Beijing notes on the categorical local Langlands conjecture

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A fool can ask more questions in a minute
than a wise man can answer in an hour.

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What these notes are trying to do

Fix a prime p, and a nonarchimedean local field E with uniformizer ϖ and residue field $\mathbf{F}_q$ of characteristic p. Let G/E be a connected reductive group, which we assume for simplicity is quasisplit. The *local Langlands conjecture,* in its vaguest form, seeks to parametrize the irreducible smooth representations of $G(E)$ in terms of Galois-theoretic data. A huge amount of effort has been expended on making this conjecture precise, and proving it in many cases. In its modern form, the conjecture proposes that irreducible smooth representations of $G(E)$ can be parametrized by (suitable) pairs (ϕ, ρ) where $\phi : W_E \times \mathrm{SL}_2 \to {}^L G$ is an L-parameter and ρ is an irreducible algebraic representation of a group closely related to the centralizer group $S_\phi = \mathrm{Cent}_{\hat{G}}(\phi)$. Moreover, this parametrization should depend only on a choice of Whittaker datum for G, and should satisfy many good properties. We refer to [Kal16] for a precise formulation.

In recent years, the subject has undergone a major transformation, thanks to Fargues's amazing discovery that ideas from *geometric Langlands* can be imported into the study of local Langlands, by reinterpreting the basic structures in local Langlands in terms of G-bundles on the Fargues–Fontaine curve [Far16, Far14]. The foundations for this rebuilding of the field were then laid in the revolutionary manuscripts [FS21, Sch17]. Among other things, they succeed in constructing a canonical map $\pi \mapsto \varphi_\pi$ from irreducible smooth representations towards semisimple L-parameters, which should be the semisimplification of the "true" local Langlands parametrization.

In a parallel stream, the idea emerged that the derived category of all smooth $G(E)$-representations should embed fully faithfully into ind-coherent sheaves on the stack of L-parameters. This was made precise (independently) by Hellmann [Hel23], Zhu [Zhu21], and Ben-Zvi–Chen–Helm–Nadler. It is then natural to conjecture that some form of the geometric Langlands conjecture holds in this setting, namely that the category of all sheaves on Bun_G should be equivalent to the category of ind-coherent sheaves on the stack of L-parameters. A formal conjecture along these lines was proposed by Fargues–Scholze [FS21, Conjecture I.10.2], without however pinning down the functor which should give the equivalence. An alternative formulation was proposed by Zhu, working instead with sheaves on the stack of G-isocrystals [Zhu21].

These notes should be regarded as a minor supplement to the works mentioned above. More concretely, we are guided by the following motivating questions, in the setting of Fargues-Scholze:

1. Can we give an unconditional formulation of the categorical local Langlands conjecture?
2. What properties and compatibilities should the categorical conjecture enjoy? What additional conjectures do these properties suggest?
3. How does the categorical conjecture encode the classical local Langlands conjecture?
4. How do the non-basic strata in Bun_G fit into the picture?

These notes are *not* intended as a detailed introduction to the structures required to formulate these questions. More specifically, we will assume the reader is somewhat familiar with the modern expectations regarding the local Langlands correspondence, as outlined in (say) [Kal16] and [Kal22]. We will also assume some familiarity with [FS21], and with the philosophy of the "classical" geometric Langlands program.

There are exercises scattered throughout the text. They vary widely in difficulty, but none of them are open problems.

Notation and conventions

We will introduce various notations throughout the text. Here we briefly mention several perhaps non-standard conventions. If X is a disjoint union of finite type (derived) Artin stacks, we write $\mathrm{Coh}(X)$ for the category of bounded coherent complexes on X with X regarded as an ind-algebraic stack. In particular, any object in $\mathrm{Coh}(X)$ automatically has quasicompact support. On the other hand, we write $\mathrm{Perf}(X)$ for all dualizable objects in $\mathrm{QCoh}(X)$. This leads to the slightly unfortunate fact that Perf may not be contained in Coh.

If X is an object in a triangulated or stable ∞-category $\mathcal{D}$, we say X admits a filtration with graded pieces $A_1, \ldots, A_n$ if there exists a sequence of maps $0 = X_0 \to X_1 \to X_2 \to \cdots \to X_n = X$ with $A_j \simeq \mathrm{cone}(X_{j-1} \to X_j)$ for all $1 \leq j \leq n$. Similar remarks apply to functors between stable ∞-categories.

1 The categorical conjecture

Fix a prime p, and a nonarchimedean local field E with uniformizer ϖ and residue field $\mathbf{F}_q$ of char. p. Fix also a prime $\ell \neq p$. Let Λ be a $\mathbf{Z}_\ell$-algebra containing a fixed choice of $\sqrt{q}$.

Let G be a connected reductive group over E. We often assume that G is quasisplit, both for simplicity and because the categorical conjecture requires this assumption. Let Q be the minimal finite quotient of W_E which acts nontrivially on $\hat{G}$, and put ${}^L G = \hat{G} \rtimes Q$, which we regard as a linear algebraic group over $\mathbf{Z}_\ell[\sqrt{q}]$. We write $\mathrm{Rep}_\Lambda({}^L G)$ and $\mathrm{Rep}_\Lambda(\hat{G})$ for the evident tensor categories of algebraic representations on finite projective Λ-modules.

1.1 The automorphic side

The basic geometric object is the stack Bun_G of G-bundles on the Fargues-Fontaine curve. We will not review this in detail here, since it is now a classical object.[1] The key feature of its geometry is

[1] I am joking.

that it has a Harder-Narasimhan stratification by locally closed substacks Bun_G^b indexed by elements of the Kottwitz set $b \in B(G)$ [Far20]. Each stratum is a classifying stack for a certain group v-sheaf $\tilde{G}_b$, which is an extension of the constant group sheaf $\underline{G_b(E)}$ by a connected unipotent group. By a deep theorem of Viehmann [Vie23], the topology on $|\mathrm{Bun}_G| \cong B(G)$ is exactly the Newton partial order topology on $B(G)$. We write $b \preceq b'$ if $b' \in \overline{\{b\}}$, so the minimal elements are exactly the basic b.

On the automorphic side, the key player is the category $D(\mathrm{Bun}_G, \Lambda) = D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ of sheaves on Bun_G. This is glued semi-orthogonally from analogous categories of sheaves attached to each Harder-Narasimhan stratum, which turn out to be much simpler. More precisely, for each stratum we have a canonical t-exact tensor equivalence

$$D(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda),$$

where the right-hand side is simply the derived category of the abelian category of smooth $\Lambda[G_b(E)]$-modules. This equivalence is induced by the functors $s_{b\natural}$ and s_b^* associated with the tautological map $s_b : [*/\underline{G_b(E)}] \to \mathrm{Bun}_G^b$, which turn out to be mutually inverse t-exact tensor equivalences. We will *always* identify $D(\mathrm{Bun}_G^b, \Lambda)$ and $D(G_b(E), \Lambda)$ in this way.

For any $b \in B(G)$, there are functors $i_b^* : D(\mathrm{Bun}_G, \Lambda) \to D(G_b(E), \Lambda)$ and $i_{b!} : D(G_b(E), \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$, which do "exactly what you think they do." The functor $i_{b!}$ has a right adjoint $i_b^!$, while i_b^* has a right adjoint i_{b*} and also a left adjoint $i_{b\sharp}$. This left adjoint is a special feature of the situation. The three pushforwards are linked by natural transformations

$$i_{b\sharp} \to i_{b!} \to i_{b*}$$

which induce the identity after applying i_b^*. Note that $i_{b*}A$ can only have nonzero stalks at points which are specializations of b, while (more strangely) $i_{b\sharp}A$ can only have nonzero stalks at points which are generizations of A.

Let us briefly recall the construction of these functors, especially the pushforwards $i_{b\sharp}$ and $i_{b!}$ and the natural transformations mentioned above. For any $b \in B(G)$, Fargues-Scholze construct an auxiliary diagram of small v-stacks

$$\begin{tikzcd} \mathcal{M}_b \arrow[r, "\pi_b"] \arrow[d, "q_b"] & \mathrm{Bun}_G \\ {[*/\underline{G_b(E)}]} & \end{tikzcd}$$

where π_b is cohomologically smooth and surjects onto the open substack $\mathrm{Bun}_G^{\preceq b}$ of points which are "more semistable" than b. Moreover, the map q_b has a canonical section given by a closed immersion $[*/\underline{G_b(E)}] \to \mathcal{M}_b$. Puncturing $\mathcal{M}_b$ along this section we get an auxiliary diagram

$$\begin{tikzcd} \mathcal{M}_b^\circ \arrow[r, "\pi_b^\circ"] \arrow[d, "q_b^\circ"] & \mathrm{Bun}_G \\ {[*/\underline{G_b(E)}]} & \end{tikzcd}$$

together with a natural open immersion $j_b : \mathcal{M}_b^\circ \to \mathcal{M}_b$. The functor $i_{b\sharp}$ turns out to be given by the formula $i_{b\sharp} = \pi_{b\natural} q_b^*$. The natural transformation $j_{b\natural} j_b^* \to \mathrm{id}$ then induces a natural transformation

$\pi^{\circ}_{b\natural}q_b^{\circ *} \to \pi_{b\natural}q_b^*$, and we define $i_{b!}$ as the cofiber of this map.[2] From this construction the map $i_{b\sharp} \to i_{b!}$ is tautological. An easy application of base change shows that $i_b^* i_{b!} \cong \mathrm{id}$, so by adjunction we get the claimed transformation $i_{b!} \to i_{b*}$. Finally, $i_b^!$ is defined as the right adjoint of $i_{b!}$.

For our purposes, it will be very convenient to "renormalize" these functors. To make the relevant definition, we note that G_b is an inner form of a Levi subgroup $M_{\{b\}} \subset G$, namely the centralizer of the Newton cocharacter ν_b. Let $P_{\{b\}}$ be the dynamic parabolic of ν_b, and let $\delta_b : M_{\{b\}}(E) \to q^{\mathbf{Z}}$ be the usual modulus character. Since G_b and $M_{\{b\}}$ have canonically isomorphic cocenters, we can regard δ_b as a cocharacter $G_b(E) \to q^{\mathbf{Z}}$. Since we have fixed a choice of $\sqrt{q}$ in our coefficient ring, we can take the square root to get a character $\delta_b^{1/2} : G_b(E) \to \Lambda^\times$.

Definition 1.1.1 (Renormalized functors)**.** For $? \in \{\sharp, !, *\}$, we define $i_{b?}^{\mathrm{ren}} : D(G_b(E), \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$ by the formula

$$i_{b?}^{\mathrm{ren}} A = i_{b?}(A \otimes \delta_b^{1/2})[-\langle 2\rho_G, \nu_b \rangle].$$

Similarly, for $? \in \{*, !\}$ we define $i_b^{?\mathrm{ren}} A = (\delta_b^{-1/2} \otimes i_b^? A)[\langle 2\rho_G, \nu_b \rangle]$.

Note that by design, we still have adjunctions $i_{b\sharp}^{\mathrm{ren}} \vdash i_b^{*\mathrm{ren}} \vdash i_{b*}^{\mathrm{ren}}$ and $i_{b!}^{\mathrm{ren}} \vdash i_b^{!\mathrm{ren}}$, as well as natural transformations $i_{b\sharp}^{\mathrm{ren}} \to i_{b!}^{\mathrm{ren}} \to i_{b*}^{\mathrm{ren}}$ which induce the identity after applying $i_b^{*\mathrm{ren}}$. Note also that when b is basic, we have not changed the functors at all. The importance of this renormalization will become clear later. For now let us just remark that the renormalized functors interact cleanly with duality (Proposition 1.1.4), have good (semi)perversity properties (Exercise 1.2.2), are closely related to geometric Eisenstein series (Remark 1.4.8), and preserve L-parameters (Theorem 1.4.2).

Proposition 1.1.2. *One has the following a priori vanishing results:*

i. $R\mathrm{Hom}(i_{b\sharp}A, i_{b'\sharp}A') = 0$ *unless* $b \preceq b'$,

ii. $R\mathrm{Hom}(i_{b!}A, i_{b'!}A') = 0$ *unless* $b' \preceq b$,

iii. $R\mathrm{Hom}(i_{b\sharp}A, i_{b'!}A') = 0$ *unless* $b = b'$,

iv. $R\mathrm{Hom}(i_{b\sharp}A, i_{b'*}A') = 0$ *unless* $b' \preceq b$.

The obvious variants hold for renormalized pushforwards.

Proof. Parts i., iii. and iv. are exercises using various adjunctions together with the support properties of i_{b*}, $i_{b!}$ and $i_{b\sharp}$ mentioned above, and ii. can be deduced from i. by duality. ☐

1.1.1 Finiteness and duality

There are two natural finiteness conditions one can impose on objects of $D(G_b(E), \Lambda)$. On one hand, we can consider the *compact objects* in this category. It turns out that any compact object is built from basic compact objects, i.e. representations of the form $\mathrm{ind}_K^{G_b(E)}\Lambda$ where $K \subset G_b(E)$ is a pro-p compact open subgroup and $\mathrm{ind}_K^{G_b(E)}$ denotes compact induction, by finitely many shifts, cones and retracts. It is not hard to see that $D(G_b(E), \Lambda)$ is compactly generated.

Remark 1.1.3. When Λ is a field of characteristic zero, a difficult theorem of Bernstein asserts that $A \in D(G_b(E), \Lambda)$ is compact iff it has bounded cohomological amplitude and each $H^n(A)$ is a finitely generated representation. As a particular consequence of this theorem, we note that the standard truncation functors on $D(G_b(E), \Lambda)$ preserve compact objects when Λ is a field of characteristic zero. We will use this fact later in the construction of the hadal t-structure.

[2]This definition may seem surprisingly complicated, but the issue is that in the D_{lis} formalism, we only have a priori access to $\natural$-pushforwards along cohomologically smooth maps. Consequently, we are forced to construct $i_{b!}$ indirectly since i_b itself is not cohomologically smooth.

On the other hand, we can consider *ULA objects*, namely those objects $A \in D(G_b(E), \Lambda)$ for which A^K is a perfect complex of Λ-modules for all pro-p open compacts K. Note that if Λ is a field and A has bounded cohomological amplitude, A is ULA exactly when each $H^n(A)$ is an admissible representation in the usual sense. One should think of the ULA condition as a very mild variant on admissibility, which is somehow more conceptual.

These two finiteness conditions come with two associated dualities. On the subcategory $D(G_b(E), \Lambda)^{\mathrm{ULA}}$ of ULA objects, we have the operation of *smooth duality* $\mathbf{D}_{\mathrm{sm}}$ sending A to $R\mathscr{H}\mathrm{om}(A, \Lambda)$, where $R\mathscr{H}\mathrm{om}(-, \Lambda)$ is the internal hom towards the trivial representation in $D(G_b(E), \Lambda)$. This operation makes sense for any A, but on the subcategory of ULA objects it restricts to an involutive anti-equivalence. When Λ is a field and $A = \pi$ is an admissible representation concentrated in degree zero, $\mathbf{D}_{\mathrm{sm}}\pi$ is just the usual smooth dual $\pi^\vee$, i.e. the smooth vectors in the abstract dual $\mathrm{Hom}_\Lambda(\pi, \Lambda)$.

On the subcategory $D(G_b(E), \Lambda)^\omega$ of compact objects, we also have Bernstein's *cohomological duality* $\mathbf{D}_{\mathrm{coh}}$, which by definition sends A to the total external derived hom $R\mathrm{Hom}(A, \mathcal{C}_c^\infty(G_b(E), \Lambda))$. It is a fun exercise to check that $\mathbf{D}_{\mathrm{coh}}$ sends any basic compact object $\mathrm{ind}_K^{G_b(E)}\Lambda$ to itself, but noncanonically so (the isomorphism depending on a choice of Λ-valued Haar measure on $G_b(E)$), and therefore defines an involutive anti-equivalence on compact objects.

These finiteness conditions and their associated dualities have perfect adaptations to sheaves on Bun_G. Namely, smooth duality matches with Verdier duality, which is given by exactly the same formula, namely $\mathbf{D}_{\mathrm{Verd}}(A) = R\mathscr{H}\mathrm{om}(A, \Lambda)$ where the internal hom is now computed in $D(\mathrm{Bun}_G, \Lambda)$. This operation makes sense for all sheaves, and on the subcategory $D(\mathrm{Bun}_G, \Lambda)^{\mathrm{ULA}}$ of ULA sheaves it induces an involutive antiequivalence. We omit the *definition* of ULA sheaves on Bun_G, and simply note that they are (non-definitionally) characterized by the property that their $*$-restriction to each stratum is ULA in $D(G_b(E), \Lambda)$.

Cohomological duality matches with the more exotic operation of *Bernstein-Zelevinsky duality*, denoted $\mathbf{D}_{\mathrm{BZ}}$. This is only defined on compact sheaves, where it again yields an involutive anti-equivalence. This duality is characterized by the formula $R\mathrm{Hom}(\mathbf{D}_{\mathrm{BZ}}(A), B) = \pi_\natural(A \otimes B)$ where $\pi : \mathrm{Bun}_G \to *$ is the structure map and $\pi_\natural$ is the left adjoint of π^*. Compact sheaves also have a useful concrete characterization, by the property that their $*$-restriction to each stratum is compact and is identically zero on all but finitely many strata.

The finiteness conditions and dualities on Bun_G and on the individual strata are linked as follows.

