Z}_{\geq 2} \cap \mathfrak{U}(\mathbf{Q}_p)$, the diagram*

$$\begin{tikzcd} \lim K_2(Y(Mp^j, Np^j)) \arrow[r, "\mathrm{Ch}^{s}_{L,M}(\mathfrak{U},r)"] \arrow[dr, "\mathrm{Ch}_{L,M}(k,r,k-1)"'] & H^1\left(G_{\mathbf{Q}}, \mathbf{V}^{s,\circ}_{\mathfrak{U}}(L,M)(-r)\right) \arrow[d] \\ & H^1\left(G_{\mathbf{Q}}, V^{\circ}_{k}(L,M)(k-r)\right) \end{tikzcd}$$

commutes, where $\mathrm{Ch}_{L,M}(k, r, r')$ is Kato's Chern class map and the vertical arrow is induced by the integration map in weight k.

Proof. For brevity, set $Y_j = Y(Lp^j, Mp^j)$. For some large fixed m, the diagram

$$\begin{tikzcd}
\varprojlim K_2(Y_{j+m}) \arrow[d] & \\
\varprojlim H^2_{\text{ét}}\left(Y_{j+m}, (\mathbf{Z}/p^j)(2)\right) \arrow[d, "\cup X^{k-2}\otimes\zeta_{p^j}^{\otimes -r}"] \arrow[dr, "\cup\mu_{\mathrm{Dir}}\otimes\zeta_{p^{j+m}}^{\otimes -r}"] & \\
\varprojlim H^2_{\text{ét}}\left(Y_{j+m}, (\mathscr{L}_k^\circ/p^j)(2-r)\right) \arrow[d, "\mathrm{tr}"] & \varprojlim H^2_{\text{ét}}\left(Y_{j+m}, (\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^{j+m-s}\mathcal{D}_{\mathfrak{U}}^{s,\circ})(2-r)\right) \arrow[d, "\mathrm{tr}"] \\
\varprojlim H^2_{\text{ét}}\left(Y(L,M), (\mathscr{L}_k^\circ/p^j)(2-r)\right) \arrow[d, "d"] & \varprojlim H^2_{\text{ét}}\left(Y(L,M), (\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^{j+m-s}\mathcal{D}_{\mathfrak{U}}^{s,\circ})(2-r)\right) \arrow[d, "d"] \\
\varprojlim H^1\left(G_{\mathbf{Q}}, H^1_{\text{ét}}\left(Y(L,M)_{/\overline{\mathbf{Q}}}, (\mathscr{L}_k^\circ/p^j)\right)(2-r)\right) \arrow[d, "\wr"] & \varprojlim H^1\left(G_{\mathbf{Q}}, H^1_{\text{ét}}\left(Y(L,M)_{/\overline{\mathbf{Q}}}, \mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^{j+m-s}\mathcal{D}_{\mathfrak{U}}^{s,\circ}\right)(2-r)\right) \arrow[d, "\wr"] \\
H^1\left(G_{\mathbf{Q}}, V_k^\circ(L,M)(k-r)\right) & H^1\left(G_{\mathbf{Q}}, \mathbf{V}_{\mathfrak{U}}^{s,\circ}(L,M)(-r)\right) \arrow[l, "i_k"']
\end{tikzcd}$$

commutes for any $k \in \mathfrak{U} \cap \mathbf{Z}_{\geq 2}$. Here all inverse limits are taken over j, the topmost vertical arrow is the étale Chern class map explained in Kato, the arrows labeled "tr" are induced by the trace map on cohomology, and the arrows labeled "d" are the edge maps in the Hochschild-Serre spectral sequence. Crucially, the diagonal arrow is well-defined, since by Lemma 2.1.1 $K(p^{j+m})$ acts trivially on $\mathcal{D}_{\mathfrak{U}}^{s,\circ}/\mathrm{Fil}^{j+m-s}\mathcal{D}_{\mathfrak{U}}^{s,\circ}$. Kato defines $\mathrm{Ch}_{L,M}(k,r,k-1)$ as the composite of all the maps in the lefthand column. We simply define $\mathrm{Ch}^s_{L,M}(\mathfrak{U},r)$ as the composite map from the upper left to the lower right. $\square$

Proposition 3.1.2. *The map $\mathrm{Ch}^s_{L,M}(\mathfrak{U},r)$ satisfies the equivariance formulas*

$$\langle a \mid b\rangle\, \mathrm{Ch}^s_{L,M}(\mathfrak{U},r) = \chi_{\mathfrak{U}}(a)(ab)^{-r}\mathrm{Ch}^s_{L,M}(\mathfrak{U},r)\,\langle a \mid b\rangle$$

and

$$T'_m\mathrm{Ch}^s_{L,M}(\mathfrak{U},r) = \chi_{\mathfrak{U}}(m)\mathrm{Ch}^s_{L,M}(\mathfrak{U},r)T'_m,$$

for any integers a, b, m with $(m, Lp) = (a, Lp) = (b, Mp) = 1$.

