Quotients of adic spaces by finite groups

David Hansen*

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Abstract

We prove that if X is an analytic adic space with an action of a finite group G, a categorical adic space quotient X/G exists under a mild hypothesis, in analogy with a classical result for schemes. We also show that if X is perfectoid, then X/G is perfectoid in many cases.

1 Introduction

Let X be a scheme with a (right) action of a finite group G. It's a classical result ([SGA03], §V.1) that a *quotient scheme* X/G exists if X admits a G-invariant open covering by affine subschemes. Here we define X/G, if it exists, as the unique scheme with a morphism $q : X \to X/G$ such that

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{Sch}}(X/G, Y) & = & \mathrm{Hom}_{\mathrm{Sch}}(X, Y)^G \\ f & \mapsto & f \circ q \end{array}$$

functorially in arbitrary schemes Y. We note that a categorical quotient X/G always exists in the larger category of ringed spaces, and the existence of a G-invariant covering by affines can be interpreted as a criterion for X/G to be a scheme.

In this note we prove a similar result in the context of nonarchimedean analytic geometry. Due to its scope and generality, we work in the language of adic spaces [Hub94, Hub96, Sch12, SW13, KL15]. Precisely, let X be an adic space with a (right) action of a finite group G. When does a quotient adic space X/G exist? By definition, adic spaces form a full subcategory of a certain category $\mathscr{V}$ of *v-ringed spaces*, and there's a natural general candidate for a quotient space X/G in $\mathscr{V}$, whose construction we give in §2 below. It's easy to check that X/G is always categorical *in the category* $\mathscr{V}$: if $Y \in \mathscr{V}$ is arbitrary and G acts on $\mathrm{Hom}_{\mathscr{V}}(X, Y)$ by precomposition, then

$$\mathrm{Hom}_{\mathscr{V}}(X, Y)^G = \mathrm{Hom}_{\mathscr{V}}(X/G, Y).$$

Our main result is a criterion for X/G to be an adic space.

Theorem 1.1. *Let X be an analytic adic space with an action of a finite group G. Suppose that X has a covering by G-stable open affinoid subspaces, and that $|G|$ is invertible in $\mathcal{O}_X(X)$. Then X/G is an adic space with underlying topological space $|X|/G$.*

The key point is the following result, which treats the case of an affinoid adic space.

*Department of Mathematics, Columbia University, 2990 Broadway, New York NY 10027; hansen@math.columbia.edu

Theorem 1.2. *Let (A, A^+) be a Tate-Huber pair with an action of a finite group G. Then the map $|\mathrm{Spa}(A, A^+)| \to |\mathrm{Spa}(A^G, A^{+G})|$ induces a natural homeomorphism*

$$|\mathrm{Spa}(A, A^+)|/G \cong |\mathrm{Spa}(A^G, A^{+G})|$$

where we give the left-hand side the quotient topology. If (A, A^+) is sheafy and $|G|$ is invertible in A, then the Tate-Huber pair (A^G, A^{+G}) is sheafy and the map $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(A^G, A^{+G})$ induces a natural identification

$$\mathrm{Spa}(A, A^+)/G = \mathrm{Spa}(A^G, A^{+G})$$

in the category $\mathscr{V}$.

Note the absence of any Noetherianity or finite type assumption on A. On the other hand, it seems very likely that the annoying hypothesis on $|G|$ is unnecessary, but we don't presently see how to remove it.

Combining the proof of this result with some small additional arguments, we get some useful special cases. In particular, we get a result for rigid analytic spaces with no hypothesis on $|G|$:

Theorem 1.3. *Let X be a rigid analytic space over a nonarchimedean field K, with a K-linear action of a finite group G. Suppose one of the following conditions holds:*

i. *X has an admissible covering by G-stable open affinoid subspaces.*

ii. *X is separated and the G-orbit of any (adic) point of X is contained in an open affinoid subspace.*

Then X/G is a rigid analytic space over K and the map $X \to X/G$ is finite. If X is affinoid then X/G is affinoid.