Proposition 1.1.4. o. *The functors $i_{b!}$, $i_{b\sharp}$, and i_b^* preserve compact objects. The functors $i_{b!}$, i_{b*}, and $i_b^!$ preserve ULA objects. Likewise for the renormalized functors.*

i. *For any $A \in D(G_b(E), \Lambda)$ we have $\mathbf{D}_{\mathrm{Verd}} i_{b!}^{\mathrm{ren}} A \simeq i_{b*}^{\mathrm{ren}} \mathbf{D}_{\mathrm{sm}} A$, and if A is ULA we also have $\mathbf{D}_{\mathrm{Verd}} i_{b*}^{\mathrm{ren}} A \simeq i_{b!}^{\mathrm{ren}} \mathbf{D}_{\mathrm{sm}} A$.*

ii. *For any $A \in D(G_b(E), \Lambda)^\omega$ we have $\mathbf{D}_{\mathrm{BZ}} i_{b!}^{\mathrm{ren}} A \simeq i_{b\sharp}^{\mathrm{ren}} \mathbf{D}_{\mathrm{coh}} A$ and $\mathbf{D}_{\mathrm{BZ}} i_{b\sharp}^{\mathrm{ren}} A \simeq i_{b!}^{\mathrm{ren}} \mathbf{D}_{\mathrm{coh}} A$.*

iii. *For any $B \in D(\mathrm{Bun}_G, \Lambda)$ we have $\mathbf{D}_{\mathrm{sm}} i_b^{*\mathrm{ren}} B \simeq i_b^{!\mathrm{ren}} \mathbf{D}_{\mathrm{Verd}} B$, and if B is ULA we also have $\mathbf{D}_{\mathrm{sm}} i_b^{!\mathrm{ren}} B \simeq i_b^{*\mathrm{ren}} \mathbf{D}_{\mathrm{Verd}} B$*

Note that the duality compatibilies here would be much uglier to state with the naive pushforwards.

Sketch. The key point, via some calesthenics with the appropriate diagram

$$\begin{tikzcd}
{[*/\underline{G_b(E)}]} \arrow[d, "s_b"] \arrow[dr, "i_b'"] & \\
\mathrm{Bun}_G^b \arrow[r, "i_b"] \arrow[d, "f_b"] & \mathrm{Bun}_G \arrow[dl, "\pi"] \\
* &
\end{tikzcd}$$

is the observation that $i_b^!\Lambda \simeq f_b^!\Lambda \simeq \delta_b[-2\langle 2\rho_G, \nu_b\rangle]$. Getting the shift correct here is not hard, but pinning down the twist is much harder, and I only know how to do it by an indirect argument using [FS21, Proposition IX.5.3 and Theorem IX.7.2]. In some cases a direct argument is also possible, as observed by Hamann [Ham22]. I am also informed that Hamann and Imai have worked out a direct proof which covers all cases [HI23]. □

Remark 1.1.5. It is instructive to recall that if Λ is a field of characteristic zero and $A = \pi$ is an irreducible admissible $G(E)$-representation concentrated in degree zero, then π is both compact *and* ULA, so both dualities make sense.[3] The smooth dual of π is again an irreducible admissible representation, and the operation of smooth duality is expected to interact very cleanly with the local Langlands parametrization [Kal13]. The effect of cohomological duality is much less obvious: it follows from deep work of Aubert/Bernstein/Schneider-Stuhler/Zelevinsky that $\mathbf{D}_{\mathrm{coh}}(\pi) \simeq \mathrm{Zel}(\pi)[-d_\pi]$ for some nonnegative integer d_π and some irreducible admissible representation $\mathrm{Zel}(\pi)$. The operation $\pi \mapsto \mathrm{Zel}(\pi)$ is the Aubert-Zelevinsky involution.[4] The integer d_π is just the dimension of the component of the Bernstein variety containing π, but the representation $\mathrm{Zel}(\pi)$ is mysterious in general, and it is hard to say much about it beyond the simple observation that it has the same supercuspidal support as $\pi^\vee$.[5]

As a sample of what can happen, we recall that $\mathrm{Zel}(-)$ interchanges the trivial representation $\mathbf{1}$ with the Steinberg representation St. Although St has the same semisimple L-parameter as the trivial representation, the associated monodromy operators are very different (in fact maximally different). In particular, cohomological duality interacts in a somewhat mysterious way with the local Langlands parametrization. Nevertheless, it will transpire that the categorical local Langlands equivalence should intertwine Bernstein-Zelevinsky duality on Bun_G with (twisted) Grothendieck-Serre duality on the stack of L-parameters.

1.2 t-structures

First we briefly recall the perverse t-structure, which makes sense for any coefficient ring Λ. The existence of this t-structure has been well-understood by experts for years (see e.g. [FS21, Remark I.10.3]), although it doesn't seem to be recorded in the literature.

Proposition 1.2.1. *There is a perverse t-structure* ${}^pD^{\leq 0}$, ${}^pD^{\geq 0}$ *on* $D(\mathrm{Bun}_G, \Lambda)$*, with abelian heart denoted* $\mathrm{Perv}(\mathrm{Bun}_G, \Lambda)$*, uniquely characterized by the condition that* $A \in D(\mathrm{Bun}_G, \Lambda)$ *lies in* ${}^pD^{\leq 0}$ *(resp.* ${}^pD^{\geq 0}$*) if* i_b^*A *sits in cohomological degrees* $\leq \langle 2\rho_G, \nu_b\rangle$ *for all* b *(resp. if* $i_b^!A$ *sits in cohomological degrees* $\geq \langle 2\rho_G, \nu_b\rangle$ *for all* b*). When* Λ *is cohomologically regular in the sense that* $\mathrm{Perf}(\Lambda)$

[3] Again, compactness is not obvious, and follows from the theorem of Bernstein mentioned earlier.

[4] Some authors would say that $\mathrm{Zel}(\pi)^\vee$ is the Aubert-Zelevinsky involution.

[5] It is at least true that $\mathrm{Zel}(\pi) = \pi^\vee$ when π is supercuspidal.

is preserved by standard truncation, the perverse truncation functors preserve the ULA property. When Λ is a field, Verdier duality exchanges ${}^pD^{\leq 0} \cap D^{\mathrm{ULA}}$ and ${}^pD^{\geq 0} \cap D^{\mathrm{ULA}}$.

Here it is again cleaner to think in terms of the renormalized functors: $A \in D(\mathrm{Bun}_G, \Lambda)$ lies in ${}^pD^{\leq 0}$ (resp. ${}^pD^{\geq 0}$) if and only if $i_b^{*\mathrm{ren}}A$ (resp. $i_b^{!\mathrm{ren}}A$) sits in cohomological degrees ≤ 0 (resp. ≥ 0) for all b.

Exercise 1.2.2. i. Check that $i_{b!}^{\mathrm{ren}}$ sends $D^{\leq 0}(G_b(E), \Lambda)$ into ${}^pD^{\leq 0}(\mathrm{Bun}_G, \Lambda)$, and that i_{b*}^{ren} sends $D^{\geq 0}(G_b(E), \Lambda)$ into ${}^pD^{\geq 0}(\mathrm{Bun}_G, \Lambda)$. Check that ${}^pD^{\leq 0}(\mathrm{Bun}_G, \Lambda)$ is generated by all $i_{b!}^{\mathrm{ren}}D^{\leq 0}(G_b(E), \Lambda)$ under extensions and colimits.

ii. Prove that if Λ is a field, there is a natural bijection between irreducible objects $A \in \mathrm{Perv}(\mathrm{Bun}_G, \Lambda)$ and pairs (b, π) where $b \in B(G)$ and $\pi \in \Pi(G_b)$ is an irreducible smooth representation, defined by sending a pair (b, π) to the intermediate extension sheaf

$$i_{b!*}^{\mathrm{ren}}\pi \overset{\mathrm{def}}{=} \mathrm{im}\left({}^pH^0(i_{b!}^{\mathrm{ren}}\pi) \to {}^pH^0(i_{b*}^{\mathrm{ren}}\pi)\right).$$

Prove that $\mathbf{D}_{\mathrm{Verd}}(i_{b!*}^{\mathrm{ren}}\pi) \simeq i_{b!*}^{\mathrm{ren}}\pi^\vee$.

Now we come to the first really new construction in these notes.

Warning. In what follows, we assume that Λ is a field *of characteristic zero*. However, the following construction works verbatim for any coefficient ring Λ with the property that the standard truncation functors on $D(G_b(E), \Lambda)$ preserve the subcategory of compact objects for all $b \in B(G)$.[6]

Define ${}^hD^{\leq 0}(\mathrm{Bun}_G, \Lambda)^\omega$ to be the full subcategory of $D(\mathrm{Bun}_G, \Lambda)^\omega$ generated under finite extensions by objects of the form $i_{b\sharp}^{\mathrm{ren}}A$ with $A \in D^{\leq 0}(G_b(E), \Lambda)^\omega$. Likewise, define ${}^hD^{\geq 0}(\mathrm{Bun}_G, \Lambda)^\omega$ to be the full subcategory of $D(\mathrm{Bun}_G, \Lambda)^\omega$ generated under finite extensions by objects of the form $i_{b!}^{\mathrm{ren}}A$ with $A \in D^{\geq 0}(G_b(E), \Lambda)^\omega$.

Theorem 1.2.3. *The pair*

$$\left({}^hD^{\leq 0}(\mathrm{Bun}_G, \Lambda)^\omega, {}^hD^{\geq 0}(\mathrm{Bun}_G, \Lambda)^\omega\right)$$

defines a t-structure on $D(\mathrm{Bun}_G, \Lambda)^\omega$, called the hadal[7] t-structure. *We write ${}^h\tau^{\leq n}$ and ${}^h\tau^{\geq n}$ for the truncation functors associated with this t-structure, and we write* $\mathrm{Had}(\mathrm{Bun}_G, \Lambda)$ *for the abelian category of* hadal sheaves *defined as its heart.*

Of course we also write

$${}^hH^n = {}^h\tau^{\leq n} \circ {}^h\tau^{\geq n} \cong {}^h\tau^{\geq n} \circ {}^h\tau^{\leq n} : D(\mathrm{Bun}_G, \Lambda)^\omega \to \mathrm{Had}(\mathrm{Bun}_G, \Lambda)$$

for the nth hadal cohomology functor.

Proof. If $U \subset \mathrm{Bun}_G$ is any open substack, set $D(U) := D(U, \Lambda)^\omega$ for brevity. Define ${}^hD^{\leq 0}(U)$ and ${}^hD^{\geq 0}(U)$ inside $D(U)$ analogously with the definition for $U = \mathrm{Bun}_G$, but only allowing $b \in |U|$ in the specification of the generators. It is clear that if $j : U \to V$ is any inclusion of open substacks of Bun_G, $j_!$ carries ${}^hD^{\leq 0}(U)$ fully faithfully into ${}^hD^{\leq 0}(V)$, and likewise for ${}^hD^{\geq 0}$. It is also clear from the definitions that ${}^hD^{\leq 0}(U)[1] \subset {}^hD^{\leq 0}(U)$ and ${}^hD^{\geq 0}(U)[-1] \subset {}^hD^{\geq 0}(U)$. Finally, we observe

[6] For any fixed G, this condition holds with $\Lambda = \overline{\mathbf{F}_\ell}$ or $\overline{\mathbf{Z}_\ell}$ for all but finitely many ℓ. It would be interesting to explicate this finite set of bad primes when $G = \mathrm{GL}_n$.

[7] Pronounced "HEY-dull". See https://en.wikipedia.org/wiki/Hadal_zone. Unlike perverse sheaves, which propagate downwards from the maximal points in their support, hadal sheaves begin their life deep in the Newton strata of Bun_G, and then bubble up to the surface (i.e. the basic b's).

that ${}^hD^{\geq 1}(U)$ is exactly the right orthogonal of ${}^hD^{\leq 0}(U)$ inside $D(U)$. This is an easy but very enlightening exercise with the definitions, which we leave to the reader.

It is now clearly enough to prove that if $U \subset \mathrm{Bun}_G$ is a *quasicompact* open substack, the pair

$$\left({}^hD^{\leq 0}(U), {}^hD^{\geq 0}(U)\right)$$

defines a t-structure on $D(U)$. We will prove this by induction on the maximal length of any chain of specializations inside $|U|$. When there are no nontrivial such chains, U is a finite disjoint union of open strata $[*/\underline{G_b(E)}]$ with b basic. In this case the result is clear, since for basic b

$$\left({}^hD^{\leq 0}([*/\underline{G_b(E)}]), {}^hD^{\geq 0}([*/\underline{G_b(E)}])\right)$$

is just the standard t-structure on $D([*/\underline{G_b(E)}]) \cong D(G_b(E), \Lambda)^\omega$.

We now proceed by induction. Fix a quasicompact open substack U, and let $b \in |U|$ be the closed point in a chain of maximal length. Let $j : V \to U$ be the inclusion of the open substack with $|V| = |U| \smallsetminus b$, and let $i_b : \mathrm{Bun}_G^b \to U$ be the inclusion of the closed substack associated with b as usual. By the induction hypothesis, we already have access to the hadal t-structure and its truncation functors ${}^h\tau_V^{\leq n}$ and ${}^h\tau_V^{\geq n}$ on $D(V)$. By the observations in the first paragraph of the proof, our only remaining task is to show that any given $A \in D(U)$ can be fit into a distinguished triangle

$$E \to A \to C \xrightarrow{[1]}$$

with $E \in {}^hD^{\leq 0}(U)$ and $C \in {}^hD^{\geq 1}(U)$. For this, we define objects $B, C \in D(U)$ inductively by requiring that they sit in distinguished triangles

$$i^{\mathrm{ren}}_{b\sharp}\tau^{\leq 0} i_b^{*\mathrm{ren}}A \to A \to B \xrightarrow{[1]}$$

and

$$j_!\, {}^h\tau_V^{\leq 0} j^* B \to B \to C \xrightarrow{[1]} .$$

By the octahedral axiom, we then get an object $E \in D(U)$ together with a diagram

$$\begin{tikzcd}
i^{\mathrm{ren}}_{b\sharp}\tau^{\leq 0} i_b^{*\mathrm{ren}}A \arrow[d] & & \\
E \arrow[r] \arrow[d] & A \arrow[r] & C \arrow[r, "{[1]}"] & {} \\
j_!\, {}^h\tau_V^{\leq 0} j^* B \arrow[d, "{[1]}"] & & \\
{} & &
\end{tikzcd}$$

where the row and column are both distinguished triangles. It is clear that E sits in ${}^hD^{\leq 0}(U)$, since it is an extension of two objects in this category, so we just need to check that C sits in ${}^hD^{\geq 1}(U)$, which we've already noted is the right orthogonal of ${}^hD^{\leq 0}(U)$. It is therefore enough to check that $\mathrm{Hom}(i^{\mathrm{ren}}_{b'\sharp}F, C) = 0$ for any $b' \in |U|$ and any $F \in D^{\leq 0}(G_{b'}(E), \Lambda)^\omega$. To check this we divide into two disjoint cases:

Case 1: $b' = b$. In this case

$$\begin{aligned}\mathrm{Hom}(i^{\mathrm{ren}}_{b\sharp}F, C) &= \mathrm{Hom}(i^{\mathrm{ren}}_{b\sharp}F, B)\\ &\simeq \mathrm{Hom}(F, i^{*\mathrm{ren}}_b B)\\ &= 0\end{aligned}$$

where the first isomorphism follows from the triangle defining C and the fact that $\mathrm{Hom}(i^{\mathrm{ren}}_{b\sharp}F, j_!-) = 0$, and the final vanishing follows from the fact that $i^{*\mathrm{ren}}_b B \simeq \tau^{\geq 1} i^{*\mathrm{ren}}_b A$ by consideration of the triangle defining B.

Case 2: $b' \in |V|$. In this case $i^{\mathrm{ren}}_{b'\sharp}F = j_! i^{\mathrm{ren}}_{b'\sharp}F \in j_!\, {}^hD^{\leq 0}(V)$, with the evident abuse of notation, so

$$\mathrm{Hom}_U(i^{\mathrm{ren}}_{b'\sharp}F, C) = \mathrm{Hom}_V(i^{\mathrm{ren}}_{b'\sharp}F, j^*C).$$

But $j^*C \simeq {}^h\tau^{\geq 1}_V j^*B$ by consideration of the triangle defining C, and $i^{\mathrm{ren}}_{b'\sharp}F \in {}^hD^{\leq 0}(V)$, so we get the desired vanishing. □

Our next goal is Theorem 1.2.7, which gives an explicit classification of irreducible hadal sheaves. This requires several preparatory lemmas.

Lemma 1.2.4. *Fix $b \in B(G)$, and let π be any finitely generated smooth $G_b(E)$-representation.*

i. *The hadal sheaf ${}^hH^0(i^{\mathrm{ren}}_{b\sharp}\pi)$ does not have any nonzero hadal quotient supported on $\mathrm{Bun}_G^{\prec b}$.*

ii. *The hadal sheaf ${}^hH^0(i^{\mathrm{ren}}_{b!}\pi)$ does not have any nonzero hadal subobject supported on $\mathrm{Bun}_G^{\prec b}$.*

Proof. We prove the first claim; the second is analogous. Let $j : \mathrm{Bun}_G^{\prec b} \to \mathrm{Bun}_G$ be the evident open immersion. Let F be a hadal sheaf supported on $\mathrm{Bun}_G^{\prec b}$, so $F \cong j_!j^*F$. Since $i^{\mathrm{ren}}_{b\sharp}$ is right t-exact for the hadal t-structure, we compute that

$$\begin{aligned}\mathrm{Hom}({}^hH^0(i^{\mathrm{ren}}_{b\sharp}\pi), j_!j^*F) &\cong \mathrm{Hom}(i^{\mathrm{ren}}_{b\sharp}\pi, j_!j^*F)\\ &\cong \mathrm{Hom}(\pi, i^{*\mathrm{ren}}_b j_!j^*F)\\ &= 0,\end{aligned}$$

since $i^*_b j_! = 0$. □

Lemma 1.2.5. *Let $A \in \mathrm{Had}(\mathrm{Bun}_G, \Lambda)$ be a hadal sheaf, and let b be a maximally special point in the support of A, i.e. we assume that $i^*_bA \not\simeq 0$ and that $i^*_{b'}A = 0$ for all $b \prec b'$ in the Newton partial order. Then $i^{*\mathrm{ren}}_b A \in D(G_b(E), \Lambda)$ is concentrated in degree zero.*

Proof. Since $i^{*\mathrm{ren}}_b$ is left t-exact with respect to the hadal t-structure on Bun_G and the standard t-structure on $D(G_b(E), \Lambda)$, we already know that $i^{*\mathrm{ren}}_b A$ is concentrated in degrees ≥ 0. To see that it is also concentrated in degrees ≤ 0, pick a quasicompact open substack $U \subset \mathrm{Bun}_G$ containing $\mathrm{supp}A$, and such that $b \in U$ is a closed point in U. Write $V = U \smallsetminus \mathrm{Bun}_G^b$ as in the proof of Theorem 1.2.3, whose notation we will refer to in what follows. As in the proof of Theorem 1.2.3, we may regard A as a hadal sheaf in $D(U)$. Since $A \simeq {}^h\tau^{\leq 0}_U A$ by assumption, the constructions in that proof give a distinguished triangle

$$i^{\mathrm{ren}}_{b\sharp}\tau^{\leq 0} i^{*\mathrm{ren}}_b A \to A \to j_!\, {}^h\tau^{\leq 0}_V j^*B \xrightarrow{[1]},$$

upon noting that the object E we constructed there is exactly the connective truncation of A. Then $i^{*\mathrm{ren}}_b j_! = 0$ and $i^{*\mathrm{ren}}_b i^{\mathrm{ren}}_{b\sharp} \cong \mathrm{id}$, so applying $i^{*\mathrm{ren}}_b$ to this triangle gives

$$i^{*\mathrm{ren}}_b A \simeq \tau^{\leq 0} i^{*\mathrm{ren}}_b A,$$

as desired. $\square$

Lemma 1.2.6. *Fix any $b \in B(G)$. Then the functor on finitely generated $G_b(E)$-representations given by $M \mapsto i_b^{*\mathrm{ren}}\, {}^hH^0(i_{b\sharp}^{\mathrm{ren}} M)$ is naturally isomorphic to the identity functor.*

In particular, if $M \to N$ is a nonzero map of finitely generated $G_b(E)$-representations, the induced map ${}^hH^0(i_{b\sharp}^{\mathrm{ren}} M) \to {}^hH^0(i_{b\sharp}^{\mathrm{ren}} N)$ is necessarily nonzero.