These maps, much like adult mayflies, exist for exactly one purpose. Let

$$({}_{c,d}z_{Lp^j,Mp^j})_{j\geq 1} \in \varprojlim_j K_2(Y_j)$$

be Kato's norm-compatible system of zeta elements (§2 of [Kat04]).

Definition 3.1.3. *We define*

$$\begin{aligned}
{}_{c,d}\mathfrak{z}^s_{L,M}(\mathfrak{U},r) &= \mathrm{Ch}^s_{L,M}(\mathfrak{U},r)\left(({}_{c,d}z_{Lp^j,Mp^j})_{j\geq 1}\right)\\
&\in H^1\left(G_{\mathbf{Q}}, \mathbf{V}_{\mathfrak{U}}^{s,\circ}(L,M)(-r)\right)
\end{aligned}$$

for integers c, d with $(c, 6pL) = (d, 6pM) = 1$.

By Theorem 3.1.1, we immediately deduce

Proposition 3.1.4. For any $k \in \mathbf{Z}_{\geq 2} \cap \mathfrak{U}$, we have an equality

$$i_k \left({}_{c,d}\mathfrak{z}^s_{L,M}(\mathfrak{U}, r) \right) = {}_{c,d}\mathbf{z}^{(p)}_{L,M}(k, r, k-1)$$

in $H^1(G_{\mathbf{Q}}, V_k^{\circ}(M, N)(k-r))$ where ${}_{c,d}\mathbf{z}^{(p)}_{L,M}(k, r, k-1)$ is the p-adic zeta element defined in §8.4 of *[Kat04]*.

By the argument in §8.5-8.6 of [Kat04], the class ${}_{c,d}\mathfrak{z}^s_{L,M}(\mathfrak{U}, r)$ is unramified away from p.

3.2 Cyclotomic zeta elements

In this subsection we p-adically interpolate the construction given in §8.9 of [Kat04]. Let m, A be positive integers with $p \nmid A$, let $a(A)$ denote a residue class modulo A, and let c, d be any integers with $(cd, 6pmA) = (d, N) = 1$. Choose any integers L, M with $mA|L$, $L|M$, $N|M$, $\mathrm{prime}(L) = \mathrm{prime}(mpA)$, and $\mathrm{prime}(M) = \mathrm{prime}(mpAN)$, and let

$$t_{m,a(A)} : \mathbf{V}^{s,\circ}_{\mathfrak{U}}(L, M) \to \mathbf{V}^{s,\circ}_{\mathfrak{U}}(N) \otimes_{\mathbf{Z}} \mathbf{Z}[G_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}]$$

be the trace map induced by the morphism $Y(L, M) \to Y_1(Np) \otimes \mathbf{Q}(\zeta_m)$ as in Kato. Define ${}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U}, r, a(A))$ as the image of ${}_{c,d}\mathfrak{z}^s_{L,M}(\mathfrak{U}, r)$ under the homomorphism

$$\begin{aligned} H^1\left(G_{\mathbf{Q}}, \mathbf{V}^{s,\circ}_{\mathfrak{U}}(L, M)(-r)\right) &\overset{t_{m,a(A)}}{\to} H^1\left(G_{\mathbf{Q}}, \mathbf{V}^{s,\circ}_{\mathfrak{U}}(N)(-r) \otimes_{\mathbf{Z}} \mathbf{Z}[G_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}]\right) \\ &\overset{\sim}{\to} H^1\left(G_{\mathbf{Q}(\zeta_m)}, \mathbf{V}^{s,\circ}_{\mathfrak{U}}(N)(-r)\right). \end{aligned}$$

Proposition 3.2.1. *For any $k \in \mathbf{Z}_{\geq 2} \cap \mathfrak{U}$, we have an equality*

$$i_k \left({}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U}, r, a(A)) \right) = {}_{c,d}\mathbf{z}^{(p)}_{1,Np,m}(k, r, k-1, a(A), \mathrm{prime}(mpA))$$

in $H^1\left(G_{\mathbf{Q}(\zeta_m)}, V_k^{\circ}(Np)(k-r)\right)$, where ${}_{c,d}\mathbf{z}^{(p)}_{1,Np,m}(k, r, k-1, a(A), \mathrm{prime}(mpA))$ is the p-adic zeta element in equation (8.1.2) of [Kat04].