This result is presumably well-known to experts; however, when we needed it in [CHJ15], we were unable to find a reference in the literature. In [CHJ15] we gave a proof of Theorem 1.3 which relied on some special features of the rigid analytic situation; however, the first version of that proof contained a small gap (as pointed out to us by Judith Ludwig). We found the general argument for Theorem 1.1 presented here while trying to fix this gap.[1] We also remark that Theorem 1.3 is essentially optimal: X/G exists as a rigid space with $q : X \to X/G$ *finite* if and only if condition i. holds (compare [SGA03], Prop. V.1.6).

We also show that if X is perfectoid, then X/G is perfectoid in many cases.

Theorem 1.4. *Let X be a perfectoid space with an action of a finite group G. Suppose that X has a covering by G-stable open affinoid perfectoid subspaces, and that one of the following conditions holds:*

i. *$|G|$ is invertible in $\mathcal{O}_X(X)$.*

ii. *X lives over $\mathrm{Spa}(K, K^+)$ for some perfectoid field K, and the G-action is K-linear.*

Then X/G is a perfectoid space.

Note that in condition ii. here, we allow the situation where $\mathrm{char} K = p > 0$ and $p||G|$.

Notation

If A is a topological ring, we write A° for the subring of power-bounded elements, and $A^{\circ\circ}$ for the subset of topologically nilpotent elements. Group actions on rings are left actions, and group actions on topological or ringed spaces are right actions.

[1] However, only a very small portion of the argument in this paper was actually necessary to fix the gap in [CHJ15].

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2 The candidate space; proof of Theorem 1.1

We very briefly recall the theory of adic spaces and the structure of the ambient category they live in. This material was developed in [Hub93, Hub94, Hub96], and §2 of [Sch12] is a nice overview of the theory. (We also recommend [KL15].)

Recall that a *Huber ring*[2] is a topological ring A containing an open subring A_0 whose induced topology is the I-adic topology for some finitely generated ideal $I \subset A_0$. Any such pair (A_0, I) is a *couple of definition.* A Huber ring is *Tate* if for some (equivalently, any) couple of definition (A_0, I) we have $I = (\varpi)$ for some element $\varpi \in A_0^{\circ\circ} \cap A^\times$; any such ϖ is a *pseudouniformizer of* A. A *Huber pair* is a pair (A, A^+) where A is a Huber ring and $A^+ \subset A$ is an open and integrally closed subring consisting of power-bounded elements (any such A^+ is a *ring of integral elements*). A *Tate-Huber pair* (A, A^+) is a Huber pair where A is a Tate ring.

Given any Huber pair, we get an associated *affinoid pre-adic space* $X = \mathrm{Spa}(A, A^+)$. This is a topological space whose points are given by equivalence classes of continuous valuations on A taking values ≤ 1 on A^+, roughly, together with a structure presheaf $\mathcal{O}_X$ of complete topological rings; for any $x \in |X|$ with associated valuation v_x, the stalk $\mathcal{O}_{X,x}$ is local and v_x extends to a valuation on $\mathcal{O}_{X,x}$. By definition, (A, A^+) is *sheafy* if $\mathcal{O}_X$ is a sheaf, in which case $X = \mathrm{Spa}(A, A^+)$ is an *affinoid adic space*. These naturally live in the following category:

Definition 2.1. An object in the category $\mathscr{V}$ of *v-ringed spaces* is a triple $X = \left(|X|, \mathcal{O}_X, \{v_x\}_{x\in|X|}\right)$ where $|X|$ is a topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings on $|X|$, and v_x is a(n equivalence class of) valuation(s) on the stalk $\mathcal{O}_{X,x}$ for each $x \in |X|$. A morphism $f : X \to Y$ in $\mathscr{V}$ is a pair of morphisms $(f, f^\sharp)$ where $f : |X| \to |Y|$ is a continuous map and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a continuous morphism of sheaves of topological rings such that $v_{f(x)}$ and $v_x \circ f^\sharp$ are equivalent as valuations on $\mathcal{O}_{Y,f(x)}$ for all $x \in |X|$.

By definition, an adic space is an object in $\mathscr{V}$ which admits an open covering by affinoid adic spaces, and a morphism of adic spaces is a morphism in $\mathscr{V}$. An adic space X is *analytic* if the valuation topology on $K(x) = \mathrm{Frac}(\mathcal{O}_{X,x}/\ker v_x)$ is nondiscrete for any point $x \in X$; this is equivalent to X having a covering by affinoid adic spaces associated with Tate-Huber pairs.