Proof. Write $U = \mathrm{Bun}_G^{\preceq b}$ and $V = \mathrm{Bun}_G^{\prec b}$ as in the proof of Theorem 1.2.3, whose notation we will partly reuse. Since $A := i_{b\sharp}^{\mathrm{ren}} M$ is connective for the hadal t-structure, ${}^hH^0(A) \simeq {}^h\tau_U^{\geq 0} A$. Rerunning the proof of Theorem 1.2.3 with the appropriate shifts, we inductively define distinguished triangles

$$i_{b\sharp}^{\mathrm{ren}} \tau^{\leq -1} i_b^{*\mathrm{ren}} A \to A \to B \xrightarrow{[1]}$$

and

$$j_!\, {}^h\tau_V^{\leq -1} j^* B \to B \to {}^hH^0(A) \xrightarrow{[1]} .$$

Applying $i_b^{*\mathrm{ren}}$ to the second triangle we get $i_b^{*\mathrm{ren} h}H^0(A) \simeq i_b^{*\mathrm{ren}} B$, so then using this and applying $i_b^{*\mathrm{ren}}$ to the first triangle we get a distinguished triangle

$$i_b^{\mathrm{ren}*} i_{b\sharp}^{\mathrm{ren}} \tau^{\leq -1} i_b^{*\mathrm{ren}} A \to i_b^{*\mathrm{ren}} A \to i_b^{*\mathrm{ren} h}H^0(A) \xrightarrow{[1]} .$$

But $i_b^{*\mathrm{ren}} A \cong M$, so the first term vanishes identically, giving the result. $\square$

Theorem 1.2.7. *For every pair (b, π) with $b \in B(G)$ and π an irreducible smooth $G_b(E)$-representation, there is a unique* irreducible *hadal sheaf $\mathscr{G}_{b,\pi}$ characterized by the requirements that* $\mathrm{supp}\,\mathscr{G}_{b,\pi} \subseteq \mathrm{Bun}_G^{\preceq b}$ *and $i_b^{*\mathrm{ren}} \mathscr{G}_{b,\pi} \simeq \pi$, and given explicitly by the formula*

$$\mathscr{G}_{b,\pi} \simeq \mathrm{im}\left({}^hH^0(i_{b\sharp}^{\mathrm{ren}} \pi) \to {}^hH^0(i_{b!}^{\mathrm{ren}} \pi) \right) .$$

Moreover, every irreducible hadal sheaf arises from a uniquely associated pair (b, π) in this way.

We will sometimes write $i_{b\sharp !}^{\mathrm{ren}} \pi$ for the sheaf $\mathscr{G}_{b,\pi}$, in analogy with the notation $i_{b!*}^{\mathrm{ren}}$ for intermediate extension of perverse sheaves.

Proof. Let A be an irreducible hadal sheaf. Pick b a maximally special point in the support of A, so $i_b^{*\mathrm{ren}} A$ is concentrated in degree zero by Lemma 1.2.5. Then $i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} A$ is connective for the hadal t-structure, so via the adjunction $i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} \to \mathrm{id}$ we get a distinguished triangle

$$i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} A \to A \to K \xrightarrow{[1]}$$

of connective objects, such that $b \notin \mathrm{supp} K$. Since the hadal truncation functors are compatible with !-extension along open substacks, taking hadal cohomology gives a long exact sequence

$$0 \to {}^hH^{-1}(K) \to {}^hH^0(i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} A) \xrightarrow{\alpha} A \to {}^hH^0(K) \to 0$$

of hadal sheaves whose outer terms have support disjoint from b. Then α is nonzero (e.g. by noting that it induces an isomorphism between nonzero objects after applying $i_b^{*\mathrm{ren}}$), so the irreducibility hypothesis on A implies that α is surjective. Since the source of α has support contained in $\mathrm{Bun}_G^{\preceq b}$, we then deduce additionally that $\mathrm{supp} A \subseteq \mathrm{Bun}_G^{\preceq b}$.

Next, we show that the point b is uniquely determined. Let b' be any other maximally special point in the support of A. By a dual version of the argument in the first paragraph of the proof, we get an injective map

$$A \xrightarrow{\alpha'} {}^hH^0(i_{b'!}^{\mathrm{ren}} i_{b'}^{*\mathrm{ren}} A).$$

Composing this with the surjective map α constructed in the first paragraph, we get a nonzero map

$${}^hH^0(i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} A) \xrightarrow{\alpha' \circ \alpha} {}^hH^0(i_{b'!}^{\mathrm{ren}} i_{b'}^{*\mathrm{ren}} A).$$

But $i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} A$ is connective and $i_{b'!}^{\mathrm{ren}} i_{b'}^{*\mathrm{ren}} A$ is coconnective, by two applications of Lemma 1.2.5, so this is the same as the datum of a nonzero map

$$i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} A \to i_{b'!}^{\mathrm{ren}} i_{b'}^{*\mathrm{ren}} A.$$

But if such a nonzero map exists, then necessarily $b = b'$ by Proposition 1.1.2.iii.

Next we show that $i_b^{*\mathrm{ren}} A$, which is concentrated in degree zero by Lemma 1.2.5, is actually an irreducible smooth representation. Let $B \subseteq i_b^{*\mathrm{ren}} A$ be any subrepresentation. Consider the maps

$${}^hH^0(i_{b\sharp}^{\mathrm{ren}} B) \xrightarrow{\beta} {}^hH^0(i_{b\sharp}^{\mathrm{ren}} i_b^{*\mathrm{ren}} A) \xrightarrow{\alpha} A.$$

By the surjectivity of α and some general nonsense, we get an exact sequence

$$0 \to \ker \beta \to \ker \alpha \circ \beta \to \ker \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \alpha \circ \beta \to 0.$$

From the first paragraph of the proof, we already know that $\ker \alpha$ has support contained in $\mathrm{Bun}_G^{\prec b}$, so then also any quotient of $\ker \alpha$ has the same support property. Now, since A is irreducible, $\operatorname{coker} \alpha \circ \beta$ is either $\simeq 0$ or $\simeq A$. If it is $\simeq 0$, then

$$\operatorname{coker}\beta \simeq {}^hH^0(i_{b\sharp}^{\mathrm{ren}}(i_b^{*\mathrm{ren}} A/B))$$

is also quotient of $\ker \alpha$ and thus is supported in $\mathrm{Bun}_G^{\prec b}$, so it must vanish identically, which then forces $B = i_b^{*\mathrm{ren}} A$ by Lemma 1.2.6. If it is $\simeq A$, then $\mathrm{im}\beta \subseteq \ker \alpha$, and we get an exact sequence

$$0 \to \ker \beta \to {}^hH^0(i_{b\sharp}^{\mathrm{ren}} B) \xrightarrow{\tau} \ker \alpha \to \operatorname{coker} \beta \to A \to 0.$$

But we already know that $\ker \alpha$ is supported in $\mathrm{Bun}_G^{\prec b}$, while ${}^hH^0(i_{b\sharp}^{\mathrm{ren}} B)$ cannot have any quotient with this support property by Lemma 1.2.4. This forces $\tau = 0$ and then $\ker \beta \xrightarrow{\sim} {}^hH^0(i_{b\sharp}^{\mathrm{ren}} B)$, so then β is the zero map. But then the natural inclusion map $B \to i_b^{*\mathrm{ren}} A$ giving rise to β is also the zero map by Lemma 1.2.6, and thus $B \simeq 0$. Therefore, either $B = 0$ or $B = i_b^{*\mathrm{ren}} A$, so $i_b^{*\mathrm{ren}} A$ is irreducible.

Summarizing our efforts so far, we have produced from the irreducible hadal sheaf A a canonical pair (b, π) as in the statement of the theorem, with b the unique maximally special point in the support of A, and with $\pi = i_b^{*\mathrm{ren}} A$ irreducible. To reconstruct A from this datum, observe that in the course of our arguments, we obtained maps

$${}^hH^0(i_{b\sharp}^{\mathrm{ren}} \pi) \twoheadrightarrow A \hookrightarrow {}^hH^0(i_{b!}^{\mathrm{ren}} \pi)$$

whose composite is the canonical map induced by the natural transformation $i_{b\sharp}^{\mathrm{ren}} \to i_{b!}^{\mathrm{ren}}$. This implies that the association

$$(b, \pi) \mapsto \mathscr{G}_{b,\pi} \overset{\mathrm{def}}{=} \mathrm{im}\left({}^hH^0(i_{b\sharp}^{\mathrm{ren}} \pi) \to {}^hH^0(i_{b!}^{\mathrm{ren}} \pi)\right)$$

defines an inverse to our recipe for extracting the pair (b, π) from A.

Finally, we need to see that for *every* pair (b, π), the sheaf $\mathscr{G}_{b,\pi}$ defined in the previous paragraph is irreducible. Fix any such pair. Pick any irreducible hadal sheaf A together with a nonzero map $f : A \to \mathscr{G}_{b,\pi}$. By our arguments so far, we already know that $A \simeq \mathscr{G}_{b',\pi'}$ for some pair (b', π'). Composing the maps

$$ {}^hH^0(i^{\mathrm{ren}}_{b'\sharp}\pi') \twoheadrightarrow \mathscr{G}_{b',\pi'} \simeq A \xrightarrow{f} \mathscr{G}_{b,\pi} \hookrightarrow {}^hH^0(i^{\mathrm{ren}}_{b!}\pi) $$

gives a nonzero map ${}^hH^0(i^{\mathrm{ren}}_{b'\sharp}\pi') \to {}^hH^0(i^{\mathrm{ren}}_{b!}\pi)$. Arguing as in the second paragraph of the proof, the datum of such a nonzero map is equivalent to the datum of a nonzero map $i^{\mathrm{ren}}_{b'\sharp}\pi' \to i^{\mathrm{ren}}_{b!}\pi$. But if such a nonzero map exists, then necessarily $b = b'$ by Proposition 1.1.2.iii. The various adjunctions easily imply a general isomorphism $\mathrm{Hom}(i^{\mathrm{ren}}_{b\sharp}A, i^{\mathrm{ren}}_{b!}B) \cong \mathrm{Hom}(A, B)$, so now we also get a nonzero map $\pi' \to \pi$, and thus an isomorphism $\pi \simeq \pi'$. Therefore, $\mathscr{G}_{b,\pi} \simeq \mathscr{G}_{b',\pi'}$ is irreducible. □

Remark 1.2.8. Just as the perverse t-structure interacts well with Verdier duality, the hadal t-structure should interact cleanly with Bernstein-Zelevinsky duality. However, the precise statement is subtle, and we refer to Remark 2.3.5 for a more detailed explanation of what we expect in this direction.

Exercise 1.2.9. Check that the hadal t-structure on $D(\mathrm{Bun}_G, \Lambda)^\omega$ is bounded and nondegenerate.

1.3 The spectral side

Here we give a very brief recollection on the stack of L-parameters and its coarse moduli, mostly just to set notation; we regard this as the "easy" side of the categorical conjecture. For a much more detailed treatment, see [DHKM20], [Zhu21], or [FS21, Chapter VIII].

Let Λ be any $\mathbf{Z}_\ell[\sqrt{q}]$-algebra. We write $\mathrm{Par}_{G,\Lambda} = Z^1(W_E, \hat{G})_\Lambda/\hat{G}$ for the stack of ℓ-adically continuous L-parameters regarded as an Artin stack locally of finite type over $\mathrm{Spec}\,\Lambda$, with its tautological map $\tau_G : \mathrm{Par}_{G,\Lambda} \to B\hat{G}_\Lambda$. Note that we can define an analogous stack for any linear algebraic group, but for non-reductive groups the correct object turns out to be a *derived* Artin stack (see [Zhu21, Section 2.3]). We will only need this extra generality for parabolic subgroups, in the definition of spectral Eisenstein series.

We will primarily be interested in the case when $\Lambda = \overline{\mathbf{Q}_\ell}$, in which case we will drop Λ from the notation. When $\Lambda = \overline{\mathbf{Q}_\ell}$ we write X^{spec}_G for the coarse quotient of Par_G, with its canonical map $q : \mathrm{Par}_G \to X^{\mathrm{spec}}_G$. The $\overline{\mathbf{Q}_\ell}$-points of Par_G parametrize isomorphism classes of ℓ-adically continuous L-parameters $\phi : W_E \to {}^LG(\overline{\mathbf{Q}_\ell})$, which we will simply call L-parameters. If ϕ is any L-parameter, we write $S_\phi = \mathrm{Cent}_{\hat{G}}(\phi)$ for its centralizer group, and we write $i_\phi : BS_\phi \to \mathrm{Par}_G$ for the associated locally closed immersion. Note that by [Zhu21, Lemma 3.1.8], we can also regard Par_G as a moduli stack of Weil-Deligne parameters.

The closed points of X^{spec}_G are in canonical bijection with *semisimple* L-parameters $W_E \to {}^LG(\overline{\mathbf{Q}_\ell})$. Here we say an L-parameter is semisimple if it is Frobenius-semisimple and has open kernel. Note that X^{spec}_G is a disjoint union of affine varieties over $\mathrm{Spec}\,\overline{\mathbf{Q}_\ell}$, and $\mathcal{O}(X^{\mathrm{spec}}_G) \cong \mathcal{O}(\mathrm{Par}_G)$. Moreover, each component of X^{spec}_G is a quotient of a torus by a finite group, and in particular is Cohen-Macaulay. Note that any L-parameter has a unique semisimplification, corresponding to the image of the point $\phi \in \mathrm{Par}_G$ along q. Conversely, if ϕ is a semisimple parameter, the fiber of q over the associated closed point $x_\phi \in X^{\mathrm{spec}}_G$ is a moduli space of L-parameters with constant semisimplification ϕ. Each such fiber contains a unique *closed* $\overline{\mathbf{Q}_\ell}$-point, corresponding to the actual parameter ϕ. In particular, q induces a bijection from the *closed* $\overline{\mathbf{Q}}_\ell$-points of Par_G onto the closed points of X_G^{spec}.

It is instructive to understand the fibers of q more explicitly.

Proposition 1.3.1. *Fix a semisimple L-parameter ϕ. The reduced fiber of q over the associated closed point $x_\phi \in X_G^{\mathrm{spec}}$ admits the explicit presentation*

$$q^{-1}(x_\phi)^{\mathrm{red}} \simeq \left\{(u,N) \in \mathcal{U}_{S_\phi} \times \mathfrak{g}^{\mathrm{ad}\phi(I_E)} \mid \mathrm{ad}\phi(\mathrm{Fr})\cdot N = q^{-1}N, \mathrm{ad}u \cdot N = N\right\}/S_\phi$$

where S_ϕ acts by simultaneous conjugation.

Here $\mathcal{U}_{S_\phi}$ denotes the variety of unipotent elements in S_ϕ. Note that the closed substack cut out by $u = 1$ is exactly the Vogan variety

$$V_\phi = \left\{N \in \mathfrak{g}^{\mathrm{ad}\phi(I_E)} \mid \mathrm{ad}\phi(\mathrm{Fr})\cdot N = q^{-1}N\right\}/S_\phi$$

parametrizing Frobenius-semisimple L-parameters with semisimplification ϕ. We also note that the associated closed immersion $V_\phi \to q^{-1}(x_\phi)^{\mathrm{red}}$ has a natural retraction, given by forgetting u. In some cases, e.g. when $\phi(\mathrm{Fr})$ is regular semisimple, the Vogan variety is the entire fiber. At the other extreme, if ϕ is the trivial L-parameter (so $S_\phi = \hat{G}$), then only $N = 0$ can occur, but *any* u can occur, and the fiber is the entire quotient $\mathcal{U}_{\hat{G}}/\hat{G}$. In general, the geometry of the fiber involves variation of both u and N. Note that in the "classical" local Langlands correspondence, only Frobenius-semisimple L-parameters are relevant.

Exercise 1.3.2 (Hellmann)**.** Take $G = \mathrm{GL}_4$, and let ϕ be the semsimple parameter which is trivial on inertia and with $\phi(\mathrm{Fr}) = \mathrm{diag}(1,q,q,q^2)$. Explicate the finite topological space $|q^{-1}(x_\phi)|$ as a set, and draw all of the nontrivial specializations within it.