Note that Kato defines ${}_{c,d}\mathbf{z}^{(p)}_{1,N,m}(k, r, k-1, a(A), \mathrm{prime}(mpA))$ without assuming $p \nmid A$, but we require this since $t_{m,a(A)}$ does not respect $\Gamma_0(p)$-structures when $p|A$.

Proposition 3.2.2. *Notation as above, let ℓ be any prime with $(\ell, cd) = 1$. Then*

$$\mathrm{Cor}_{\mathbf{Q}(\zeta_{m\ell})/\mathbf{Q}(\zeta_m)} \left({}_{c,d}\mathfrak{z}^s_{N,\ell m}(\mathfrak{U}, r, a(A)) \right) = \begin{cases} {}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U}, r, a(A)) & \text{if } \ell | mpA \\ P_\ell(\sigma_\ell^{-1}) \cdot {}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U}, r, a(A)) & \text{if} \ell \nmid mpA, \end{cases}$$

where

$$P_\ell(\sigma_\ell^{-1}) = (1 - T'_\ell \ell^{-r} \sigma_\ell^{-1} + \delta_{\ell \nmid N} \langle \ell \rangle^{-1} \chi_{\mathscr{W}}(\ell) \ell^{-1-2r} \sigma_\ell^{-2}) \in \mathscr{O}(\mathfrak{U}) \otimes_{\mathbf{Z}} \mathbf{T}[G_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}].$$

Proof. Follows formally from the properties of all the maps defined so far and Kato's analysis of the Euler system relations in K_2. □

Suppose now that $(\mathfrak{U},h)$ is a slope datum. Define ${}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U},r,a(A))_h$ as the image of ${}_{c,d}\mathfrak{z}^s_{N,m}(\mathfrak{U},r,a(A))$ under the natural map

$$H^1\left(G_{\mathbf{Q}(\zeta_m)},\mathbf{V}^{s,\circ}_{\mathfrak{U}}(N)(-r)\right)\to H^1\left(G_{\mathbf{Q}(\zeta_m)},\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)(-r)\right).$$

Now define

$$\begin{aligned}H^1_{\mathrm{Iw}}(G_{\mathbf{Q}(\zeta_m)},\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)(-r)) &= \varprojlim_{j} H^1(G_{\mathbf{Q}(\zeta_{mp^j})},\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)(-r))\\ &= H^1(G_{\mathbf{Q}(\zeta_m)},\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)(-r)\otimes_{\mathscr{O}(\mathfrak{U})}\mathscr{O}(\mathfrak{U})\widehat{\otimes}\mathbf{Z}_p[[\Gamma_m]])^{\iota}.\end{aligned}$$

By Proposition 3.2.2, the inverse system $\left({}_{c,d}\mathfrak{z}^s_{N,mp^j}(\mathfrak{U},r,a(A))_h\right)_{j\geq 1}$ defines an element

$${}_{c,d}\mathfrak{Z}^s_{N,m}(\mathfrak{U},r,a(A))_h\in H^1_{\mathrm{Iw}}(G_{\mathbf{Q}(\zeta_m)},\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)(-r)).$$

Proposition 3.2.3. *The classes ${}_{c,d}\mathfrak{Z}^s_{N,m}(\mathfrak{U},r,a(A))_h$ satisfy the compatibility*

$$\mathrm{Tw}_{r_1-r_2}\left({}_{c,d}\mathfrak{Z}^s_{N,m}(\mathfrak{U},r_1,a(A))_h\right)={}_{c,d}\mathfrak{Z}^s_{N,m}(\mathfrak{U},r_2,a(A))_h$$

under the twisting isomorphism

$$\mathrm{Tw}_{r_1-r_2}:H^1_{\mathrm{Iw}}(G_{\mathbf{Q}(\zeta_m)},\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)(-r_1))\simeq H^1_{\mathrm{Iw}}(G_{\mathbf{Q}(\zeta_m)},\mathbf{V}^{s,\circ}_{\mathfrak{U},h}(N)(-r_2)).$$

Inverting p and passing to the inverse limit over s yields classes

$${}_{c,d}\mathfrak{z}_{N,m}(\mathfrak{U},r,a(A))_h\in H^1\left(G_{\mathbf{Q}(\zeta_m)},\mathscr{V}_{\mathfrak{U},h}(N)(-r)\right)$$

and

$$\begin{aligned}{}_{c,d}\mathfrak{Z}_{N,m}(\mathfrak{U},r,a(A))_h &\in H^1_{\mathrm{Iw}}(G_{\mathbf{Q}(\zeta_m)},\mathscr{V}_{\mathfrak{U},h}(N)(-r))\\ &:= H^1(G_{\mathbf{Q}(\zeta_m)},\mathscr{V}_{\mathfrak{U},h}(N)(-r)\otimes_{\mathscr{O}(\mathfrak{U}^{\mathrm{rig}})^b}\mathscr{O}(\mathfrak{U}^{\mathrm{rig}}\times\mathscr{X}_m)^b)^{\iota}\end{aligned}$$

(here $\mathscr{X}_m=\mathrm{Spf}\mathbf{Z}_p[[\Gamma_m]]$).