Definition 2.2. Let $X = \left(|X|, \mathcal{O}_X, \{v_x\}_{x\in|X|}\right)$ be an adic space with a right action of a finite group G. Then we define an object $X/G = \left(|X/G|, \mathcal{O}_{X/G}, \{v_y\}_{y\in|X/G|}\right)$ in the category $\mathscr{V}$ as follows:

- $|X/G| := |X|/G$ with the quotient topology. Let $q : |X| \to |X/G|$ be the evident map.
- $\mathcal{O}_{X/G} := (q_*\mathcal{O}_X)^G$. This is clearly a sheaf (since $(-)^G$ is left-exact) of complete topological rings.
- v_y is the valuation on the stalk $\mathcal{O}_{X/G,y}$ induced by $\mathcal{O}_{X/G,y} \to \mathcal{O}_{X,x} \overset{v_x}{\to} \Gamma_{v_x} \cup \{0\}$ where $x \in q^{-1}(y)$ is any preimage of y.

[2] Also called an *f-adic ring* in [Hub93, Hub94, SW13].

By construction X/G defines an object in $\mathscr{V}$ with a canonical morphism $q : X \to X/G$ in $\mathscr{V}$, G-equivariant for the trivial action on X/G. The following proposition is easy and left to the reader.

Proposition 2.3. *The quotient X/G is categorical in $\mathscr{V}$.*

Proof of Theorem 1.1. Let X and G be as in Theorem 1.1, and let $\{U_i = \mathrm{Spa}(A_i, A_i^+) \subset X\}$ be an open covering of X by G-stable affinoids. Since $q : X \to X/G$ is a quotient map on topological spaces and

$$\begin{aligned} U_i &= q^{-1}(q(U_i)) \\ &= q^{-1}(U_i/G) \end{aligned}$$

is open in X, each U_i/G is open in X/G. By Theorem 1.2, each U_i/G is an affinoid adic space; since these give an open covering of X/G, we immediately deduce that X/G is an adic space. □

3 Proof of Theorems 1.2-1.4

In this section we prove Theorem 1.2, as the natural combination of Theorems 3.1 and 3.3 below. Theorems 1.3 and 1.4 follow immediately from (the proof of) Theorem 1.1 combined with Theorems 3.4 and 3.5-3.6 below.

3.1 The statement on topological spaces

In this section only, $\mathrm{Spa}(A, A^+)$ refers to the underlying topological space.

Theorem 3.1. *Let (A, A^+) be a Tate-Huber pair with an action of a finite group G. Then the map $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(A^G, A^{+G})$ induces a natural homeomorphism*

$$\mathrm{Spa}(A, A^+)/G \cong \mathrm{Spa}(A^G, A^{+G})$$

where we give the left-hand side the quotient topology.

We prove this in four steps:

Step 1: $\mathrm{Cont}(A) \to \mathrm{Cont}(A^G)$ **is surjective, and the fibers are G-orbits.**

Fix a choice of G-invariant pseudouniformizer $\varpi \in A^G$ (so it also does the job for A). We recall a useful criterion for continuity:

Proposition. *If R is a Tate ring with pseudouniformizer ϖ, then an arbitrary valuation $v : R \to \Gamma_v \cup \{0\}$ is continuous if and only if $v(\varpi)$ is cofinal for Γ_v and $v(\varpi x) < 1$ for all $x \in R_0$, where R_0 is any choice of an open and bounded subring of R containing ϖ.*

Proof. This is a special case of Corollary 9.3.3 in Conrad's notes from the Seminaire Scholze, which is deduced in turn as a corollary of Theorem 3.1 of [Hub93]. □

Next, we note that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Cont}(A) \arrow[r, "\phi"] \arrow[d] & \mathrm{Cont}(A^G) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "\phi'"] & \mathrm{Spec}(A^G) \end{tikzcd}$$

where the vertical maps send a valuation v to its support $\mathfrak{p} = \ker v$. By the going-up theorem, ϕ' is surjective.