1.4 Categorical conjecture

We can now start to put the two sides together. For simplicitly, let L be an algebraic extension of $\mathbf{Q}_\ell(\sqrt{q})$, and let $\Lambda \in \{L, \mathcal{O}_L\}$. In all that follows, we assume either that $\Lambda = L$ or that ℓ is a very good prime for G in the sense of [FS21].

The essential carriers of information in Fargues-Scholze are the Hecke operators, which we briefly recall (see [FS21, Section IX.2]).

Theorem 1.4.1. *For any $V \in \mathrm{Rep}_\Lambda({}^LG)$, there is a naturally associated functor*

$$T_V : D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)^{BW_E}$$

where $D(\mathrm{Bun}_G, \Lambda)^{BW_E}$ denotes the appropriate category of W_E-equivariant objects in $D(\mathrm{Bun}_G, \Lambda)$. More generally, for any finite set I and any $V \in \mathrm{Rep}_\Lambda(({}^LG)^I)$, there is a naturally associated functor

$$T_V : D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)^{BW_E^I}.$$

Composing with the forgetful functor to $D(\mathrm{Bun}_G, \Lambda)$, the induced endofunctor

$$T_V : D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$$

depends only on the restriction of V to the diagonally embedded copy of $\hat{G}$ in ${}^LG^I$. As an endofunctor of $D(\mathrm{Bun}_G, \Lambda)$, T_V preserves compact and ULA objects, and has left and right adjoint given by $T_{V^\vee}$.

The first key construction in Fargues-Scholze linking the the spectral and automorphic worlds is a natural map

$$\mathcal{O}(\mathrm{Par}_{G,\Lambda}) \to \mathfrak{Z}(D(\mathrm{Bun}_G, \Lambda))$$

which they construct using the Hecke operators together with V. Lafforgue's excursion operator formalism [FS21, Theorem IX.5.2]. Note that for any $b \in B(G)$, the functor $i_{b!}^{\mathrm{ren}}$ defines a fully faithful embedding $D(G_b(E), \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$, which induces a map in the other direction on Bernstein centers. Post-composing with the above map, we obtain a canonical map $\mathcal{O}(\mathrm{Par}_{G,\Lambda}) \overset{\Psi_G^b}{\to} \mathfrak{Z}(G_b(E), \Lambda)$.[8] When $b = 1$, we simply write Ψ_G for this map. The following is a renormalized version of [FS21, Theorem IX.7.2].

Theorem 1.4.2. *For any G and any $b \in B(G)$, the diagram*

$$\begin{tikzcd}
\mathcal{O}(\mathrm{Par}_{G,\Lambda}) \arrow[r, "\Psi_G^b"] \arrow[d] & \mathfrak{Z}(G_b(E), \Lambda) \\
\mathcal{O}(\mathrm{Par}_{G_b,\Lambda}) \arrow[ur, "\Psi_{G_b}"'] &
\end{tikzcd}$$

commutes. Here the left vertical arrow is induced by the canonical finite map $\mathrm{Par}_{G_b,\Lambda} \to \mathrm{Par}_{G,\Lambda}$ associated with the canonical L-embedding ${}^L G_b \to {}^L G$.

Note that for non-basic b, our definition of Ψ_G^b differs from that of Fargues-Scholze, and is in fact simpler to use, as reflected in the fact that we have the canonical L-embedding appearing in the previous theorem rather than the twisted embedding used in Fargues-Scholze.

Exercise 1.4.3. Check that the map Ψ_G^b is unaltered upon replacing $i_{b!}^{\mathrm{ren}}$ by $i_{b\sharp}^{\mathrm{ren}}$ in its definition.

When $\Lambda = \overline{\mathbf{Q}_\ell}$, it can be helpful to think more geometrically: the datum of the map Ψ_G is equivalent to the datum of a map of (ind-)varieties $\Psi_G^{\mathrm{geom}} : X_G \to X_G^{\mathrm{spec}}$. From this perspective, the Fargues-Scholze construction of L-parameters is extremely transparent: any irreducible smooth representation π determines a closed point in X_G, and thus a closed point in X_G^{spec} via the map Ψ_G^{geom}. But closed points in X_G^{spec} correspond exactly to semisimple L-parameters.

However, Fargues-Scholze go much further than this, and prove the following substantial upgrade of the above construction.

Theorem 1.4.4. *There is a canonical Λ-linear $\otimes$-action of $\mathrm{Perf}(\mathrm{Par}_{G,\Lambda})$ on $D(\mathrm{Bun}_G, \Lambda)$ compatible with the action of Hecke operators and preserving the subcategory $D(\mathrm{Bun}_G, \Lambda)^\omega$ of compact objects.*

We write $C * (-) \circlearrowright D(\mathrm{Bun}_G, \Lambda)$ for the spectral action of $C \in \mathrm{Perf}(\mathrm{Par}_{G,\Lambda})$. Compatibility with Hecke operators means that the diagram

$$\begin{tikzcd}
 & \mathrm{End}\left(D(\mathrm{Bun}_G, \Lambda)\right) \\
\mathrm{Rep}_\Lambda({}^L G) \arrow[r] \arrow[ur, "V \mapsto T_V"] & \mathrm{Rep}_\Lambda(\hat{G}) \arrow[u, "V \mapsto \tau_G^* V * (-)"']
\end{tikzcd}$$

[8] We write $\mathfrak{Z}(G_b(E), \Lambda)$ for the center of the category of smooth $\Lambda[G_b(E)]$-modules. When $\Lambda = \overline{\mathbf{Q}_\ell}$, this is the usual Bernstein center, and we drop Λ from the notation. In that case we also write X_{G_b} for the Bernstein variety, so $\mathcal{O}(X_{G_b}) \cong \mathfrak{Z}(G_b(E))$.

commutes.

Fargues-Scholze then formulate a deep conjectural refinement of these constructions. To state this, we need to assume that G is quasisplit. We also choose a Whittaker datum, i.e. a Borel subgroup $B \subset G$ and a generic character $\psi : U(E) \to \Lambda^\times$, where $U \subset B$ is the unipotent radical. We will typically use ψ as shorthand for the choice of Whittaker datum. We may then define the space

$$W_\psi = \mathrm{ind}_U^G(\psi)$$

of compactly supported Whittaker functions with coefficients in Λ. In other words, $W_\psi \subset \mathcal{C}(G(E), \Lambda)$ is the space of functions such that $f(ug) = \psi(u)f(g)$ for all $u \in U(E)$ and $g \in G(E)$, f is right-invariant by some open compact subgroup of $G(E)$, and the support of f has compact image in $U(E)\backslash G(E)$. Note that W_ψ is a "large" $G(E)$-representation, but nonetheless has excellent properties (which we will recall in Appendix A).

Conjecture 1.4.5. *Let Λ be as above, and containing all p-power roots of unity. Then there is a natural Λ-linear equivalence of categories*

$$\mathbf{L}_\psi^G : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda}) \xrightarrow{\sim} D(\mathrm{Bun}_G, \Lambda)^\omega$$

which is compatible with the spectral action, and which (after ind-completion) sends the structure sheaf $\mathcal{O}_{\mathrm{Par}_{G,\Lambda}}$ to $i_{1!}W_\psi$.

Compatibility with the spectral action means that we should have $\mathbf{L}_\psi^G(A \otimes B) \simeq A * \mathbf{L}_\psi^G(B)$ for all $A \in \mathrm{Perf}(\mathrm{Par}_{G,\Lambda})$ and $B \in \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda})$. Applying this compatibility with $A \in \mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_{G,\Lambda})$ and $B = \mathcal{O}$, in combination with the expectation that $\mathbf{L}_\psi^G(\mathcal{O}) = i_{1!}W_\psi$, we deduce that necessarily we should have $\mathbf{L}_\psi^G(A) = A * i_{1!}W_\psi$ for $A \in \mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_{G,\Lambda})$.[9] In particular, we see that $A \mapsto A * i_{1!}W_\psi$ should map $\mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_{G,\Lambda})$ towards *compact* objects in $D(\mathrm{Bun}_G, \Lambda)$. This is not obvious! When Λ is a field, this is closely related to showing that the map $\pi \mapsto \varphi_\pi$ has finite fibers. We also see that $A \mapsto A * i_{1!}W_\psi$ should be *fully faithful* as a functor from Perf towards $D(\mathrm{Bun}_G, \Lambda)$, which again is far from obvious.

What further conditions do we expect $\mathbf{L}_\psi^G$ to satisfy?

Compatibility with the central grading

Using the inclusion $Z(\hat{G})^\Gamma \subset \hat{G}$, we get a decomposition

$$\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda}) \cong \oplus_{\chi \in X^*(Z(\hat{G})^\Gamma)} \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda})^{Z(\hat{G})^\Gamma = \chi}.$$

On the other hand, by a classic theorem of Kottwitz we have a canonical bijection

$$\pi_0(\mathrm{Bun}_G) = \pi_1(G)_\Gamma \cong X^*(Z(\hat{G})^\Gamma)$$
$$\alpha \mapsto \chi_\alpha$$

which induces a corresponding decomposition

$$D(\mathrm{Bun}_G, \Lambda)^\omega \cong \oplus_{\alpha \in \pi_1(G)_\Gamma} D(\mathrm{Bun}_{G,\alpha}, \Lambda)^\omega.$$

Writing $\mathrm{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Bun}_{G,\alpha}$, we expect that $\mathbf{L}_\psi^G$ should restrict to compatible equivalences

$$\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda})^{Z(\hat{G})^\Gamma = \chi_\alpha} \xrightarrow{\sim} D(\mathrm{Bun}_{G,\alpha}, \Lambda)^\omega$$

for all $\alpha \in \pi_1(G)_\Gamma$.

[9] Fargues likes to advocate the perspective that $A \mapsto A * i_{1!}W_\psi$ is a kind of non-abelian Fourier transform, and $\mathbf{L}_\psi^G$ should be some kind of "continuous" extension of it from Perf to Coh.

Compatibility with duality

We let $\mathbf{D}_{\mathrm{GS}} = R\mathscr{H}\mathrm{om}(-, \omega)$ denote Grothendieck-Serre duality functor on $\mathrm{Coh}(\mathrm{Par}_{G,\Lambda})$. (We note for later use that $\omega = \mathcal{O}_{\mathrm{Par}_{G,\Lambda}}$ canonically.) Let $c : \mathrm{Par}_{G,\Lambda} \xrightarrow{\sim} \mathrm{Par}_{G,\Lambda}$ be the involution defined by composition with the Chevalley involution at the level of L-parameters. Note that $\mathbf{D}_{\mathrm{GS}}$ and c^* commute, so *twisted Grothendieck-Serre duality*

$$\mathbf{D}_{\mathrm{twGS}} = c^* \mathbf{D}_{\mathrm{GS}}$$

still defines an involutive anti-equivalence on $\mathrm{Coh}(\mathrm{Par}_{G,\Lambda})$. The compatibility of categorical local Langlands with duality can now be formulated as follows.

Conjecture 1.4.6. *There is a natural equivalence of functors*

$$\mathbf{L}^G_{\psi^{-1}} \circ \mathbf{D}_{\mathrm{twGS}} \simeq \mathbf{D}_{\mathrm{BZ}} \circ \mathbf{L}^G_{\psi}$$

from $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda})$ *towards* $D(\mathrm{Bun}_G, \Lambda)^{\omega}$.

Note that this is again a meta-conjecture, since we have not actually given a candidate for the functor $\mathbf{L}^G_{\psi}$.

Compatibility with Eisenstein series

To explain this, we need to first formulate our expectations for Eisenstein series. A much more detailed discussion will appear in [HHS].

Expectation. For any parabolic $P = MU \subset G$, there is a canonically associated functor

$$\mathrm{Eis}_P = \mathrm{Eis}^G_P : D(\mathrm{Bun}_M, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$$

with the following properties.

1. There is a natural equivalence of functors $\mathrm{Eis}_P \circ i^M_{1!} \simeq i_{1!} \circ i^G_P$, where $i_{1!}$ and $i^M_{1!} : D(M(E), \Lambda) \to D(\mathrm{Bun}_M, \Lambda)$ are the appropriate extension by zero functors, and i^G_P denotes the functor of normalized parabolic induction. More generally, if $b \in B(M)$ is any element whose image in $B(G)$ is basic, we expect that $\mathrm{Eis}_P \circ i^M_{b!} \simeq i_{b!} \circ i^{G_b}_{P_b}$.

2. Eis_P is compatible with composition: for any inclusion of parabolics $P_1 = M_1U_1 \subset P_2 = M_2U_2$, $P_1 \cap M_2$ is a parabolic in M_2 with Levi M_1, and there should be a natural equivalence $\mathrm{Eis}^G_{P_1} \simeq \mathrm{Eis}^G_{P_2} \circ \mathrm{Eis}^{M_2}_{P_1 \cap M_2}$. This equivalence should be compatible with triple composition in the evident sense.

3. Eis_P is compatible with any extension of scalars $\Lambda \to \Lambda'$.

4. Eis_P commutes with all direct sums, and preserves compact objects.

5. Eis_P preserves ULA objects with quasicompact support.

6. When Λ is killed by a power of ℓ (so $D_{\mathrm{lis}} = D_{\mathrm{\acute{e}t}}$), Eis_P is the functor $p_!(\mathrm{IC}_{\mathrm{Bun}_P, \mathbf{Z}_\ell[\sqrt{q}]} \otimes_{\mathbf{Z}_\ell[\sqrt{q}]} q^*(-))$, where

$$\begin{array}{ccc} \mathrm{Bun}_P & \xrightarrow{p} & \mathrm{Bun}_G \\ \downarrow{\scriptstyle q} & & \\ \mathrm{Bun}_M & & \end{array}$$

is the usual diagram, and $\mathrm{IC}_{\mathrm{Bun}_P, \mathbf{Z}_\ell[\sqrt{q}]} \in D_{\mathrm{\acute{e}t}}(\mathrm{Bun}_P, \mathbf{Z}_\ell[\sqrt{q}])$ is a certain explicit invertible object (a square root of the dualizing complex on Bun_P, which can be described explicitly [HI23]).

Of course, when Λ is a torsion ring, the formula in 6. can and should be taken as the definition of Eis_P. With this definition, properties 1.-3. are relatively easy, but 4. and 5. seem to lie significantly deeper. The formula in 6. should in fact be applicable for any coefficient ring, once the sheaf-theoretic machinery is sufficiently developed.

We now expect the following compatibility with categorical local Langlands.

Conjecture 1.4.7. *For all standard parabolics $P = MU \subset G$, there is an equivalence $\mathbf{L}_\psi^G \circ \mathrm{Eis}_P^{\mathrm{spec}} \simeq \mathrm{Eis}_P \circ \mathbf{L}_{\psi_M}^M$, where $\mathrm{Eis}_P^{\mathrm{spec}} = p_*^{\mathrm{spec}} \circ q^{\mathrm{spec}*}$ is the spectral Eisenstein functor associated with the diagram*

$$\begin{tikzcd} \mathrm{Par}_{P,\Lambda} \arrow[r, "p^{\mathrm{spec}}"] \arrow[d, "q^{\mathrm{spec}}"] & \mathrm{Par}_{G,\Lambda} \\ \mathrm{Par}_{M,\Lambda} & \end{tikzcd}$$

of (derived) Artin stacks. This equivalence is compatible with composition in the evident sense.

Note: Zhu defines the spectral Eisenstein functor by the formula $p_*^{\mathrm{spec}} \circ q^{\mathrm{spec}!}$. However, there is an isomorphism $q^{\mathrm{spec}!} \simeq q^{\mathrm{spec}*}$ because q is quasismooth, hence Gorenstein by [AG15, Corollary 2.2.7], so $q^{\mathrm{spec}!}\mathcal{F} \simeq q^{\mathrm{spec}*}\mathcal{F} \otimes q^{\mathrm{spec}!}\mathcal{O}$ [Gai13, Remark 7.2.7], and one can show that $q^{\mathrm{spec}!}\mathcal{O} \simeq \mathcal{O}$ (see [Zhu21], Remark 2.3.8 and the comments after the proof of Proposition 2.3.9).

Remark 1.4.8. The functors $i_{b!}^{\mathrm{ren}}$ and $i_{b\sharp}^{\mathrm{ren}}$ are closely related to Eisenstein series. More precisely, let $b \in B(G)$ be any element, and let $M \subset G$ be the centralizer of its Newton point ν_b; we may assume $b \in M(\breve{E})$, so $M_b = G_b$. Let P and $\overline{P}$ be the attracting and repelling dynamic parabolics associated with ν_b. We then expect (for any coefficient ring) that there are natural isomorphisms

$$i_{b!}^{\mathrm{ren}} \simeq \mathrm{Eis}_P^G \circ i_{b!}^M$$

and

$$i_{b\sharp}^{\mathrm{ren}} \simeq \mathrm{Eis}_{\overline{P}}^G \circ i_{b!}^M$$

as functors $D(M_b(E), \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$. When Λ is a torsion ring, this is an easy exercise (modulo the identification of $\mathrm{IC}_{\mathrm{Bun}_P, \mathbf{Z}_\ell[\sqrt{q}]}$). See [FS21, Example V.3.4] for a hint.

1.5 More conjectures

In this section we illustrate how the categorical conjecture leads naturally to various additional conjectures.

Let us begin with the following question: How do Eisenstein series interact with Hecke operators? More precisely, we could ask: for fixed $P = MU \subset G$ and $V \in \mathrm{Rep}({}^L G)$, is there an intellegent way of rewriting the composite functor $T_V \mathrm{Eis}_P$? This seems rather difficult at first glance, but we can ask the same question on the spectral side. Here things become simpler, since the spectral analogue of T_V is tensoring with the tautological vector bundle $\tau_G^* V$. We are now asking whether the functor $\tau_G^* V \otimes \mathrm{Eis}_P^{\mathrm{spec}}$ can be rewritten in an intelligent way. This turns out to have a very satisfying answer.