Now everything glues over an admissible cover of the type constructed in Lemma 2.3.4, and we get global sections ${}_{c,d}\mathfrak{z}_{N,m}(r,a(A))$ and ${}_{c,d}\mathfrak{Z}_{N,m}(r,a(A))$ of the appropriate Galois cohomology sheaves.

Let ν be as in the introduction. Set

$$\mu_{c,d}(r,\nu)=(c^2-c^r\chi_{\mathscr{W}}(c)^{-1}\nu(c)^{-1}\sigma_c)(d^2-d^r\nu(d)\sigma_d)\in\left(\mathscr{O}(\mathscr{W})\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(\nu)\right)[\mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q})],$$

and choose c,d such that $c\equiv 1 \bmod mp$, $d\equiv 1\bmod Nmp$, and $\nu(c)=\nu(d)=-1$; the existence of c,d with these properties follows immediately from the Chinese remainder theorem together with our assumption that ν has even order. The quantity $\mu_{c,d}$ is then a unit in $\left(\mathscr{O}(\mathscr{W})\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(\nu)\right)[\mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q})]$ and

$$\mathfrak{z}_{N,m}(r,\nu)=\mu_{c,d}(r,\nu)^{-1}\sum_{a\in(\mathbf{Z}/A\mathbf{Z})^{\times}}\nu(a)_{c,d}\mathfrak{z}_m(r,a(A))$$

is well-defined independently of choosing c,d. We have analogously defined Iwasawa classes $\mathfrak{Z}_{N,m}(r,\nu)$.

4 Applications to p-adic L-functions

4.1 The regulator map

In this subsection we prove Theorem 1.2.2. Let

$$\mathscr{R}_{\mathbf{Q}_p} = \left\{ f \in \mathbf{Q}_p[[T, T^{-1}]] \mid f \,\text{convergent on}\, 1 > |T| > r \,\text{for some}\, r = r_f < 1 \right\}$$

be the Robba ring over $\mathbf{Q}_p$, and set $\mathscr{R}_L = \mathscr{R}_{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} L$ and $\mathscr{R}_L^+ = \mathscr{R}_L \cap L[[T]]$. For background on (φ, Γ)-modules over $\mathscr{R}_L$, we refer the reader to [Ber11, CC99, KPX14, Pot12].

Notation as in the theorem, fix an affinoid $U \subset \mathscr{C}$. We define $\mathrm{Log}|_U$ as follows: For any $\mathbf{z} \in H^1_{\mathrm{Iw}}(\mathbf{Q}_p, \mathscr{V}_0(U))$ and $v \in \mathscr{D}^*_{\mathrm{crys}}(U)$, $\mathrm{Log}(\mathbf{z})(v)$ is the image of $\mathbf{z}$ under the sequence of maps

$$\begin{aligned} H^1_{\mathrm{Iw}}(\mathbf{Q}_p, \mathscr{V}_0(U)) \hookrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p, \mathscr{V}_0(U)) \otimes_{\Lambda(U)} \mathscr{O}(U \times \mathscr{X}) \xrightarrow{\sim} \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_0(U))^{\psi=1} \xrightarrow{v^*} \mathscr{R}(U)^{+,\psi=(\alpha^c)^{-1}} \\ \xrightarrow{1-\varphi\circ\psi} \mathscr{R}(U)^{+,\psi=0} \xrightarrow{\mathscr{M}^{-1}} \mathscr{O}(U)\widehat{\otimes}\mathbf{D}_{\mathrm{an}}(\Gamma) \xrightarrow{\mathscr{A}} \mathscr{O}(U \times \mathscr{X}). \end{aligned}$$

Let us explain these arrows one-by-one. The first arrow follows from the identification $\Lambda(U) \cong \mathscr{O}(U \times \mathscr{X})^b$ together with the faithful flatness of $\mathscr{O}(U \times \mathscr{X})$ over $\mathscr{O}(U \times \mathscr{X})^b$. The second arrow follows upon composing the canonical isomorphisms

$$\begin{aligned} H^1_{\mathrm{Iw}}(\mathbf{Q}_p, \mathscr{V}_0(U)) \otimes_{\Lambda(U)} \mathscr{O}(U \times \mathscr{X}) &\cong H^1_{\mathrm{an.Iw}}(\mathbf{Q}_p, \mathscr{V}_0(U)) \\ &\cong \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_0(U))^{\psi=1}. \end{aligned}$$

We refer to Corollary 4.4.11 of [KPX14] for details.