Now choose $v \in \mathrm{Cont}(A^G)$, with associated support $\mathfrak{p}$, so v factors through a continuous valuation of $K = \mathrm{Frac}(A^G/\mathfrak{p})$ (continuous relative to the nonarchimedean field topology on K induced by the unique rank one generization of v in $\mathrm{Cont}(A)$). By standard scheme theory ([SGA03], Prop. V.1.1), G acts transitively on the (nonempty) fiber $\phi'^{-1}(\mathfrak{p})$, and for any $\mathfrak{q} \in \phi'^{-1}(\mathfrak{p})$ with associated stabilizer $G_{\mathfrak{q}}$, $L = \mathrm{Frac}(A/\mathfrak{q})$ is an algebraic and quasi-Galois extension of K and the induced map $G_{\mathfrak{q}} \to \mathrm{Aut}(L/K)$ is surjective. Fixing a choice of $\mathfrak{q}$, the surjectivity portion of Step 1 will be done if we can show that any choice w of a valuation of L extending v is continuous.

The extension $K \to L$ factors as $K \to K' \to L$ where $K \to K' = L^{\mathrm{Aut}(L/K)}$ is purely inseparable and $K' \to L$ is a finite Galois extension with Galois group $\mathrm{Aut}(L/K)$. Since valuations extend uniquely along purely inseparable extensions, we get a unique valuation v' of K' extending v. We claim that v' is continuous. Indeed, since $v(\varpi) \in \Gamma_v$ is cofinal for Γ_v and $\Gamma_{v'}/\Gamma_v$ is a torsion group, an easy argument shows that $v(\varpi) = v'(\varpi) \in \Gamma_{v'}$ is cofinal for $\Gamma_{v'}$. For any given $x \in K'^{\circ}$, let n be such that $x^{p^n} \in K^{\circ}$ (where $p = \mathrm{char} K$). Then

$$v'(\varpi x)^{p^n} = v'(\varpi^{p^n} x^{p^n}) = v(\varpi^{p^n} x^{p^n}) < 1,$$

so $v'(\varpi x) < 1$ and thus v' is continuous by the criterion above.

Replacing K by K' and v by v', we may now assume L/K is a finite algebraic Galois extension. Let w be any extension of v to L. Since L/K is a finite algebraic extension, Γ_w/Γ_v is a finite group. In particular, $w(\varpi) < 1$ is cofinal for Γ_w. Now choose some element $x \in L$ with $L = K[x]$. For large $n \gg 0$, the element $\varpi^n x \in L^{\circ}$ will satisfy a monic polynomial with coefficients in K°, and then $L_{0,n} = K^{\circ}[\varpi^n x]$ is an open and bounded subring of L containing ϖ and K° and generated as a module over K° by $(\varpi^n x)^i, 0 \leq i \leq [L:K]-1$. Since $w(\varpi)$ is cofinal for Γ_w, we have $w(\varpi^n x) < 1$ for $n \gg 0$, so then an easy calculation shows that $w(\varpi a) < 1$ for all $a \in L_{0,n}$, and thus w is continuous by the criterion above.

Finally, we need to show that the fibers of ϕ are G-orbits. Choose any $v \in \mathrm{Cont}(A^G)$ with support $\mathfrak{p}$. We've already seen that G acts transitively on the possible supports $\mathfrak{q}$ of valuations in the fiber $\phi^{-1}(v)$. Fixing a choice of $\mathfrak{q}$ and writing L and K as above, it suffices to see that $G_{\mathfrak{q}}$ acts transitively on the set of valuations of L extending v; this is exactly Corollary VI.7.3 in [ZS75].

Step 2: $\mathrm{Cont}(A)/G \to \mathrm{Cont}(A^G)$ **is a homeomorphism.**

Consider the diagram

$$\begin{tikzcd} \mathrm{Cont}(A) \arrow[rr, "\phi"] \arrow[dr, "q"'] & & \mathrm{Cont}(A^G) \\ & \mathrm{Cont}(A)/G \arrow[ur, "r"'] & \end{tikzcd}$$

of topological spaces. By Step 1, ϕ is surjective, so r is surjective as well. Again by Step 1, r is also injective, hence bijective. It suffices to show that r is continuous and open. For continuity, note that q is a quotient map by definition; moreover, since ϕ is a surjective generalizing spectral map of spectral spaces, Lemma 3.2 below implies that ϕ is a quotient map as well. Thus if $U \subset \mathrm{Cont}(A^G)$ is open, then $r^{-1}(U)$ is open iff $q^{-1}(r^{-1}(U)) = \phi^{-1}(U)$ is open, which it is because ϕ is a quotient map. This shows that r is continuous. Finally, we need to show that r is open. Given any open $V \subset \mathrm{Cont}(A)/G$, we have

$$q^{-1}(V) = \phi^{-1}(r(V))$$

since r is bijective, but $q^{-1}(V)$ is open, and thus $r(V)$ is as well since ϕ is a quotient map.