Proposition 1.5.1. *Choose a finite filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V|\hat{P}_\Lambda$ such that the $\hat{U}_\Lambda$-action on each graded piece $W_i = V_i/V_{i-1}$ is trivial, i.e. such that each W_i is naturally inflated from $\mathrm{Rep}(\hat{M}_\Lambda)$. Then $\tau_G^* V \otimes \mathrm{Eis}_P^{\mathrm{spec}}(-)$ admits a corresponding finite filtration with graded pieces $\mathrm{Eis}_P^{\mathrm{spec}}(\tau_M^* W_i \otimes -)$.*

Proof. To begin, observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Par}_{M,\Lambda} \arrow[d, "\tau_M"] & \mathrm{Par}_{P,\Lambda} \arrow[l, "q^{\mathrm{spec}}"'] \arrow[r, "p^{\mathrm{spec}}"] \arrow[d, "\tau_P"] & \mathrm{Par}_{G,\Lambda} \arrow[d, "\tau_G"] \\
B\hat{M}_\Lambda & B\hat{P}_\Lambda \arrow[l] \arrow[r] & B\hat{G}_\Lambda
\end{tikzcd}$$

of derived Artin stacks. We can now use the projection formula to write

$$\begin{aligned}
\tau_G^* V \otimes \mathrm{Eis}_P^{\mathrm{spec}}(-) &= \tau_G^* V \otimes p_*^{\mathrm{spec}} q^{\mathrm{spec}*}(-) \\
&\simeq p_*^{\mathrm{spec}}(p^{\mathrm{spec}*}\tau_G^* V \otimes q^{\mathrm{spec}*}(-)) \\
&\simeq p_*^{\mathrm{spec}}(\tau_P^*(V|\hat{P}_\Lambda) \otimes q^{\mathrm{spec}*}(-)).
\end{aligned}$$

Then by assumption $\tau_P^*(V|\hat{P}_\Lambda)$ has a finite filtration with graded pieces $q^{\mathrm{spec}*}\tau_M^* W_i$, so the functor $p_*^{\mathrm{spec}}(\tau_P^*(V|\hat{P}_\Lambda) \otimes q^{\mathrm{spec}*}(-))$ acquires a finite filtration with graded pieces

$$\begin{aligned}
p_*^{\mathrm{spec}}(q^{\mathrm{spec}*}\tau_M^* W_i \otimes q^{\mathrm{spec}*}(-)) &\simeq p_*^{\mathrm{spec}} q^{\mathrm{spec}*}(\tau_M^* W_i \otimes -) \\
&= \mathrm{Eis}_P^{\mathrm{spec}}(\tau_M^* W_i \otimes -),
\end{aligned}$$

using that $q^{\mathrm{spec}*}$ is symmetric monoidal. □

But now we can turn this into a conjecture on the automorphic side.

Conjecture 1.5.2. *Choose a finite filtration* $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V|\hat{P}_\Lambda$ *such that the* $\hat{U}_\Lambda$*-action on each graded piece* $W_i = V_i/V_{i-1}$ *is trivial, i.e. such that each* W_i *is naturally inflated from* $\mathrm{Rep}(\hat{M}_\Lambda)$*. Then* $T_V \mathrm{Eis}_P(-)$ *admits a corresponding finite filtration with graded pieces* $\mathrm{Eis}_P(T_{W_i}-)$*.*

We emphasize that while this conjecture takes place purely on the automorphic side, it is forced on us by the categorical conjecture and the previous proposition. This conjecture has been proved in many cases by Hamann [Ham22], and a similar argument should work in general when Λ is a torsion ring, and for arbitrary coefficients once the sheaf-theoretic machinery improves.

We can also start combining our predictions in more artful ways. For instance, recall that we conjectured a natural equivalence $\mathbf{L}_\psi^G \circ \mathrm{Eis}_P^{\mathrm{spec}} \simeq \mathrm{Eis}_P \circ \mathbf{L}_{\psi_M}^M$. How does this interact with duality? Applying $\mathbf{D}_{\mathrm{BZ}}$ to both sides, we compute

$$\begin{aligned}
\mathbf{D}_{\mathrm{BZ}} \circ \mathrm{Eis}_P \circ \mathbf{L}_{\psi_M}^M &\simeq \mathbf{D}_{\mathrm{BZ}} \circ \mathbf{L}_\psi^G \circ \mathrm{Eis}_P^{\mathrm{spec}} \\
&\simeq \mathbf{L}_{\psi^{-1}}^G \circ \mathbf{D}_{\mathrm{twGS}} \circ \mathrm{Eis}_P^{\mathrm{spec}} \\
&\overset{!}{\simeq} \mathbf{L}_{\psi^{-1}}^G \circ \mathrm{Eis}_{\overline{P}}^{\mathrm{spec}} \circ \mathbf{D}_{\mathrm{twGS}}^M \\
&\simeq \mathrm{Eis}_{\overline{P}} \circ \mathbf{L}_{\psi_M^{-1}}^M \circ \mathbf{D}_{\mathrm{twGS}}^M \\
&\simeq \mathrm{Eis}_{\overline{P}} \circ \mathbf{D}_{\mathrm{BZ}}^M \circ \mathbf{L}_{\psi_M}^M.
\end{aligned}$$

Here we used several times the expected compatibility of the categorical equivalence with duality, along with the equivalence $\mathbf{L}_\psi^G \circ \mathrm{Eis}_P^{\mathrm{spec}} \simeq \mathrm{Eis}_P \circ \mathbf{L}_{\psi_M}^M$ and its variant $\mathbf{L}_{\psi^{-1}}^G \circ \mathrm{Eis}_{\overline{P}}^{\mathrm{spec}} \simeq \mathrm{Eis}_{\overline{P}} \circ \mathbf{L}_{\psi_M^{-1}}^M$. The only point which requires further analysis is the isomorphism labelled "!". We leave this as an enlightening exercise to the reader, with the key hint being that the Chevalley involution exchanges $\hat{P}$ and $\hat{\overline{P}}$. Anyway, recall now that $\mathbf{L}_{\psi_M}^M$ is supposed to be an equivalence of categories, in which case we may "cancel it out" from the first and last expressions. We have thus arrived at the following conjecture, which again lives purely on the automorphic side.

Conjecture 1.5.3. *For any given parabolic P with opposite $\overline{P}$, there is a natural equivalence of functors $\mathbf{D}_{\mathrm{BZ}} \circ \mathrm{Eis}_P \simeq \mathrm{Eis}_{\overline{P}} \circ \mathbf{D}^M_{\mathrm{BZ}}$ from $D(\mathrm{Bun}_M, \Lambda)^\omega$ to $D(\mathrm{Bun}_G, \Lambda)^\omega$.*

This is a geometric analogue of the well-known fact that $\mathbf{D}_{\mathrm{coh}} \circ i_P^G \simeq i_{\overline{P}}^G \circ \mathbf{D}^M_{\mathrm{coh}}$ with $\mathbf{C}$- or $\overline{\mathbf{Q}_\ell}$-coefficients, which was first observed by Bernstein and is in fact *equivalent* to his famous second adjointness theorem. With general coefficients, this isomorphism follows from recent work of Dat-Helm-Kurinczuk-Moss [DHKM22].

Exercise 1.5.4. 1) Prove that Conjecture 1.5.3 implies second adjointness.

2) (Difficult.) Fix a torsion coefficient ring Λ and a parabolic P. Prove that the following two statements are *equivalent*:

i. The functor Eis_P preserves compact objects, and Conjecture 1.5.3 is true.

ii. There is a natural equivalence of "constant term" functors $q_* p^! \simeq \overline{q}_! \overline{p}^*$. (Here $\overline{p}$ and $\overline{q}$ refer to the obvious maps in the defining diagram for $\mathrm{Eis}_{\overline{P}}$.)

Recall the map $\mathcal{O}(X_G^{\mathrm{spec}}) \cong \mathcal{O}(\mathrm{Par}_G) \to \mathfrak{Z}(D(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell}))$ discussed at the beginning of section 1.4. By the very nature of the categorical center, this induces a canonical and functorial map $\mathcal{O}(X_G^{\mathrm{spec}}) \to \mathrm{End}(\mathcal{F})$ for any $\mathcal{F} \in D(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$.

Exercise 1.5.5. Show that the categorical conjecture and Conjecture 1.4.7 together imply the following: for any parabolic $P = MU \subset G$ and any $\mathcal{F} \in D(\mathrm{Bun}_M, \overline{\mathbf{Q}_\ell})$, the diagram

$$\begin{tikzcd} \mathrm{End}(\mathcal{F}) \arrow[r] & \mathrm{End}(\mathrm{Eis}_P(\mathcal{F})) \\ \mathcal{O}(X_M^{\mathrm{spec}}) \arrow[u] & \mathcal{O}(X_G^{\mathrm{spec}}) \arrow[l] \arrow[u] \end{tikzcd}$$

commutes.

1.6 Finiteness conditions and spectral decomposition of sheaves

Now we set $\Lambda = \overline{\mathbf{Q}_\ell}$ and drop it from the notation. Before continuing our discussion, it is very useful to analyze how the various finiteness conditions on objects in $D(\mathrm{Bun}_G)$ interact, and how they interact with "spectral decomposition" of sheaves.

We begin with the following observations. For any semisimple parameter ϕ, we can formally define the full subcategory $D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi \subset D(\mathrm{Bun}_G)^{\mathrm{ULA}}$ of ϕ-*local* ULA sheaves spanned by objects A such that for all b and n, every irreducible subquotient of $H^n(i_b^{*\mathrm{ren}} A)$ has Fargues-Scholze parameter ϕ. It is easy to see that $D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi$ is a thick triangulated subcategory stable under Hecke operators, and one can also prove that it is stable under the perverse truncation functors. In fact, one can prove that $D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi$ is canonically a direct factor of $D(\mathrm{Bun}_G)^{\mathrm{ULA}}$, in the sense that any ULA sheaf A has a canonical and functorial decomposition $A \cong A_\phi \oplus A^\phi$ where $A_\phi \in D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi$ and

$$\mathrm{Hom}(B, A^\phi) = \mathrm{Hom}(A^\phi, B) = 0$$

for all $B \in D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi$ (see [Han23b] for a detailed statement and proof). One can also prove that Verdier duality induces an involutive anti-equivalence

$$\mathbf{D}_{\mathrm{Verd}} : D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi \xrightarrow{\sim} D(\mathrm{Bun}_G)^{\mathrm{ULA}}_{\phi^\vee}.$$

At the level of perverse sheaves, we get an evident category $\mathrm{Perv}(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}}$ of ϕ-local perverse ULA sheaves, which is a direct factor of $\mathrm{Perv}(\mathrm{Bun}_G)^{\mathrm{ULA}}$, and Verdier duality induces an exact anti-equivalence of abelian categories

$$\mathbf{D}_{\mathrm{Verd}} : \mathrm{Perv}(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}} \xrightarrow{\sim} \mathrm{Perv}(\mathrm{Bun}_G)_{\phi^\vee}^{\mathrm{ULA}}.$$

Exercise 1.6.1. Check that $\mathrm{Perv}(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}}$ is a Serre subcategory of $\mathrm{Perv}(\mathrm{Bun}_G)^{\mathrm{ULA}}$.

It will also be very useful to consider the following more restrictive finiteness condition.

Definition 1.6.2. A sheaf $A \in D(\mathrm{Bun}_G)$ is *finite* if it is both compact and ULA. Equivalently, A is finite if it has quasicompact support and $\oplus_n H^n(i_b^* A)$ is a finite length $G_b(E)$-representation for every b.

Finite sheaves clearly form a thick triangulated subcategory of $D(\mathrm{Bun}_G)$, which we denote $D(\mathrm{Bun}_G)_{\mathrm{fin}}$. The name is meant to suggest that such objects have finite length in some sense. Indeed, one easily checks that $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ is the thick triangulated subcategory of $D(\mathrm{Bun}_G)$ consisting of sheaves which can be obtained from objects of the form $i_{b!}^{\mathrm{ren}}\pi$ (with π any irreducible $G_b(E)$-representation) via finitely many shifts, cones and retracts. This category is stable under Hecke operators, since Hecke operators preserve compactness and ULAness separately. We also note that for any two finite sheaves A, B, $R\mathrm{Hom}(A, B)$ is a perfect complex of $\overline{\mathbf{Q}_\ell}$-vector spaces, and in particular $\mathrm{End}(A)$ is an Artinian $\overline{\mathbf{Q}_\ell}$-algebra. This is a special case of the more general fact that $R\mathrm{Hom}(A, B)$ is perfect whenever A is compact and B is ULA, which follows from [FS21, Prop. VII.7.4 and Prop. VII.7.9].

Warning. Finite sheaves are *not* stable under Verdier duality, except when G is a torus. Indeed, for non-toral groups it is easy to see that $i_{1!}\overline{\mathbf{Q}_\ell}$ is a finite sheaf whose Verdier dual $i_{1*}\overline{\mathbf{Q}_\ell}$ does not have quasicompact support. However, we have the following important result.

Theorem 1.6.3. *If $\pi \in \Pi(G_b)$ is irreducible, then $i_{b\sharp}^{\mathrm{ren}}\pi$ is finite. In particular, finite sheaves are stable under Bernstein-Zelevinsky duality.*

Proof. By Proposition 1.1.4 and an easy induction on support, it is enough to prove the first statement. This immediately reduces to proving that $i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}}\pi$ is compact and admissible for any $b' \preceq b$. Compactness is clear, since both $i_{b\sharp}^{\mathrm{ren}}$ and $i_{b'}^{*\mathrm{ren}}$ preserve compactness. For admissibility, we apply the following criterion: a compact object $B \in D(G_{b'}(E), \overline{\mathbf{Q}_\ell})$ is admissible iff its support in the Bernstein variety $X_{G_{b'}}$ is zero-dimensional. The "only if" direction is clear; for the "if" direction, note that any compact B is admissible "over the Bernstein center" [Ber84], so true admissibility follows if the action of the Bernstein center factors over an Artinian quotient.

To apply this criterion, we note that the composite map

$$c : X_{G_{b'}} \xrightarrow{\Psi_{G_{b'}}^{\mathrm{geom}}} X_{G_{b'}}^{\mathrm{spec}} \to X_G^{\mathrm{spec}}$$

has discrete fibers, and in fact it is a finite morphism after restricting to any connected component of the source. This is clear for the second map (see [DHKM22] for a more general statement which also works integrally), and for the first map it is Lemma 1.6.4 below. Writing $x_\pi \in X_G^{\mathrm{spec}}$ for the closed point corresponding to the semisimple L-parameter of π, we conclude by observing that the support of $i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}}\pi$ is contained (set-theoretically) in $c^{-1}(x_\pi)$, which follows from Theorem 1.4.2. The admissibility criterion then applies, giving the desired result. $\square$

Lemma 1.6.4. *For any G, the Fargues-Scholze map $\Psi_G^{\mathrm{geom}} : X_G \to X_G^{\mathrm{spec}}$ has discrete fibers, and in fact is a finite morphism after restricting to any connected component of X_G.*

Proof. This can be checked one component at a time in X_G. Then the usual explicit description of individual Bernstein components (in terms of a fixed pair $[M, \sigma]$ with σ supercuspidal) plus compatibility of everything with parabolic induction and twisting reduces us to the special case of a cuspidal component D. In this case, one simply uses the fact that the map from D to the spectral Bernstein variety is compatible with unramified twists, and unramified twisting has finite stabilizers in both settings and is transitive on D. (Alternatively - but it's basically the same argument in different clothing - in the case of a cuspidal component one can reduce the claim for G to the analogous claims for G^{der} and Z_G° separately, using compatibility with products and central isogenies. But for G^{der} it is trivial, since cuspidal components of semisimple groups are zero-dimensional, and for Z_G° it is clear since one has $X_T = X_T^{\mathrm{spec}}$ for T any torus.) □

Remark 1.6.5. I do not know how to prove this theorem with $\overline{\mathbf{F}_\ell}$-coefficients. The problem is that I have no idea how to prove Lemma 1.6.4 with $\overline{\mathbf{F}_\ell}$-coefficients in general, although the same argument as written above works for ℓ outside an explicit finite set of bad primes.

Exercise 1.6.6. Prove that for any compact objects $A, B \in D(\mathrm{Bun}_G)^\omega$, the Fargues-Scholze map $\mathcal{O}(\mathrm{Par}_G) \to \mathfrak{Z}(D(\mathrm{Bun}_G))$ naturally makes $\mathrm{Hom}(A, B)$ into a *finitely generated* $\mathcal{O}(\mathrm{Par}_G)$-module. Hint: Reduce to the case where A, B are !- or $\sharp$-pushforwards from individual strata, then use Lemma 1.6.4 and Theorem 1.4.2 together with Bernstein's basic finiteness theorems.

Now suppose A is a finite sheaf. Then A is ULA, so for any semisimple parameter we have a decomposition $A \cong A_\phi \oplus A^\phi$ as before, where A_ϕ and A^ϕ are both finite. Since A is finite, it is easy to see that $A_\phi = 0$ for all but finitely many ϕ, so we get a canonical direct sum decomposition

$$A \cong \bigoplus_\phi A_\phi$$

in $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ where only finitely many summands are nonzero. By the functoriality of this decomposition, we even get a canonical direct sum decomposition of categories

$$D(\mathrm{Bun}_G)_{\mathrm{fin}} \cong \bigoplus_\phi D(\mathrm{Bun}_G)_{\mathrm{fin},\phi}$$

where of course we set $D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \overset{\mathrm{def}}{=} D(\mathrm{Bun}_G)_{\mathrm{fin}} \cap D(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}}$.