For the third arrow, recall that

$$\begin{aligned} \mathscr{D}^*_{\mathrm{crys}}(U) &= \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_0^*(U))^{\Gamma=1, \varphi=\alpha^c} \\ &\subset \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_0^*(U)) \\ &\cong \mathrm{Hom}_{\mathscr{R}(U)}\left(\mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_0(U)), \mathscr{R}(U)\right), \end{aligned}$$

so v defines a linear functional v^* into $\mathscr{R}(U)$. To calculate the ψ-action on the image of this functional evaluated on some $d \in \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_0(U))^{\psi=1}$, note that the φ-action on this Hom space is characterized by the equation $(\varphi \cdot f)(\varphi(d)) = \varphi(f(d))$, and likewise the ψ-action is characterized by $(\psi \cdot f)(d) = \psi(f(\varphi(d)))$. Let $f_v = v^*$, so in particular $\varphi(f_v(d)) = \alpha^c f_v(\varphi(d))$ and $f_v(d) = \alpha^c \psi(f_v(\varphi(d)))$. By the definition of an étale φ-module over $\mathscr{R}(U)$ we may (locally on U) write $d = \sum_{i \in I} a_i \varphi(d_i)$ for some $d_i \in \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_0(U))$ and $a_i \in \mathscr{R}(U)$, and since $\psi(d) = d$ we have $\sum_{i \in I} a_i \varphi(d_i) = \sum_{i \in I} \psi(a_i) d_i$. Now we calculate

$$\begin{aligned} \psi\left(f_v(d)\right) &= \psi\left(f_v\left(\sum a_i \varphi(d_i)\right)\right) \\ &= \psi\left(\sum a_i f_v(\varphi(d_i))\right) \\ &= (\alpha^c)^{-1} \psi\left(\sum a_i \varphi(f_v(d_i))\right) \\ &= (\alpha^c)^{-1} \sum \psi(a_i) f_v(d_i) \\ &= (\alpha^c)^{-1} f_v\left(\sum \psi(a_i) d_i\right) \\ &= (\alpha^c)^{-1} f_v(d), \end{aligned}$$

as claimed.

To check that the third arrow really has image in the claimed subspace, note that for $L/\mathbf{Q}_p$ finite and any $a \in \mathcal{O}_L$, there is a natural inclusion $\mathscr{R}_L^{\psi=a^{-1}} \subset \mathscr{R}_L^+$ unless $a = 1$, in which case $\mathscr{R}_L^{\psi=1} \subset \mathscr{R}_L^+ \oplus L \cdot \frac{1}{T} \subset \mathscr{R}_L$. (This is a special case of [Col10], Prop. I.11.) In particular, given a reduced affinoid U and a function $f \in \mathscr{O}(U)^\circ$ with $f \neq 1$, there is an inclusion

$$\left(\mathscr{R}_{\mathbf{Q}_p} \widehat{\otimes} \mathscr{O}(U)\right)^{\psi=f^{-1}} \subset \mathscr{R}_{\mathbf{Q}_p}^+ \widehat{\otimes} \mathscr{O}(U).$$

Now at any crystalline point x of weight $k_x \in \mathbf{Z}_{\geq 2}$, the specialization α_x^c is a p-Weil number of weight $k_x - 1$, and such points are dense in $\mathscr{C}$, so the zero locus of $\alpha^c - 1$ is nowhere dense in $\mathscr{C}$.

The fifth arrow is the inverse of the Mellin transform isomorphism

$$\mathscr{M} : \mathbf{D}_{\mathrm{an}}(\Gamma) \widehat{\otimes} \mathscr{O}(U) \xrightarrow{\sim} \mathscr{R}(U)^{+,\psi=0}$$

induced by the isomorphism

$$\begin{array}{rcl} \mathbf{D}_{\mathrm{an}}(\Gamma) & \xrightarrow{\sim} & \mathscr{R}_{\mathbf{Q}_p}^{+,\psi=0} \\ \mu & \mapsto & \displaystyle\int (1+T)^{\chi_{\mathrm{cyc}}(\gamma)} \mu(\gamma), \end{array}$$

where $\mathbf{D}_{\mathrm{an}}(\Gamma)$ denotes the ring of locally analytic distributions on Γ.

The sixth arrow is the Amice transform.

The comparison with the Perrin-Riou regulator map follows from Berger's alternate construction [Ber03] of the latter (cf. the discussion following Theorem A in [LLZ11]).