Step 3: The natural G-equivariant map $\mathrm{Spa}(A, A^+) \to \mathrm{Cont}(A) \times_{\mathrm{Cont}(A^G)} \mathrm{Spa}(A^G, A^{+G})$ is a homeomorphism.

In what follows, B^{icl} (for a given subring $B \subset A$) means "the integral closure of B in A". Recall that for any Huber ring R, the ring $R^{\min}$ defined as the integral closure in R of the subring $\mathbf{Z} + R^{\circ\circ}$ is the minimal ring of integral elements, and $\mathrm{Cont}(R) \cong \mathrm{Spa}(R, R^{\min})$. Using this observation, we calculate the fiber product

$$\mathrm{Cont}(A) \times_{\mathrm{Cont}(A^G)} \mathrm{Spa}(A^G, A^{+G})$$

in the category of adic spectra. By standard theory [Hub94], this fiber product is represented by $\mathrm{Spa}(D, D^+)$, where $D = A \otimes_{A^G} A^G = A$ and

$$D^+ = \left(\mathrm{im}(A^{\min} \otimes_{A^{G\min}} A^{+G} \to A)\right)^{\mathrm{icl}}$$

We claim that $D^+ = A^+$. To see this, we first note that $A^+ = (A^{+G})^{\mathrm{icl}}$: indeed, elements of A^+ are integral over A^{+G}, so $A^+ \subseteq (A^{+G})^{\mathrm{icl}}$, but on the other hand $A^{+G} \subseteq A^+$ so $(A^{+G})^{\mathrm{icl}} \subseteq (A^+)^{\mathrm{icl}} = A^+$. Now we have the chain of inclusions

$$A^{+G} \subseteq \mathrm{im}(A^{\min} \otimes_{A^{G\min}} A^{+G} \to A) \subseteq A^+$$

of subrings of A, so applying $(-)^{\mathrm{icl}}$ we get that D^+ is trapped between A^+ and A^+, and hence $D^+ = A^+$.

Step 4: The natural map $\mathrm{Spa}(A, A^+)/G \to \mathrm{Spa}(A^G, A^{+G})$ is a homeomorphism.

By the previous step, we have a natural G-equivariant homeomorphism

$$\mathrm{Spa}(A, A^+) \to \mathrm{Cont}(A) \times_{\mathrm{Cont}(A^G)} \mathrm{Spa}(A^G, A^{+G}).$$

Applying $-/G$ to both sides, we get

$$\begin{aligned} \mathrm{Spa}(A, A^+)/G &\cong \mathrm{Cont}(A)/G \times_{\mathrm{Cont}(A^G)} \mathrm{Spa}(A^G, A^{+G}) \\ &\cong \mathrm{Spa}(A^G, A^{+G}) \end{aligned}$$

using Step 2 to get the second line, and the claim follows. □

For completeness, we include a proof of the following lemma, which is presumably well-known to experts.

Lemma 3.2. *Let $f : Y \to X$ be a surjective generalizing spectral map of spectral spaces. Then f is a quotient map.*

Proof. It suffices to show that if $Z \subset X$ is any subset such that $f^{-1}(Z)$ is closed in Y, then Z is closed in X. Fix such a subset Z. Since $f^{-1}(Z) \subset Y$ is closed, the inclusion $f^{-1}(Z) \to Y$ is a spectral map, and then the composite map $f^{-1}(Z) \to Y \to X$ is a spectral map as well. Since the image of this composite map is exactly Z, [Sta16, Tag 0A2S] now implies that Z is closed in the constructible topology on X, so then by [Sta16, 0903] we deduce that Z is closed if it is stable under specialization.