Exercise 1.6.7. Check that Bernstein-Zelevinsky duality induces an involutive anti-equivalence

$$\mathbf{D}_{\mathrm{BZ}} : D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \overset{\sim}{\to} D(\mathrm{Bun}_G)_{\mathrm{fin},\phi^\vee}.$$

Finally, recall that have constructed the hadal t-structure (Theorem 1.2.3), which is a certain t-structure on $D(\mathrm{Bun}_G)^\omega$ with heart denoted $\mathrm{Had}(\mathrm{Bun}_G)$. We then write

$$\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}} \overset{\mathrm{def}}{=} \mathrm{Had}(\mathrm{Bun}_G) \cap D(\mathrm{Bun}_G)_{\mathrm{fin}}$$

for the subcategory of finite hadal sheaves. Using Theorem 1.6.3, one can check that the hadal truncation functors preserve $D(\mathrm{Bun}_G)_{\mathrm{fin}}$, so the pair

$$\left({}^hD^{\leq 0}(\mathrm{Bun}_G, \Lambda)^\omega \cap D(\mathrm{Bun}_G)_{\mathrm{fin}}, {}^hD^{\geq 0}(\mathrm{Bun}_G, \Lambda)^\omega \cap D(\mathrm{Bun}_G)_{\mathrm{fin}}\right)$$

defines a t-structure on $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ with heart $\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}}$. Finally, by the functoriality of the direct sum decompositions above, we get a canonical decomposition

$$\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}} \cong \bigoplus_{\phi} \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin},\phi}.$$

Exercise 1.6.8. i. Check that $\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}}$ is a Serre subcategory of $\mathrm{Had}(\mathrm{Bun}_G)$.

ii. Check that every object in $\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}}$ is of finite length.

Warning. When G is not a torus, the interplay between these finiteness conditions can be very subtle. In particular, we highlight the following phenomena:

1) The perverse truncation functors do not always preserve the property of being finite. An explicit example is given in the discussion after Conjecture 2.5.1.

2) There are examples of irreducible perverse sheaves which are not finite sheaves. For instance, the constant sheaf $\overline{\mathbf{Q}_\ell}$ on one component of Bun_G has this property.

3) There are examples of finite sheaves which are perverse, but whose Jordan-Holder series in the perverse category is infinite. For instance, the sheaf $i^{\mathrm{ren}}_{b_n!}\pi_n$ appearing at the end of section 2.2 has this property.

1.7 The categorical conjecture over $\overline{\mathbf{Q}_\ell}$, unconditionally

We continue to fix $\Lambda = \overline{\mathbf{Q}_\ell}$ as our coefficient ring. This leads to several simplifications in the formulation of the categorical conjecture:

1) the nilpotent singular support condition on $\mathrm{Coh}(\mathrm{Par}_G)$ is automatic [FS21, Prop. VIII.2.11], and

2) each connected component of Par_G has the property that IndPerf = QCoh [BZFN10, Corollary 3.22].

Using 2), we can formally upgrade the spectral action to a monoidal action of $\mathrm{QCoh}(\mathrm{Par}_G)$ on $D(\mathrm{Bun}_G)$. Acting on the Whittaker sheaf in particular yields a functor

$$\begin{aligned} a_\psi : \mathrm{QCoh}(\mathrm{Par}_G) &\to D(\mathrm{Bun}_G) \\ \mathcal{F} &\mapsto \mathcal{F} * i_{1!}W_\psi \end{aligned}$$

where we choose to write "a" for *a*ction. As noted in the discussion after Conjecture 1.4.5, we expect that a_ψ is fully faithful: under the hoped-for equivalence $D(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}(\mathrm{Par}_G)$, a_ψ should correspond to the natural fully faithful inclusion $\mathrm{QCoh} \overset{\Xi}{\to} \mathrm{IndCoh}$. As noted earler, we also expect that a_ψ carries $\mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_G)$ into $D(\mathrm{Bun}_G)^\omega$. This is known unconditionally for many groups: more precisely, for those G which are *reasonable* in the sense of Definition A.1.1.

Warning. The category IndCoh contains two copies of Coh: the "native" copy coming from the tautological inclusion $\mathcal{C} \subset \mathrm{Ind}(\mathcal{C})$, and a second "phantom" copy $\Xi(\mathrm{Coh})$. These are not the same, and their overlap is actually just Perf. In particular, non-perfect objects in the "phantom" copy are not compact when viewed as objects of IndCoh. Translating into the above picture, we see that for $\mathcal{F}$ a coherent complex on Par_G which is not perfect, $a_\psi(\mathcal{F})$ should not be compact. In particular, a_ψ cannot be the functor realizing the categorical local Langlands equivalence.

However, there is a closely related functor which *should* realize this equivalence.

Proposition 1.7.1. *There is a (unique) functor*

$$c_\psi : D(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{Par}_G)$$

such that for all $A \in D(\mathrm{Bun}_G)$ and all $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Par}_G)$, there is a natural isomorphism

$$R\mathrm{Hom}(i_{1!}W_\psi, \mathcal{F} * A) \simeq R\Gamma(\mathrm{Par}_G, \mathcal{F} \otimes c_\psi(A)).$$

The functor c_ψ is motivated by classical geometric Langlands, where the analogous beast is usually called the functor of *enhanced Whittaker coefficient*, and its construction in our setting is exactly the same as in classical geometric Langlands (see [FR22, Section 10.2] and [Ras22]). As justification for the name, note that

$$\begin{aligned} R\Gamma(\mathrm{Par}_G, c_\psi(A)) &\simeq R\mathrm{Hom}(i_{1!}W_\psi, A) \\ &= R\mathrm{Hom}(W_\psi, i_1^* A) \end{aligned}$$

is exactly the space of Whittaker models of i_1^*A. Note also that c_ψ is QCoh-linear, in the sense that $c_\psi(\mathcal{G} * A) \simeq \mathcal{G} \otimes c_\psi(A)$ for all $A \in D(\mathrm{Bun}_G)$ and $\mathcal{G} \in \mathrm{QCoh}(\mathrm{Par}_G)$. In particular, for all $V \in \mathrm{Rep}({}^L G)$ we have $c_\psi(T_V A) \simeq V \otimes c_\psi(A)$ and consequently

$$R\Gamma(\mathrm{Par}_G, V \otimes c_\psi(A)) \simeq R\mathrm{Hom}(i_{1!}W_\psi, T_V A).$$

Proposition 1.7.2. *The functor c_ψ is right adjoint to a_ψ.*

Proof. For $\mathcal{F} \in \mathrm{Perf}(\mathrm{Par}_G)$ and $A \in D(\mathrm{Bun}_G)$ we compute

$$\begin{aligned} \mathrm{Hom}(a_\psi(\mathcal{F}), A) &= \mathrm{Hom}(\mathcal{F} * i_{1!}W_\psi, A) \\ &\cong \mathrm{Hom}(i_{1!}W_\psi, \mathbf{D}_{\mathrm{GS}}\mathcal{F} * A) \\ &\cong \mathrm{Hom}(\mathcal{O}, \mathbf{D}_{\mathrm{GS}}\mathcal{F} \otimes c_\psi(A)) \\ &\cong \mathrm{Hom}(\mathcal{F}, c_\psi(A)). \end{aligned}$$

The only nontrivial point here is the second line, which follows from the fact that for any given $\mathcal{F} \in \mathrm{Perf}(\mathrm{Par}_G)$, the endofunctor $A \mapsto \mathcal{F} * A$ on $D(\mathrm{Bun}_G)$ is both left and right adjoint to the endofunctor $A \mapsto \mathbf{D}_{\mathrm{GS}}\mathcal{F} * A$. Since both sides of this isomorphism convert colimits in $\mathcal{F}$ into limits, it formally extends to an isomorphism valid for any $\mathcal{F} \in \mathrm{IndPerf} = \mathrm{QCoh}$. ☐

Since c_ψ is right adjoint to a_ψ, under the hoped-for equivalence $D(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}(\mathrm{Par}_G)$ it should correspond to the natural functor $\mathrm{IndCoh} \xrightarrow{\Psi} \mathrm{QCoh}$ right adjoint to Ξ. Now, it is easy to see that Ψ defines an equivalence from the "native" copy of Coh in IndCoh onto the usual copy of Coh inside QCoh.[10] We are thus led to the following conjecture.

Conjecture 1.7.3. *The functor c_ψ restricts to an equivalence of categories*

$$c_\psi : D(\mathrm{Bun}_G)^\omega \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G).$$

In other words, c_ψ should realize the categorical local Langlands equivalence. Note that this is a precise and unconditional conjecture! However, very little is obvious here. For instance, implicit in this conjecture is the expectation that c_ψ carries compact objects in $D(\mathrm{Bun}_G)$ to coherent complexes, which is already far from obvious.

Exercise 1.7.4. Prove that c_ψ is compatible with the central grading.

[10] This follows, for instance, from the observation that the sequence of functors $\mathrm{QCoh} \xrightarrow{\Xi} \mathrm{IndCoh} \xrightarrow{\Psi} \mathrm{QCoh}$ is obtained by ind-completing the tautological sequence of functors $\mathrm{Perf} \to \mathrm{Coh} \to \mathrm{QCoh}$.

Warning. If Conjecture 1.7.3 is true, the equivalence postulated there can be formally ind-completed to the desired equivalence

$$(\mathbf{L}_\psi^G)^{-1} : D(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Par}_G).$$

However, one should be careful to distinguish this functor from the functor c_ψ as initially defined above: c_ψ can be recovered from $(\mathbf{L}_\psi^G)^{-1}$ by composing with the quotient functor $\Psi : \mathrm{IndCoh}(\mathrm{Par}_G) \to \mathrm{QCoh}(\mathrm{Par}_G)$. Since Ψ identifies the two copies of Coh as discussed above, there is no distinction between $(\mathbf{L}_\psi^G)^{-1}$ and c_ψ as functors on *compact* sheaves on Bun_G, but on general sheaves they do differ. As a sobering exercise, one can unconditionally check that if we take A to be the constant sheaf $\overline{\mathbf{Q}_\ell}$ on Bun_G, then $c_\psi(A) = 0$. Of course, A is not compact, so this is no contradiction. When $G = \mathrm{PGL}_2$, an explicit candidate object $\mathcal{F}$ in $\mathrm{IndCoh}(\mathrm{Par}_G)$ which should match this particular sheaf under the putative functor $\mathbf{L}_\psi^G$ has been constructed by Bertoloni Meli, and it is visibly clear from his construction that $\Psi(\mathcal{F}) = 0$.

In this setting, we can also formulate the duality compatibility unconditionally.

Conjecture 1.7.5. *There is a natural equivalence of functors*

$$\mathbf{D}_{\mathrm{twGS}} \circ c_\psi \simeq c_{\psi^{-1}} \circ \mathbf{D}_{\mathrm{BZ}}$$

from $D(\mathrm{Bun}_G)^\omega$ *towards* $\mathrm{Coh}(\mathrm{Par}_G)$.

Here as before, $c_{\psi^{-1}}$ denotes the enhanced coefficient functor associated with the dual Whittaker datum (B, ψ^{-1}). Composing with $i_{1!}$ and using $i_{1!}\mathbf{D}_{\mathrm{coh}} \simeq \mathbf{D}_{\mathrm{BZ}} i_{1!}$, we see that Hellmann's functor $R_\psi = c_\psi \circ i_{1!}$ should satisfy $\mathbf{D}_{\mathrm{twGS}} \circ R_\psi \simeq R_{\psi^{-1}} \circ \mathbf{D}_{\mathrm{coh}}$.

As some evidence for this, we have the following result. For the notion of a reasonable group, see Definition A.1.1.

Proposition 1.7.6. *If G is reasonable, there is a natural equivalence of functors*

$$\mathbf{D}_{\mathrm{BZ}} \circ a_\psi \simeq a_{\psi^{-1}} \circ \mathbf{D}_{\mathrm{twGS}}$$

from $\mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_G)$ *towards* $D(\mathrm{Bun}_G)^\omega$.

Sketch. This follows by combining Theorem A.0.1 and Proposition A.1.3.i with the following result: For any $\mathcal{F} \in \mathrm{Perf}(\mathrm{Par}_G)$ and $A \in D(\mathrm{Bun}_G)^\omega$, one has $\mathbf{D}_{\mathrm{BZ}}(\mathcal{F} * A) \simeq \mathbf{D}_{\mathrm{twGS}}(\mathcal{F}) * \mathbf{D}_{\mathrm{BZ}}(A)$. This can be deduced from [FS21, Theorem VIII.5.1 and Theorem IX.2.2]. □

Exercise 1.7.7. (Difficult.) Assuming that G is reasonable and ℓ is a very good prime in the sense of [FS21], formulate and prove the appropriate variant of Proposition 1.7.6 with $\Lambda = \overline{\mathbf{Z}_\ell}$ or with $\Lambda = \overline{\mathbf{F}_\ell}$. (Note that this can be done *without* imposing some additional condition of being "ℓ-integrally reasonable".)

1.7.1 A variant with restricted variation

Recall that we defined the category of finite sheaves $D(\mathrm{Bun}_G)_{\mathrm{fin}}$.

Conjecture 1.7.8. *The functor c_ψ restricts to an equivalence from $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ towards the full subcategory $\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} \subset \mathrm{Coh}(\mathrm{Par}_G)$ spanned by objects which are supported set-theoretically on finitely many fibers of the map $q : \mathrm{Par}_G \to X_G^{\mathrm{spec}}$.*

This is reminiscent of (and motivated by) the recent "restricted variation" variant of classical geometric Langlands. The compatibility of c_ψ with the action of the spectral Bernstein center easily implies that for any $A \in D(\mathrm{Bun}_G)_{\mathrm{fin}}$, the quasicoherent complex $c_\psi(A)$ does in fact satisfy the desired support condition. One can be more precise: for any finite sheaf A, the ring $\mathrm{End}(A)$ is Artinian, so the natural map $\mathcal{O}(X_G^{\mathrm{spec}}) \to \mathrm{End}(A)$ factors over an Artinian quotient. Writing $Z_A \subset X_G^{\mathrm{spec}}$ for the associated finite-length subscheme, the sheaf $c_\psi(A)$ then has support contained in $q^{-1}(Z_A)$. With a little more thought, it's not difficult to see that Conjecture 1.7.3 implies Conjecture 1.7.8. In fact, the converse implication also holds, in the following form.

Proposition 1.7.9. *Suppose that the functor c_ψ preserves compact objects, and that Conjecture 1.7.8 is true. Then Conjecture 1.7.3 is true.*

Proof. This follows by a straightforward adaptation of the arguments in [AGK+23, Section 21.4]. ☐

However, the coherence property of c_ψ applied to any finite sheaf does not seem easy. Here is some weak evidence in its favor.

Proposition 1.7.10. *Suppose $A \in D(\mathrm{Bun}_G)_{\mathrm{fin}}$. Then for any $\mathcal{F} \in \mathrm{Perf}(\mathrm{Par}_G)$, $R\Gamma(\mathrm{Par}_G, \mathcal{F} \otimes c_\psi(A)) \in \mathrm{Perf}(\overline{\mathbf{Q}_\ell})$.*

Sketch. Reduce to the case where $\mathcal{F}$ is a vector bundle pulled back from $V \in \mathrm{Rep}(B\hat{G})$, so then

$$\begin{aligned} R\Gamma(\mathrm{Par}_G, V \otimes c_\psi(A)) &\simeq R\mathrm{Hom}(i_{1!}W_\psi, T_V A) \\ &\simeq R\mathrm{Hom}(W_\psi, i_1^* T_V A) \end{aligned}$$

by direct examination of the definitions. Then A is finite and Hecke operators preserve finite sheaves, so $i_1^* T_V A$ has only finitely many nonzero cohomologies, each of finite length, and hence is supported on a finite union of Bernstein components. If e is the idempotent projector onto this finite union of Bernstein components, then eW_ψ is compact by a result of Bushnell-Henniart cited in the discussion before Theorem A.0.1. Then

$$R\mathrm{Hom}(W_\psi, i_1^* T_V A) \simeq R\mathrm{Hom}(eW_\psi, i_1^* T_V A)$$

is perfect, as desired. ☐

Finally, we note that it can be useful to localize even further and study Conjecture 1.7.8 "one semisimple parameter at a time". More precisely, for any fixed semisimple parameter ϕ, define $\mathrm{Coh}(\mathrm{Par}_G)_\phi$ as the full subcategory of $\mathrm{Coh}(\mathrm{Par}_G)$ spanned by objects which are supported set-theoretically on the fiber $q^{-1}(x_\phi)$. Clearly we have a direct sum decomposition of categories

$$\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} = \bigoplus_\phi \mathrm{Coh}(\mathrm{Par}_G)_\phi.$$

Then Conjecture 1.7.8 is obviously equivalent to the expectation that c_ψ induces an equivalence $D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G)_\phi$ for every ϕ.

Exercise 1.7.11. Show that if G is reasonable (Definition A.1.1), the functor a_ψ induces functors

$$\mathrm{Perf}(\mathrm{Par}_G)_{\mathrm{fin}} \to D(\mathrm{Bun}_G)_{\mathrm{fin}}$$

and

$$\mathrm{Perf}(\mathrm{Par}_G)_\phi \to D(\mathrm{Bun}_G)_{\mathrm{fin},\phi}$$

for every semisimple parameter ϕ.

2 Compatibility with classical local Langlands

We continue to fix $\Lambda = \overline{\mathbf{Q}_\ell}$ as our coefficient ring, and drop it from the notation. In what sense does the categorical local Langlands conjecture encode classical local Langlands? The answer to this question turns out to be surprisingly complicated. Let us begin with a naive guess for what might be true, closely following [FS21, Remark I.10.3].

Best Hope. The categorical equivalence is t-exact with respect to the perverse t-structure on $D(\mathrm{Bun}_G)$ and some exotic perverse coherent t-structure on $\mathrm{Coh}(\mathrm{Par}_G)$, and thus induces a bijection between irreducible objects in the hearts. On the Bun_G side, the irreducibles are indexed by pairs (b, π) where $b \in B(G)$ and $\pi \in \mathrm{Irr}(G_b(E))$. On the Par_G side, the irreducibles are indexed by pairs (ϕ, ρ) where ϕ is a Frobenius-semisimple L-parameter and ρ is an irreducible algebraic representation of the centralizer S_ϕ.

Unfortunately, this "best hope" seems slightly too simple, for several related reasons:

1. Par_G includes many points corresponding to L-parameters which are *not* Frobenius-semisimple, which play no role in the classical local Langlands correspondence.

2. The perverse t-structure on $D(\mathrm{Bun}_G)$ is *not* self-dual with respect to $\mathbf{D}_{\mathrm{BZ}}$, which however is the natural duality appearing in the categorical conjecture.