It remains to show the claimed growth property at individual points. To see this, we note that for any point $x \in U(\overline{\mathbf{Q}_p})$, we have a commutative square

$$\begin{array}{ccc} H^1_{\mathrm{Iw}}(\mathbf{Q}_p, \mathscr{V}_{0,x}) & \longrightarrow & H^1_{\mathrm{Iw}}(\mathbf{Q}_p, \mathscr{V}_{0,x}) \otimes_{\mathscr{O}(\mathscr{X})^b} \mathscr{O}(\mathscr{X}) \\ \downarrow \wr & & \downarrow \wr \\ \mathbf{D}^\dagger(\mathscr{V}_{0,x})^{\psi=1} & \longrightarrow & \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{V}_{0,x})^{\psi=1} \end{array}$$

of injective maps; here the upper horizontal and righthand vertical maps are the specializations at x of the first two maps in the definition of Log, and the lefthand vertical isomorphism is a well-known theorem of Fontaine (cf. [CC99]). Going around the lower left half of the square, the claim about growth now follows from the following result, applied to the representation $\mathscr{V}_{0,x}$.

Proposition 4.1.1. *Let V be a two-dimensional trianguline representation of $G_{\mathbf{Q}_p}$ on an L-vector space, and suppose $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ admits a triangulation of the form*

$$0 \to \mathscr{R}_L(\eta(x_0)\mu_\alpha) \to \mathbf{D}^\dagger_{\mathrm{rig}}(V) \to \mathscr{R}_L(\mu_\beta) \to 0,$$

where μ_α and μ_β are unramified characters of $\mathbf{Q}_p^\times$ and η is some character of $\mathbf{Z}_p^\times$. Let v_α, v_β be a basis of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ realizing this triangulation. Then given any element $f \in \mathbf{D}^\dagger(V) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)$, the image of f in $\mathscr{R}_L(\mu_\beta)$ is of the form $v_\beta \cdot f_\beta$ where $f_\beta \in \mathscr{R}_L$ has growth of order $\leq v(\beta^{-1})$.

Proof. This follows from (the proof of) Proposition 3.11 of [Col08]. □

4.2 Specialization at noble points

In this section we analyze the specializations of the classes $\mathfrak{z}_m(r,\nu)$ and $\mathfrak{Z}_m(r,\nu)$ at noble points and prove Theorem 1.2.3 and Theorem 1.2.1.

Let $f \in S_k(\Gamma_1(N))$ be a cuspidal newform, α a root of the pth Hecke polynomial of f, and $\mathbf{f}$ the associated p-stabilization of f. We assume that $x_{\mathbf{f}}$ is noble. Set

$$V_{\mathbf{f}} = \left(H^1_{\mathrm{et}}(Y_1(Np)_{\overline{\mathbf{Q}}}, \mathscr{L}_k)(2-k) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\mathbf{f})\right)[\ker \phi_{f,\alpha}]$$

and $V_{\mathbf{f}}^\circ$ =obvious lattice, so $V_{\mathbf{f}}(k)$ is simply the fiber of $\mathscr{V}$ at $x_{\mathbf{f}}$. Define

$$H^1_{\mathrm{Iw}}(V_{\mathbf{f}}^\circ) \quad = \quad \lim_{\leftarrow j} H^1\left(G_{\mathbf{Q}(\zeta_{p^j}),Np\infty}, V_{\mathbf{f}}^\circ \otimes (\mathbf{Z}/p^j)\right)$$

and $H^1_{\mathrm{Iw}}(V_{\mathbf{f}}) = H^1_{\mathrm{Iw}}(V_{\mathbf{f}}^\circ)[\frac{1}{p}]$. Recall that $\mathscr{X} = \mathrm{Spf}(\mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})]])^{\mathrm{an}}$, so $\mathscr{O}(\mathscr{X})^\circ = \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})]]$.

Proposition 4.2.1. *The module $H^1_{\mathrm{Iw}}(V_f)$ is locally free of rank one over $\mathscr{O}(\mathscr{X})^b$.*

Proof. The ring $\mathscr{O}(\mathscr{X})^b$ is a one-dimensional, Noetherian, regular, Jacobson ring, and in particular every prime ideal is maximal and principal. If $\mathfrak{m}$ is a maximal ideal, then the usual long exact sequence in cohomology gives a surjection

$$H^0(G_{\mathbf{Q},Np\infty}, V_f \otimes_{\mathbf{Q}_p} \mathscr{O}(\mathscr{X})^b/\mathfrak{m}) \twoheadrightarrow H^1_{\mathrm{Iw}}(V_f)[\mathfrak{m}],$$

and the H^0 vanishes by the irreducibility of $V_{\mathbf{f}}$. Therefore $H^1_{\mathrm{Iw}}(V_{\mathbf{f}})$ is torsion-free, so is locally free of some rank. The latter is one by Tate's global Euler characteristic formula and Kato's results on H^2_{Iw} (cf. Theorem 12.4 of [Kat04]). $\square$