To check that Z is stable under specialization, let $z \in Z$ be any point, and let $z_0 \prec z$ be any specialization of z in X. Suppose that $z_0 \notin Z$; then $z_0 \in X \smallsetminus Z$, so $f^{-1}(z_0) \subset Y \smallsetminus f^{-1}(Z)$. Choosing any $\tilde{z}_0 \in f^{-1}(z_0)$ and using the assumption that generalizations lift along f, we may choose some $\tilde{z} \in Y$ such that $\tilde{z}_0 \prec \tilde{z}$ and $f(\tilde{z}) = z$. But $\tilde{z}_0 \in Y \smallsetminus f^{-1}(Z)$, and $Y \smallsetminus f^{-1}(Z)$ is open in the spectral space Y, hence is stable under generalization, so $\tilde{z} \in Y \smallsetminus f^{-1}(Z)$ as well; but then $z = f(\tilde{z}) \notin Z$, which is a contradiction. This shows that Z is stable under specialization, so Z is closed as desired. □

3.2 The statement on structure sheaves

Theorem 3.3. *Let (A, A^+) be a sheafy Tate-Huber pair with an action of a finite group G. Set $X = \mathrm{Spa}(A, A^+)$ and $Y = \mathrm{Spa}(A^G, A^{+G})$, and let $q : X \to Y$ be the natural map of affinoid pre-adic spaces. If $|G|$ is invertible in A, then $\mathcal{O}_Y = (q_*\mathcal{O}_X)^G$. In particular, $\mathcal{O}_Y$ is a sheaf and $Y = X/G$ in the sense of Definition 2.2.*

Proof. By construction, there is a natural map of presheaves $\mathcal{O}_Y \to (q_*\mathcal{O}_X)^G$ on Y, where the target is a sheaf. It suffices to show that this map is an isomorphism. By construction, it suffices to check this on rational subsets of Y. Let $U = U(\frac{f_1,\ldots,f_n}{h})$ be any rational subset of Y, so $q^{-1}U$ is also a rational subset of X. We need to show that $\mathcal{O}_Y(U) \cong \mathcal{O}_X(q^{-1}U)^G$, i.e. that

$$A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle = A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle^G.$$

To reduce further, note that the usual idempotent $e_G = \frac{1}{|G|}\sum_{g\in G} g \in \mathbf{Q}[G]$ gives a continuous and G-equivariant A^G-Banach module splitting $A = A^G \oplus M$ where $M = (1 - e_G)A$ satisfies $M^G = 0$. Applying $-\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$ to this decomposition gives a G-linear direct sum decomposition

$$A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle = A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle \oplus M\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle.$$

Taking G-invariants in this identity, we see it suffices to prove that $\left(M\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle\right)^G = 0$.

Let $A_0 \subset A$ be a choice of G-stable open and bounded subring containing a G-invariant pseudouniformizer ϖ, so A_0^G is an open and bounded subring of A^G. After possibly enlarging A_0 and replacing $f_1, \ldots, f_n, h$ by $\varpi^M \cdot f_1, \ldots, \varpi^M \cdot f_n, \varpi^M \cdot h$ for some $M \gg 0$, we may assume that $f_1, \ldots, f_n, h \in A_0^G$. Let M_0 be the image of A_0 under the natural projection $A \twoheadrightarrow M$, so M_0 is a G-stable open and bounded A_0^G-submodule of M. We slightly abusively write $M_0\widehat{\otimes}_{A_0^G} A_0^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$ for the image of this completed tensor product in $M\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$. Note that $M_0\widehat{\otimes}_{A_0^G} A_0^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$ is an open, G-stable and ϖ-adically separated A_0^G-submodule of $M\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$. If we fix a choice of N such that $|G| \mid \varpi^N$ in A_0, then e_G carries $\varpi^j \cdot (M_0\widehat{\otimes}_{A_0^G} A_0^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle)$ into $\varpi^{j-N} \cdot (M_0\widehat{\otimes}_{A_0^G} A_0^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle)$ for all j. Note also that

$$M\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle = \varpi^j \cdot \left(M_0\widehat{\otimes}_{A_0^G} A_0^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle\right) + M \otimes_{A^G} A^G[\tfrac{1}{h}]$$

for any j; accordingly, for any $m \in M\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$ and any j, we write $m = m_{1,j} + m_{2,j}$ for a choice of associated (non-unique) decomposition.