3. The geometry of Par_G seems to preclude any simple direct definition of the hoped-for t-structure, especially around L-parameters with nontrivial monodromy, since perverse coherence is not (naively) a reasonable notion around these parameters.

One of our main points is that neverthless, the "best hope" should be true for most L-parameters, in a sense we will make precise. The starting point for this is a remarkable recent result of Bertoloni Meli-Oi [BMO23], which says that the bijection on irreducibles predicted by the "best hope" is actually true! We will then turn things around and use their results as a guide to formulate some precise guesses for how the categorical LLC interacts with the classical LLC.

Here is a brief and impressionistic outline of our current understanding. We will spend most of this section developing this outline into a precise collection of conjectures, and giving evidence for them.

- After localizing around a *generous* L-parameter (Definition 2.1.5), the perverse and hadal t-structures on (appropriately decorated versions of) $D(\mathrm{Bun}_G)$ should coincide, and they should match with the standard t-structure on $\mathrm{Coh}(\mathrm{Par}_G)$. Moreover, sheaves on different strata of Bun_G will not interact with each other, and Hecke operators should be t-exact.

- After localizing around a *semisimple generic* L-parameter (Definition 2.3.1), the perverse t-structure on $D(\mathrm{Bun}_G)$ should match the standard t-structure on $\mathrm{Coh}(\mathrm{Par}_G)$, and the hadal t-structure on $D(\mathrm{Bun}_G)$ should match some perverse coherent t-structure on $\mathrm{Coh}(\mathrm{Par}_G)$. Sheaves on different strata of Bun_G will interact with each other in highly nontrivial ways, but Hecke operators should still be t-exact for both t-structures.

- Around a generic L-parameter, the spectral action should encode the classical local Langlands correspondence in a precise way, although we do not yet know how to upgrade this to a t-exact matching of sheaves as in the best hope. See section 3.1 for a detailed discussion.

Beyond the semisimple generic case, we have the following very natural general question: The hadal t-structure is defined on the whole category $D(\mathrm{Bun}_G)^\omega$, so Conjecture 1.7.3 implies that it *must* correspond to *some* t-structure on $\mathrm{Coh}(\mathrm{Par}_G)$. Can we describe this t-structure intrinsically?

Remark 2.0.1. It is of significant interest to extend these ideas to integral and torsion coefficients. Some precise conjectures in these settings have been proposed by Hamann and Hamann-Lee ([Ham23, Section 3] and [HL23, Section 6.2]). We note in particular that our conditions of being "generous" and "semisimple generic" match with the conditions of being "Langlands-Shahidi type" and "weakly Langlands-Shahidi type", respectively, as formulated in [HL23, Definition 6.2]. However, we emphasize that [HL23, Definition 6.2] also makes sense with $\overline{\mathbf{F}_\ell}$-coefficients, which falls outside the scope of our discussion. I would also like to point out that I landed on the notion of generous parameters while trying to understand [Ham22]. Finally, we also refer to [Ham23, HL23] for some very striking global applications of this philosophy.

2.1 Generous L-parameters

Let us begin by recalling the $B(G)_{\mathrm{bas}}$ form of the local Langlands correspondence. Fix G quasisplit and pinned as usual, and fix a Whittaker datum ψ. For any ϕ, we set $S_\phi^\natural = S_\phi/(S_\phi \cap \hat{G}^{\mathrm{der}})^\circ$. This is a disconnected reductive group whose identity component is a torus, and there is a natural map of algebraic groups $Z(\hat{G})^\Gamma \to S_\phi^\natural$ which turns out to be an isogeny.

The following form of the local Langlands conjecture was formulated by Kottwitz in unpublished work. To the best of my knowledge it first appeared in print in [Rap95] (see the discussion preceding Conjecture 5.1). We also refer to [Kal16, Conjecture F] for a modern and slightly more precise formulation.

Conjecture 2.1.1 ($B(G)_{\mathrm{bas}}$ local Langlands correspondence)**.** *For each $b \in B(G)_{\mathrm{bas}}$, there is a natural finite-to-one map $\Pi(G_b) \to \Phi(G)$. Writing $\Pi_\phi(G_b)$ for the fiber over ϕ, there is a natural bijection*

$$\iota_\psi : \coprod_{b \in B(G)_{\mathrm{bas}}} \Pi_\phi(G_b) \xrightarrow{\sim} \mathrm{Irr}(S_\phi^\natural)$$

depending only on the choice of Whittaker datum, such that the diagram

$$\begin{tikzcd} \coprod_{b \in B(G)_{\mathrm{bas}}} \Pi_\phi(G_b) \arrow[r, "\iota_\psi"] \arrow[d] & \mathrm{Irr}(S_\phi^\natural) \arrow[d] \\ B(G)_{\mathrm{bas}} \arrow[r, "\kappa_G"] & X^*(Z(\hat{G})^\Gamma) \end{tikzcd}$$

commutes. Here the left vertical arrow is the obvious projection, the right vertical map is induced by restriction along $Z(\hat{G})^\Gamma \to S_\phi^\natural$, and κ_G is the Kottwitz isomorphism.

Of course, the word "natural" is doing a lot of heavy lifting here. However, this conjecture is now known unconditionally for many groups.

Theorem 2.1.2 (Bertoloni Meli-Oi)**.** *Suppose that the $B(G)_{\mathrm{bas}}$ LLC holds for G and all its standard Levi subgroups. Then for each $b \in B(G)$, there is a natural finite-to-one map $\Pi(G_b) \to \Phi(G)$. Writing $\Pi_\phi(G_b)$ for the fiber over ϕ, there is a natural bijection*

$$\iota_\psi : \coprod_{b \in B(G)} \Pi_\phi(G_b) \xrightarrow{\sim} \mathrm{Irr}(S_\phi)$$

depending only on the choice of Whittaker datum, such that the diagram

$$\begin{tikzcd}
\coprod_{b\in B(G)} \Pi_\phi(G_b) \arrow[r, "\iota_\psi"] \arrow[d] & \mathrm{Irr}(S_\phi) \arrow[d] \\
B(G) \arrow[r, "\kappa_G"] & X^*(Z(\hat{G})^\Gamma)
\end{tikzcd}$$

commutes. The left vertical map is the obvious projection and the right vertical map is induced by restriction along $Z(\hat{G})^\Gamma \to S_\phi$.

We like to think of this as follows: there is a natural bijection between pairs (ϕ, ρ) where ϕ is a Frobenius-semisimple L-parameter and $\rho \in \mathrm{Irr}(S_\phi)$ is an irreducible algebraic representation, and pairs (b, π) where $b \in B(G)$ and π is an irreducible smooth representation of $G_b(E)$. We will sometimes call this the BM-O bijection. We refer the reader to [BMO23] for the details behind this beautiful construction.

Warning. For non-basic b, the set $\Pi_\phi(G_b)$ appearing in Theorem 2.1.2 is usually not an L-packet. Rather, it is the (finite) union of the L-packets attached to the parameters $\phi' : W_E \times \mathrm{SL}_2 \to {}^L G_b$ such that $i \circ \phi'$ is conjugate to ϕ, for $i : {}^L G_b \to {}^L G$ the evident map of L-groups. Said differently, the set $\Pi_\phi(G_b)$ is the fiber of the composite map

$$\Pi(G_b) \to \Phi(G_b) \to \Phi(G),$$

and the second arrow is typically not injective.

Remark 2.1.3. If ϕ is discrete, then $S_\phi \cong S_\phi^\natural$, and $\Pi_\phi(G_b)$ is empty for all non-basic b. In this case the $B(G)$ LLC contains the same information as the $B(G)_{\mathrm{bas}}$ LLC. In general, the BM-O bijection extends the $B(G)_{\mathrm{bas}}$ correspondence in the evident way, by restricting to $\rho \in \mathrm{Irr}(S_\phi^\natural) \subset \mathrm{Irr}(S_\phi)$ on the right-hand side and $b \in B(G)_{\mathrm{bas}}$ on the left-hand side.

Example 2.1.4. Take $G = \mathrm{GL}_2$, and let $\phi = \chi_1 \oplus \chi_2$ be a sum of two random characters. By local class field theory we can also think of the χ_i's as smooth characters of $E^\times$. Then $S_\phi = \mathbf{G}_m^2$, so $\mathrm{Irr}(S_\phi) = \mathbf{Z}^2$. How to attach to an ordered pair $(m, n) \in \mathbf{Z}^2$ a pair (b, π)? It's easy to guess what to do for b: just take the isocrystal with slopes m and n. Now when $m = n$, $G_b = G$ and we can take $\pi = i_B^G(\chi_1 \boxtimes \chi_2)$. When $m \neq n$, $G_b = T$ is the usual maximal torus in G, and we take $\pi = \chi_1 \boxtimes \chi_2$ or $\pi = \chi_2 \boxtimes \chi_1$ according to whether $m > n$ or $m < n$. Note that $\Pi_\phi(G_b)$ contains two elements when b is not basic.

Definition 2.1.5. A semisimple L-parameter $\phi : W_E \to {}^L G(\overline{\mathbf{Q}_\ell})$ is *generous* if $\mathrm{Par}_G \times_{X_G^{\mathrm{spec}}} \{x_\phi\} \cong BS_\phi$.

The name is meant to suggest on one hand that such parameters are "generic", in two different ways - they form an open dense subset of X_G^{spec}, and they are generic in the technical sense of Proposition 2.3.1.(2) - and also that they have nice properties which make the classical and categorical local Langlands correspondence simpler at these parameters.

Example 2.1.6. Suppose $G = \mathrm{GL}_n$. Then an L-parameter is generous iff it is a direct sum $\phi \simeq \phi_1 \oplus \cdots \oplus \phi_d$ where the ϕ_i's are pairwise-distinct supercuspidal parameters with $\sum_i \dim\phi_i = n$, and with $\phi_i \not\simeq \phi_j(1)$ for any $i \neq j$. Note that the associated $\mathrm{GL}_n(E)$-representation π is generic, and is irreducibly induced from a supercuspidal representation of a Levi.

Proposition 2.1.7. 1) *A discrete L-parameter is generous iff it is supercuspidal.*

2) *If ϕ is generous, then S_ϕ° is a torus.*

3) *If ϕ is generous, then q is flat in a neighborhood of $q^{-1}(x_\phi)$, and there is a natural* regular *closed immersion*

$$i_\phi : BS_\phi = \mathrm{Par}_G \times_{X_G^{\mathrm{spec}}} \{x_\phi\} \hookrightarrow \mathrm{Par}_G,$$

and Par_G *is smooth in a neighborhood of* $\mathrm{im}\, i_\phi$.

In particular, if ϕ is generous, the pushforward $i_{\phi*}$ sends irreducible representations of S_ϕ towards perfect complexes on Par_G. We can now state the main conjecture relating classical and categorical LLC at generous parameters.

Conjecture 2.1.8. *Suppose that (ϕ, ρ) and (b, π) match under the BM-O bijection associated with the Whittaker datum ψ, where ϕ is a generous L-parameter.*

i. *There is an isomorphism*

$$a_\psi(i_{\phi*}\rho) \overset{\mathrm{def}}{=} i_{\phi*}\rho * i_{1!}W_\psi \simeq i_{b!}^{\mathrm{ren}}\pi$$

in $D(\mathrm{Bun}_G)$.

ii. *The natural maps*

$$i_{b\sharp}^{\mathrm{ren}}\pi \overset{\sim}{\to} i_{b!}^{\mathrm{ren}}\pi \overset{\sim}{\to} i_{b*}^{\mathrm{ren}}\pi$$

are isomorphisms.

This immediately suggests another conjecture, which in some cases can be verified more easily.

Conjecture 2.1.9. *Suppose that ϕ is a generous L-parameter. Then*

$$\mathscr{F}_\phi = \bigoplus_{b\in B(G), \pi\in\Pi_\phi(G_b)} i_{b!}^{\mathrm{ren}}\pi^{\oplus \dim \iota_\psi(b,\pi)}$$

is a Hecke eigensheaf with eigenvalue ϕ.

This follows immediately from Conjecture 2.1.8, since that conjecture formally implies an isomorphism

$$\mathscr{F}_\phi \simeq i_{\phi*}\mathcal{O}(S_\phi) * i_{1!}W_\psi$$

where $\mathcal{O}(S_\phi)$ is the regular representation, and the right-hand side is a Hecke eigensheaf of the stated type for completely formal reasons. Note that the second part of Conjecture 2.1.8 implies additionally that $\mathscr{F}_\phi$ is perverse.

Remark 2.1.10. We emphasize that Conjecture 2.1.8 completely describes how to compute Hecke operators on the atomic sheaves attached to generous parameters. More precisely, suppose ϕ is generous and (b, π) matches (ϕ, ρ) under the BM-O bijection. Let $V \in \mathrm{Rep}({}^L G)$ be any representation, with associated Hecke operator T_V. To compute $T_V i_{b!}^{\mathrm{ren}}\pi$, simply decompose $V|_{S_\phi} \otimes \rho \simeq \oplus_j \rho_j^{\oplus m_j}$ as a sum of irreducible S_ϕ-representations (with multiplicity). Let (b_j, π_j) be the pair matching (ϕ, ρ_j) under the BM-O bijection. Then Conjecture 2.1.8 predicts that

$$T_V i_{b!}^{\mathrm{ren}}\pi \simeq \oplus_j i_{b_j!}^{\mathrm{ren}}\pi_j^{\oplus m_j}.$$

With slightly more effort, one can also describe the Galois action on $T_V i_{b!}^{\mathrm{ren}}$ explicitly. The final outcome is the formula

$$T_V i_{b!}^{\mathrm{ren}}\pi \simeq \bigoplus_{(b',\pi')} i_{b'!}^{\mathrm{ren}}\pi' \boxtimes \mathrm{Hom}_{S_\phi}(\iota_\psi(b,\pi)^\vee \otimes \iota_\psi(b',\pi'), V\circ\phi)$$

in $D(\mathrm{Bun}_G)^{BW_E}$, where the W_E-action comes via the second factor in the evident sense. When ϕ is supercuspidal, this exactly recovers the Kottwitz conjecture.

Remark 2.1.11. If we know the first part of Conjecture 2.1.8 for ϕ and $\phi^\vee$, then in fact the second part follows. More precisely, suppose that (ϕ,ρ) and (b,π) match under the BM-O bijection for ψ. Then we expect that $(\phi^\vee, c\circ\rho^\vee)$ and $(b,\pi^\vee)$ match under the BM-O bijection for ψ^{-1} [Kal13], so the first part of Conjecture 2.1.8 implies that

$$(i_{\phi^\vee *}c\circ\rho^\vee) * i_{1!}W_{\psi^{-1}} \simeq i_{b!}^{\mathrm{ren}}\pi^\vee.$$

On the other hand, it is easy to compute that

$$\begin{aligned}\mathbf{D}_{\mathrm{twGS}}(i_{\phi*}\rho)[\dim S_\phi] &\simeq c^* i_{\phi*}\rho^\vee \\ &\simeq i_{\phi^\vee *}c\circ\rho^\vee,\end{aligned}$$

so using the compatibility of the spectral action with duality proved in Proposition 1.7.6, we compute that

$$\begin{aligned}(i_{\phi^\vee *}c\circ\rho^\vee) * i_{1!}W_{\psi^{-1}} &\simeq \mathbf{D}_{\mathrm{BZ}}\left(i_{\phi*}\rho * i_{1!}W_\psi\right)[\dim S_\phi] \\ &\simeq \mathbf{D}_{\mathrm{BZ}}\left(i_{b!}^{\mathrm{ren}}\pi\right)[\dim S_\phi] \\ &\simeq i_{b\sharp}^{\mathrm{ren}}\mathbf{D}_{\mathrm{coh}}(\pi)[\dim S_\phi]\end{aligned}$$

where we used Proposition 1.1.4 in the last line. Equating these two calculations, we get an isomorphism $i_{b!}^{\mathrm{ren}}\pi^\vee \simeq i_{b\sharp}^{\mathrm{ren}}\mathbf{D}_{\mathrm{coh}}(\pi)[\dim S_\phi]$. From this we immediately see (by taking the stalk at b) that $\pi^\vee \simeq \mathbf{D}_{\mathrm{coh}}(\pi)[\dim S_\phi] \simeq \mathrm{Zel}(\pi)$, and then that $i_{b!}^{\mathrm{ren}}\pi^\vee \simeq i_{b\sharp}^{\mathrm{ren}}\pi^\vee$. Applying $\mathbf{D}_{\mathrm{BZ}}$ to this last isomorphism and using Proposition 1.1.4 again, we get the same isomorphism for π. This implies the first isomorphism in the second part of the conjecture.

To get the second isomorphism, we argue more generally as follows. Fix a semisimple L-parameter ϕ, and suppose that $i_{b\sharp}^{\mathrm{ren}}\pi \xrightarrow{\sim} i_{b!}^{\mathrm{ren}}\pi$ is an isomorphism for all b and all $\pi \in \Pi(G_b)$ with Fargues-Scholze parameter ϕ. It then follows that $i_{b!}^{\mathrm{ren}}\pi \xrightarrow{\sim} i_{b*}^{\mathrm{ren}}\pi$ is an isomorphism for all b and all $\pi\in\Pi(G_b)$ with Fargues-Scholze parameter ϕ. Indeed, it's clearly enough to prove that $i_{b'}^{*\mathrm{ren}}i_{b*}^{\mathrm{ren}}\pi = 0$ for all $b'\neq b$. Using that $i_{b'}^{*\mathrm{ren}}i_{b*}^{\mathrm{ren}}\pi$ is admissible and left-bounded, with all of its irreducible constituents having Fargues-Scholze parameter ϕ, we easily reduce further to proving that $R\mathrm{Hom}(\tau, i_{b'}^{*\mathrm{ren}}i_{b*}^{\mathrm{ren}}\pi)=0$ for all $b'\neq b$ and all $\tau\in\Pi(G_{b'})$ with Fargues-Scholze parameter ϕ. But now we compute that

$$\begin{aligned}R\mathrm{Hom}_{G_{b'}}(\tau, i_{b'}^{*\mathrm{ren}}i_{b*}^{\mathrm{ren}}\pi) &\cong R\mathrm{Hom}(i_{b'\sharp}^{\mathrm{ren}}\tau, i_{b*}^{\mathrm{ren}}\pi) \\ &\cong R\mathrm{Hom}(i_{b'!}^{\mathrm{ren}}\tau, i_{b*}^{\mathrm{ren}}\pi) \\ &\cong R\mathrm{Hom}_{G_b}(i_b^{*\mathrm{ren}}i_{b'!}^{\mathrm{ren}}\tau, \pi) \\ &= 0\end{aligned}$$

where the first and third isomorphisms come from the obvious adjunctions, the second isomorphism follows from our assumptions, and the last line follows from the trivial vanishing $i_b^{*\mathrm{ren}}i_{b'!}^{\mathrm{ren}}\tau = 0$ for $b'\neq b$.