Kato's 'optimal Euler system' is an injective $\mathbf{Q}_p(f)$-linear map

$$\begin{array}{rcl} V_{\mathbf{f}} & \to & H^1_{\mathrm{Iw}}(V_{\mathbf{f}}). \\ \gamma & \mapsto & \mathbf{z}^{(p)}_{\gamma,\mathbf{f}} \end{array}$$

with a number of remarkable properties. Let us note that, strictly speaking, Kato constructs an analogous map $V_f \to H^1_{\mathrm{Iw}}(V_f)$ for the unrefined Galois representation $V_f \subset H^1_{\mathrm{et}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathscr{L}_k)(2-k) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f)$, but his construction extends verbatim to nobly refined forms.

Proposition 4.2.2.

i. *We have*

$$\tfrac{1-\nu(-1)\cdot\sigma_{-1}}{2}\mathfrak{Z}_1(k,\nu)_{x_{\mathbf{f}}} = \prod_{\ell | A} P_\ell(\ell^{-k}\sigma_\ell^{-1})_{x_{\mathbf{f}}} \cdot \mathbf{z}^{(p)}_{\delta(\mathbf{f},\nu)},$$

where $\delta(\mathbf{f},\nu)$ is the projection to the $\mathbf{f}$-eigenspace of $V_k(Np)$ of the cohomology class

$$\sum_{a \in (\mathbf{Z}/A)^\times} \nu(a)\delta_{1,Np}(k,k-1,a(A)).$$

ii. *Under the hypothesis of Theorem 1.1.3.iii, the class $\delta(\mathbf{f},\nu)$ is nonzero.*

Proof. This follows from Proposition 3.2.1 together with Lemma 13.11 of Kato. In fact, writing $\delta(\mathbf{f},\nu^\pm) = b^\pm \cdot \gamma^\pm$ with $\gamma^\pm$ any basis of $V_{\mathbf{f}}^\pm$, a calculation shows that $b^\pm$ is a nonzero multiple of $L_{\mathrm{alg}}(k-1, f^c \otimes \nu^{-1})$ $\square$

Proof of Theorem 1.2.3. Notation as in the theorem, write

$$\mathbf{L}_{\nu^\pm} = e^\pm \prod_{\ell | \frac{A^\mp}{(A^+,A^-)}} P_\ell(\ell^{-1}\sigma_\ell^{-1}) \cdot \mathrm{Log}(\mathrm{res}_p \mathfrak{Z}_1(1,\nu^\pm)),$$

so $\mathbf{L}_{\nu^+,\nu^-} = \mathbf{L}_{\nu^+} + \mathbf{L}_{\nu^-}$. Let

$$\log_{\mathbf{f}} : H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V_{\mathbf{f}}(k-1)) \to \mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{crys}}(V_{\mathbf{f}}(k-1))$$

be the Perrin-Riou regulator map. Kato defines $L_{p,\mathrm{alg}}(\mathbf{f}^c)$ as $\eta^* \log_{\mathbf{f}}(\mathbf{z}^{(p)}_{\gamma,\mathbf{f}}(k-1))$ for a certain $\eta \in \mathbf{D}_{\mathrm{crys}}(V_{\mathbf{f}}(k-1))^\vee \simeq \mathbf{D}_{\mathrm{crys}}(V_{\mathbf{f}^c})$ satisfying $\varphi(\eta) = \alpha^c\eta$. By Proposition 4.2.2 and a simple twisting argument, we see that

$$\begin{aligned} \prod_{\ell | A^\pm} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{x_{\mathbf{f}}} \cdot \mathbf{z}^{(p)}_{\delta(\mathbf{f},\nu^\pm)}(k-1) &= \tfrac{1+\nu(-1)(-1)^k\cdot\sigma_{-1}}{2} \mathfrak{Z}_1(1,\nu^\pm)_{x_{\mathbf{f}}} \\ &\in H^1_{\mathrm{Iw}}(V_{\mathbf{f}}(k-1)). \end{aligned}$$