Now we finish the proof as follows: Given $m \in \left(M\widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle\right)^G$, then

$$\begin{aligned} m &= e_G m \\ &= e_G m_{1,j} + e_G m_{2,j} \\ &= e_G m_{1,j} \end{aligned}$$

for all j, where the third line follows from the fact that $(M \otimes_{A^G} A^G[\frac{1}{h}])^G = M^G \otimes_{A^G} A^G[\frac{1}{h}] = 0$. But then

$$m = e_G m_{1,j} \in \varpi^{j-N} \cdot \left(M_0\widehat{\otimes}_{A_0^G} A_0^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle\right)$$

for all $j \gg 0$, so

$$m \in \bigcap_{j \gg 0} \varpi^j \cdot \left(M_0 \widehat{\otimes}_{A_0^G} A_0^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle \right) = 0.$$

□

3.3 Groups actions on classical affinoid K-algebras

Theorem 3.4. *Let A be a (classical) affinoid K-algebra over some nonarchimedean field K, with a K-linear action of a finite group G. Set $X = \mathrm{Spa}(A, A^\circ)$ and $Y = \mathrm{Spa}(A^G, A^{\circ G})$, and let $q : X \to Y$ be the natural map of affinoid adic spaces. Then $\mathcal{O}_Y = (q_* \mathcal{O}_X)^G$, so $Y = X/G$ in the sense of Definition 2.2. Furthermore, q is finite.*

Proof. (Suggested by Christian Johansson.) By [BGR84, Prop. 6.3.3/3], A^G is an affinoid K-algebra and A is module-finite and integral over A^G; in particular, $\mathcal{O}_Y$ is sheafy and q is finite. Note also that we have an exact sequence

$$0 \to A^G \to A \xrightarrow{a \mapsto (ga-a)_{g \in G}} \prod_{g \in G} A_{(g)}$$

of finite A^G-modules. Let $U = U(\frac{f_1, \ldots, f_n}{h})$ be any rational subset of Y, so $q^{-1}U$ is also a rational subset of X. Again, we need to show that $\mathcal{O}_Y(U) = \mathcal{O}_X(q^{-1}U)^G$, i.e. that $A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle = A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle^G$. Note that $A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$ is flat over A^G and that

$$A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle = A \widehat{\otimes}_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle = A \otimes_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle,$$

since A is module-finite over A^G, so then applying $- \otimes_{A^G} A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle$ to the above sequence we get

$$0 \to A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle \to A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle \xrightarrow{a \mapsto (ga-a)_{g \in G}} \prod_{g \in G} A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle_{(g)}.$$

Since the kernel of the third arrow here is $A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle^G$, we get $A^G \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle = A \left\langle \frac{f_1}{h}, \ldots, \frac{f_n}{h} \right\rangle^G$ as desired. □

Now we show Theorem 1.3.

Proof of Theorem 1.3. Taking into account Theorems 1.1, 3.1 and 3.4, it's clear that condition i. of Theorem 1.3 implies the conclusion of the theorem, so all we need to prove is that condition ii. implies condition i. Suppose ii. is true, so for any point $x \in |X|$ we may choose an open affinoid subspace $U_x \subset X$ containing the orbit xG. Setting $V_x := \bigcap_{g \in G} U_x g$, we have $xG \subset V_x$ by construction. Clearly V_x is G-stable and open, and V_x is affinoid since each $U_x g$ is affinoid and X is separated [BGR84, Prop. 9.6.1/6]. Since every orbit xG is contained in some V_x, the G-stable affinoids V_x give an open covering of X, and we're done. □

3.4 Group actions on perfectoid algebras

Theorem 3.5. *Let A be a perfectoid K-algebra for some perfectoid field K, with a K-linear action of a finite group G. Then A^G is a perfectoid K-algebra.*

Proof. Set $B = A^G$; this is a Tate ring over K. If $\mathrm{char}K = p > 0$, then B automatically is a perfectoid K-algebra: it's clearly a complete uniform Tate ring, and it's perfect since the Frobenius on A commutes with the G-action.