Therefore by Theorem 1.2.2, we have

$$\begin{aligned} \mathbf{L}_{\nu^\pm}(v)_{x_{\mathbf{f}}} &= e^\pm \prod_{\ell | \frac{A^\mp}{(A^+,A^-)}} P_\ell(\ell^{-1}\sigma_\ell^{-1}) \cdot v^*_{x_{\mathbf{f}}} \log_{\mathbf{f}} \left(\prod_{\ell | A^\pm} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{x_{\mathbf{f}}} \cdot \mathbf{z}^{(p)}_{\delta(\mathbf{f},\nu^\pm)}(k-1) \right) \\ &= e^\pm \prod_{\ell | A} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{x_{\mathbf{f}}} \cdot v^*_{x_{\mathbf{f}}} \log_{\mathbf{f}} \left(\mathbf{z}^{(p)}_{\delta(\mathbf{f},\nu^\pm)}(k-1) \right) \\ &= e^\pm \prod_{\ell | A} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{x_{\mathbf{f}}} \cdot b^\pm v^*_{x_{\mathbf{f}}} \log_{\mathbf{f}} \left(\mathbf{z}^{(p)}_{\gamma^\pm}(k-1) \right) \\ &= C_{\nu^\pm}(v_{x_{\mathbf{f}}}) e^\pm \prod_{\ell | A} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{x_{\mathbf{f}}} \cdot L_{p,\mathrm{alg}}(\mathbf{f}^c) \end{aligned}$$

as claimed. □

Proof of Theorem 1.2.1. Fix a small connected affinoid $U \subset \mathscr{C}$ containing $x_{\mathbf{f}}$ and some nowhere vanishing section $v \in \mathscr{D}^*_{\mathrm{crys}}(U)$, and choose $\gamma \in \mathscr{V}(U)$ with $\gamma^\pm_{x_{\mathbf{f}}} \neq 0$. By our previous analysis together with a deep theorem of Rohrlich [Roh84], we may choose $\nu^\pm$ such that $\mathbf{L}_{\nu^\pm}$ is not identically zero at $x_{\mathbf{f}}$. Let $\mathscr{X}^\pm$ denote either portion of $\mathscr{X}$, and consider the ratio

$$R^\pm = \frac{c^* \mathbf{L}_{\nu^\pm}(v)}{\mathbf{L}^\pm_{p,\mathrm{an}} \cdot c^* \prod_{\ell | A^+A^-} P_\ell(\ell^{-1}\sigma_\ell^{-1})} \in \mathrm{Frac}(\mathscr{O}(U \times \mathscr{X}^\pm)).$$

By the same theorem of Rohrlich, $L_{p,\mathrm{an}}(\mathbf{f})|_{\mathscr{X}_i} \neq 0$ on any connected component $\mathscr{X}_i$ of $\mathscr{X}$, so the denominator is not a zero-divisor and this ratio is well-defined. Shrinking U if necessary, we may assume the denominator of $R^\pm$ is a non-zero-divisor after specialization at any noble point $y \in U(\overline{\mathbf{Q}}_p)$. Choosing any such point y of noncritical slope, we have

$$\begin{aligned} R^\pm_y &= \frac{\mathbf{L}_{\nu^\pm}(v)_{y^c}}{L^\pm_{p,\mathrm{an}}(\mathbf{f}_y) \cdot \prod_{\ell | A^+A^-} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{y^c}} \\ &= \frac{C_{\nu^\pm}(v)_{y^c} \cdot L^\pm_{p,\mathrm{alg}}(\mathbf{f}_y) \cdot \prod_{\ell | A^+A^-} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{y^c}}{L^\pm_{p,\mathrm{an}}(\mathbf{f}_y) \cdot \prod_{\ell | A^+A^-} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{y^c}} \\ &= C^\pm_y \in E_y \end{aligned}$$

for some constant C_y by Visik's theorem. By the Zariski-density of such points, we deduce that

$$R^{\pm} \in \mathrm{Frac}(\mathscr{O}(U)) \subset \mathrm{Frac}(\mathscr{O}(U \times \mathscr{X}^{\pm})).$$

Since $L_{p,\mathrm{an}}(\mathbf{f})$ is not a zero-divisor, the polar divisor of $R^{\pm}$ is necessarily disjoint from $\{x_{\mathbf{f}}\} \times \mathscr{X}^{\pm}$, so we may specialize R at $x_{\mathbf{f}}$ and conclude that $R^{\pm}_{x_{\mathbf{f}}}$ is a constant, and is furthermore nonzero, since $C_{\nu^{\pm}}(v)_{x^c_{\mathbf{f}}}$ is a nonzero multiple of $L(k-1, f \otimes \nu^{\pm})$. Since furthermore $\prod_{\ell | A^+A^-} P_\ell(\ell^{-1}\sigma_\ell^{-1})_{y^c}$ is not a zero-divisor, we deduce that $L^{\pm}_{p,\mathrm{an}}(\mathbf{f}) = C^{\pm}_{x_{\mathbf{f}}} L_{p,\mathrm{alg}}(\mathbf{f})$, and evaluating either side at any interpolatory point $x^j\eta(x) \in \mathscr{X}^{\pm}$ for which $L(j+1, f \otimes \eta^{-1}) \neq 0$ implies $C^{\pm}_{x_{\mathbf{f}}} = 1$. □

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