For the rest of the proof, we may thus assume $\mathrm{char}K = 0$; let $p = \mathrm{char}K^\flat$. Since the operation $(-)^\flat$ is a functor on rings, we have $B^\flat = (A^G)^\flat = (A^\flat)^G$, so by the argument of the previous paragraph, $B^\flat$ is a perfectoid $K^\flat$-algebra. Choose an element $z \in W(K^{\flat\circ})$ generating the kernel of the usual theta map $\theta : W(K^{\flat\circ}) \twoheadrightarrow K^\circ$, so z is primitive of degree one in the sense of [KL15]. By tilting [KL15, Sch12], we have a canonical identification

$$A = W(A^{\flat\circ}) \otimes_{W(K^{\flat\circ})} K = W(A^{\flat\circ})[\tfrac{1}{p}]/(z).$$

Note that z is a G-invariant nonzero-divisor in any of the rings $W(K^{\flat\circ}) \subset W(B^{\flat\circ}) \subset W(A^{\flat\circ})$ and in any of these rings with p inverted.

Since the surjection $\theta : W(A^{\flat\circ})[\frac{1}{p}] \to A$ is G-equivariant, it induces a map $\theta : W(B^{\flat\circ})[\frac{1}{p}] \to B$ which we claim is surjective. Indeed, if $b \in B$ is arbitrary and $b' \in W(A^{\flat\circ})[\frac{1}{p}]$ is any preimage of b under θ, then $e_G b' \in W(A^{\flat\circ})[\frac{1}{p}]^G = W(B^{\flat\circ})[\frac{1}{p}]$ is also a preimage of b (here $e_G = \frac{1}{|G|}\sum_{g\in G} g$ as before). Therefore, taking G-invariants of the sequence

$$0 \to z \cdot W(A^{\flat\circ})[\tfrac{1}{p}] \to W(A^{\flat\circ})[\tfrac{1}{p}] \to A \to 0$$

and noting that

$$\left(z \cdot W(A^{\flat\circ})[\tfrac{1}{p}]\right)^G = z \cdot W(B^{\flat\circ})[\tfrac{1}{p}],$$

we get an identification $B = W(B^{\flat\circ})[\frac{1}{p}]/(z)$. Since $B^\flat$ is perfectoid and $z \in W(B^{\flat\circ})$ is primitive of degree one, this implies that B is perfectoid. □

In fact, something much more general is true:

Theorem 3.6 (Kedlaya-Liu [KL16, Theorem 3.3.24])**.** *If A is any perfectoid Tate ring over $\mathbf{Z}_p$ with an action of a finite group G, the Tate ring A^G is perfectoid.*

Proof of Theorem 1.4. Combining Theorem 1.1 with Theorems 3.5-3.6, we immediately see that X/G is perfectoid if condition i. holds, or if condition ii. holds with $\mathrm{char}\, K = 0$. Therefore it remains to prove that X/G is perfectoid under the hypotheses in condition ii. when $\mathrm{char}\, K = p > 0$ and p divides $|G|$. Let $(K^\sharp, \iota)$ be any fixed choice of an untilt of K, i.e. $K^\sharp$ is a perfectoid field over $\mathbf{Q}_p$ and ι is an isomorphism $\iota : K \xrightarrow{\sim} K^{\sharp\flat}$. Let $X^\sharp$ be the corresponding untilt of X over $\mathrm{Spa}(K^\sharp, K^{\sharp+})$. By functoriality of tilting and untilting over a fixed perfectoid base, G acts on $X^\sharp$ and satisfies condition i., so $X^\sharp/G$ is a perfectoid space over $K^\sharp$. Tilting back to K, one easily sees that the map $q^\sharp : X^\sharp \to X^\sharp/G$ tilts to a map $q : X \to (X^\sharp/G)^\flat$ with $(X^\sharp/G)^\flat$ a perfectoid space over K. Finally, one checks that $(X^\sharp/G)^\flat \cong X/G$ in the category $\mathscr{V}$; it suffices to check this when X is affinoid perfectoid (in which case $X^\sharp$ is affinoid perfectoid too), in which case it follows from Theorem 3.1 together with the compatibility under tilting of rational subsets and sections of the structure sheaf over them.[3] □

[3] Note that we are bypassing Theorem 3.3 in this last argument.

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