Remark 2.1.12. The logic in the last paragraph of the previous remark can also be reversed, in the evident sense.

2.1.1 Example: Supercuspidal parameters

Here we exactly recover Fargues's original eigensheaf conjecture. More precisely, for supercuspidal ϕ, Conjecture 2.1.9 was already formulated by Fargues in 2014 [Far16], and Conjecture 2.1.8 is essentially stated in [FS21]. Here some partial results are known. More precisely, Conjecture 2.1.8 is known for GL_n [ALB21, Han23a] and unramified $\mathrm{U}_{2n+1}/\mathbf{Q}_p$ [BMHN22]. Additionally, Conjecture 2.1.9 is known for GSp_4 [Ham21] and SO_{2n+1} (H., unpublished) when $E/\mathbf{Q}_p$ is unramified with $p > 2$.

2.1.2 Example: Generic toral parameters for GL_n

We now take $G = \mathrm{GL}_n$. Let $\phi = \chi_1 \oplus \cdots \oplus \chi_n$ be a direct sum of characters, which we also identify with characters $\chi_i : E^\times \to \overline{\mathbf{Q}_\ell}^\times$ through the usual reciprocity map. We assume ϕ is generous, so in particular $S_\phi \cong \mathbf{G}_m^n$ and $\mathrm{Irr}(S_\phi) \cong \mathbf{Z}^n$. Our goal here is to sketch the following result.

Theorem 2.1.13. *Notation as above, suppose also that ϕ is ℓ-integral and generous mod-ℓ. Then Conjecture 2.1.8 is true for ϕ.*

The argument relies critically on deep work of Hamann [Ham22]. The extra assumptions related to ℓ-integrality are needed in Hamann's work, and will be irrelevant once the sheaf-theoretic machinery improves.

Proof. Under the obvious bijection $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbf{Z}^n \leftrightarrow \rho_{\mathbf{j}} \in \mathrm{Irr}(S_\phi)$, set $|\mathbf{j}| = \sum_i |j_i|$, and let $(b_{\mathbf{j}}, \pi_{\mathbf{j}})$ be the pair associated with $(\phi, \rho_{\mathbf{j}})$ under the BM-O bijection. We will prove the isomorphism $i_{\phi *}\rho_{\mathbf{j}} * i_{1!}W_\psi \simeq i^{\mathrm{ren}}_{b_{\mathbf{j}}!}\pi_{\mathbf{j}}$ by induction on $|\mathbf{j}|$.

Base case $|\mathbf{j}| = 0$**.** This amounts to the assertion that

$$\sigma := W_\psi \otimes_{\mathfrak{Z}(G)} \mathfrak{Z}(G)/\mathfrak{m}_\phi \simeq i_B^G(\chi_1 \boxtimes \cdots \boxtimes \chi_n),$$

which can be proved directly. Sketch: Using Theorem A.0.1 and the fact that $X_G \cong X_G^{\mathrm{spec}}$ for GL_n [FS21], together with some standard structure theory a la Bernstein, it is easy to see that σ is a finite-length admissible representation with $\sigma^{\mathrm{ss}} \simeq i_B^G(\chi_1 \boxtimes \cdots \boxtimes \chi_n)^{\oplus m}$ for some $m \geq 1$. To show that $m \leq 1$, it is enough to see that the Jacquet module $j_G^B\sigma$ has length $\leq n!$. This can be done by noting that $j_G^B W_\psi \simeq \mathcal{C}_c^\infty(T, \overline{\mathbf{Q}_\ell})$, and then using the general fact that

$$j_G^P(- \otimes_{\mathfrak{Z}(G)} \mathfrak{Z}(G)/I) \simeq j_G^P(-) \otimes_{\mathfrak{Z}(M)} \mathfrak{Z}(M)/f(I)\mathfrak{Z}(M)$$

as functors, where $f : \mathfrak{Z}(G) \to \mathfrak{Z}(M)$ is the usual map induced by parabolic induction. In the case at hand, one concludes by noting that $\mathfrak{Z}(T)/f(\mathfrak{m}_\phi)\mathfrak{Z}(T) \simeq \oplus_{\sigma \in S_n} \mathfrak{Z}(T)/\mathfrak{m}_{\chi_{\sigma(1)} \boxtimes \cdots \boxtimes \chi_{\sigma(n)}}$.

Induction step. Suppose given $\mathbf{j}$. We can choose some $\mathbf{j}'$ with $|\mathbf{j}'| = |\mathbf{j}| - 1$ and $V \in \{\mathrm{std}, \mathrm{std}^\vee\}$ such that $\rho_{\mathbf{j}} \in V|_{S_\phi} \otimes \rho_{\mathbf{j}'}$. Note that $V|_{S_\phi} \otimes \rho_{\mathbf{j}'}$ is multiplicity free, so we can decompose it as a sum $V|_{S_\phi} \otimes \rho_{\mathbf{j}'} \simeq \rho_{\mathbf{j}_1} \oplus \cdots \oplus \rho_{\mathbf{j}_n}$ where $\mathbf{j}'$ and $\mathbf{j}_i$ have the same component except in the ith spot. We will now compute $T_V i^{\mathrm{ren}}_{b_{\mathbf{j}'}!}\pi_{\mathbf{j}'} \in D(\mathrm{Bun}_G)^{BW_E}$ in two different ways, and then use the Weil group action to break apart the results of the calculation.

First method. By the induction hypothesis we have $i^{\mathrm{ren}}_{b_{\mathbf{j}'}!}\pi_{\mathbf{j}'} \simeq i_{\phi*}\rho_{\mathbf{j}'} * i_{1!}W_\psi$, so we compute that

$$\begin{aligned} T_V i^{\mathrm{ren}}_{b_{\mathbf{j}'}!}\pi_{\mathbf{j}'} &\simeq T_V(i_{\phi*}\rho_{\mathbf{j}'} * i_{1!}W_\psi) \\ &\simeq (V \otimes i_{\phi*}\rho_{\mathbf{j}'}) * i_{1!}W_\psi \\ &\simeq i_{\phi*}(V|_{S_\phi} \otimes \rho_{\mathbf{j}'}) * i_{1!}W_\psi \\ &\simeq \oplus_{1\leq k\leqslant n}(i_{\phi*}\rho_{\mathbf{j}_k} * i_{1!}W_\psi) \boxtimes \chi_k^{\pm 1} \end{aligned}$$

where the $\pm$ sign is chosen according to whether $V = \mathrm{std}$ or $V = \mathrm{std}^\vee$. Note that the Weil group action in the fourth line comes from the tautological Weil group action on $V|_{S_\phi}$.

Second method. Hamann's results show that $i^{\mathrm{ren}}_{b_{\mathbf{j}'}!}\pi_{\mathbf{j}'} \simeq \mathrm{Eis}_B(i^T_{b_{\mathbf{j}'}!}\chi)$ for all $\mathbf{j}'$ [Ham22, Theorem 9.1]. Now using the filtered commutation of Eis with Hecke operators, together with the genericity, we compute that

$$\begin{aligned} T_V \mathrm{Eis}_B(i^T_{b_{\mathbf{j}'}!}\chi) &\simeq \mathrm{Eis}_B(T_{V|\hat{T}} i^T_{b_{\mathbf{j}'}!}\chi) \\ &\simeq \mathrm{Eis}_B(\oplus_{1\leq k\leq n} i^T_{b_{\mathbf{j}_k}!}\chi \boxtimes \chi_k^{\pm 1}) \\ &\simeq \oplus_{1\leq k\leq n}\mathrm{Eis}_B(i^T_{b_{\mathbf{j}_k}!}\chi) \boxtimes \chi_k^{\pm 1} \\ &\simeq \oplus_{1\leq k\leq n} i^{\mathrm{ren}}_{b_{\mathbf{j}_k}!}\pi_{\mathbf{j}_k} \boxtimes \chi_k^{\pm 1} \end{aligned}$$

with the same sign convention, where we have used Hamann's results again.

Equating the outcomes of these two calculations, and using the fact that $\chi_1, \dots, \chi_k$ are distinct as characters of W_E, we get the desired result. $\square$

Next, we explain how the finiteness conditions and t-structures should behave around generous parameters.

Proposition 2.1.14. *Let ϕ be a generous L-parameter, and assume Conjecture 2.1.8 for ϕ. Assume also that the BM-O bijection for ϕ exhausts all pairs (b, π) such that $i^{\mathrm{ren}}_{b!}\pi$ has Fargues-Scholze parameter ϕ.*

Then the functors $i^{\mathrm{ren}}_{b!}$ and $i^{*\mathrm{ren}}_b$ induce a canonical direct product decomposition

$$D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi \cong \prod_b D(G_b(E), \overline{\mathbf{Q}_\ell})^{\mathrm{ULA}}_\phi$$

identifying the perverse t-structure on the left-hand side with the product of the standard t-structures on the right-hand side, and the Hecke action of $\mathrm{Rep}(\hat{G})$ on $D(\mathrm{Bun}_G)^{\mathrm{ULA}}_\phi$ is perverse t-exact.

Moreover, the perverse and hadal t-structures on $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ coincide, and the same functors as above induce a canonical direct sum decomposition

$$D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \cong \bigoplus_b D(G_b(E), \overline{\mathbf{Q}_\ell})_{\mathrm{fin},\phi}$$

compatible with the decomposition above and identifying the hadal (=perverse) t-structure on the left-hand side with the sum of the standard t-structures on the right-hand side. Finally, the Hecke action of $\mathrm{Rep}(\hat{G})$ on $D(\mathrm{Bun}_G)_{\mathrm{fin},\phi}$ is hadal t-exact.

Sketch. Using the second half of Conjecture 2.1.8 together with the exhaustion assumption, it is clear that we have isomorphisms of functors $i^{\mathrm{ren}}_{b!} \simeq i^{\mathrm{ren}}_{b*} \simeq i^{\mathrm{ren}}_{b\sharp}$ and $i^{*\mathrm{ren}}_b \simeq i^{!\mathrm{ren}}_b$ on the ϕ-localized

ULA/finite categories, which forces these functors to be t-exact with respect to the perverse/hadal t-structure on Bun_G and the standard t-structures on the representation categories. Moreover, all the gluing functors $i_{b'}^{*\mathrm{ren}} i_{b*}^{\mathrm{ren}}$ between ϕ-localized ULA sheaves on different strata vanish identically. This implies the direct product decomposition and the identifications of t-structures as stated. For the statements regarding the Hecke action, it is enough to prove that any sheaf of the form $T_V i_{b!}^{\mathrm{ren}}\pi$ is perverse/hadal. This follows from the discussion in Remark 2.1.10. $\square$

Exercise 2.1.15. Fix a generous parameter ϕ. Assume that the hypotheses of Proposition 2.1.14 hold for ϕ and $\phi^\vee$, and that the BM-O bijections for ϕ and $\phi^\vee$ satisfy the expected compatibility with smooth duality as in Remark 2.1.11. How do the decompositions of categories in Proposition 2.1.14 interact with the relevant dualities?

At this point, it is hard not to state the following (unconditional!) conjecture.

Conjecture 2.1.16. *If ϕ is a generous parameter, the functor*

$$c_\psi : D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \to \mathrm{QCoh}(\mathrm{Par}_G)$$

is t-exact with respect to the hadal t-structure on the left-hand side and the standard t-structure on the right-hand side.

Of course, as noted above, we expect that the perverse and hadal t-structures should coincide on the left-hand side. However, if we said "perverse" instead of "hadal" in the formulation of this conjecture, it would no longer be entirely unconditional, because it is not clear a priori that perverse truncations preserve $D(\mathrm{Bun}_G)_{\mathrm{fin},\phi}$.

2.2 The trivial L-parameter

In this section we assume for simplicity that G is split. Let ϕ be the *trivial* L-parameter. Clearly $S_\phi = \hat{G}$, so $\mathrm{Irr}(S_\phi) \cong X^*(\hat{T})^+ = X_*(T)^+$ by usual highest weight theory. Therefore, for any $\lambda \in X_*(T)^+$, the BM-O bijection defines an associated pair (b_λ, π_λ) which can be described totally explicitly. In fact, b_λ is just the element $\lambda(\varpi) \in G(E)$, and $G_{b_\lambda} = C_G(\lambda)$ is the standard Levi subgroup of G centralizing λ. The representation π_λ turns out to be the (irreducible) normalized parabolic induction $i_B^{G_{b_\lambda}}(\mathbf{1})$ of the trivial representation, where $B \subset G_{b_\lambda}$ is any choice of Borel. Our goal here is to formulate a conjecture describing the coherent sheaves on Par_G associated with the sheaves $i_{b_\lambda !}^{\mathrm{ren}}\pi_\lambda$ and $i_{b_\lambda \sharp}^{\mathrm{ren}}\pi_\lambda$ on Bun_G. It turns out that a precise guess for these sheaves is forced on us by the expected compatibility of the categorical equivalence with Eisenstein functors and with duality. However, the situation is much more complicated than for generous L-parameters.

To begin, observe that the fiber of the map $\mathrm{Par}_G \to X_G^{\mathrm{spec}}$ over the trivial L-parameter is exactly the quotient stack $\mathcal{N}/\hat{G}$, where $\mathcal{N} \subset \hat{\mathfrak{g}}$ is the nilpotent cone with its usual $\hat{G}$-action. In particular, we have a canonical closed immersion $\nu : \mathcal{N}/\hat{G} \hookrightarrow \mathrm{Par}_G$. Now, recall the $\hat{G}$-equivariant diagram of schemes

$$\begin{tikzcd} & \tilde{\mathcal{N}} = T^*(\hat{G}/\hat{B}) \cong \hat{\mathfrak{u}} \times^{\hat{B}} \hat{G} \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \hat{G}/\hat{B} & & \mathcal{N} \end{tikzcd}$$

where $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ is the Springer resolution. For any $\lambda \in X^*(\hat{T})$, we have the usual equivariant line bundle $\mathcal{L}_\lambda$ on $\hat{G}/\hat{B}$, and we may form the associated $\hat{G}$-equivariant coherent complex $A_\lambda =$

$\pi_*\eta^*\mathcal{L}_\lambda \in \mathrm{Coh}(\mathcal{N}/\hat{G})$. The A_λ's are sometimes called *Andersen-Jantzen sheaves*, and they are very interesting objects in geometric representation theory. We refer the reader to [Ach15] for a beautiful overview of this topic. Among the highlights of this theory, we note that Bezrukavnikov [Bez03] proved that the A_λ's are *perverse* with respected to a suitable perverse coherent t-structure on $\mathrm{Coh}(\mathcal{N}/\hat{G})$, whose heart we denote $\mathrm{PCoh}(\mathcal{N}/\hat{G})$. We also note that $A_0 = \mathcal{O}_{\mathcal{N}}$, and that for any $\lambda \in X^*(\hat{T})^+$, A_λ is an honest coherent sheaf, i.e. is concentrated in cohomological degree zero. When λ is dominant, it is conventional to write $\overline{\nabla}_\lambda = A_\lambda$ and $\overline{\Delta}_\lambda = A_{w_0(\lambda)}$. We also note that for λ dominant, there is a unique (up to scalar) nonzero map $\overline{\Delta}_\lambda \to \overline{\nabla}_\lambda$ whose image in PCoh is an irreducible object IC_λ, and that this recipe gives all the irreducible objects in the heart of the perverse coherent t-structure.

Conjecture 2.2.1. *For any $\lambda \in X_*(T)^+$, there are isomorphisms*

$$c_\psi(i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda) \simeq \nu_*\overline{\nabla}_\lambda \quad \text{and} \quad c_\psi(i^{\mathrm{ren}}_{b_\lambda \sharp}\pi_\lambda) \simeq \nu_*\overline{\Delta}_\lambda.$$

Moreover, the functor c_ψ sends the canonical map

$$i^{\mathrm{ren}}_{b_\lambda \sharp}\pi_\lambda \to i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$$

to the map obtained by applying ν_* to the canonical map $\overline{\Delta}_\lambda \to \overline{\nabla}_\lambda$.

Remark 2.2.2. It is not hard to see that $\nu_*\overline{\nabla}_\lambda$ and $\nu_*\overline{\Delta}_\lambda$ are perfect complexes on Par_G, so we could also formulate an obvious variant of this conjecture using the more humane functor a_ψ instead of c_ψ.

This conjecture follows from the expected compatibilities of the categorical conjecture with Eisenstein series and duality. To see this, recall that the unramified component of Par_T is canonically $\mathrm{Par}_T^{\mathrm{nr}} \cong \hat{T} \times B\hat{T}$. We have a line bundle $\mathcal{O}_\lambda$ on $B\hat{T}$ corresponding to λ. Pushing forward along the evident closed immersion $e : B\hat{T} \to \hat{T} \times B\hat{T} \cong \mathrm{Par}_T^{\mathrm{nr}} \subset \mathrm{Par}_T$ gives a coherent sheaf $e_*\mathcal{O}_\lambda$ on Par_T. Under the known categorical equivalence for T [Zou21], it corresponds to the sheaf $i_{\lambda !}\mathbf{1}$ on $\mathrm{Bun}_T = \coprod_{X_*(T)}[*/T(E)]$ given by the !-extension of the trivial representation from the component labelled by λ.

Proposition 2.2.3. *Notation and assumptions as above, we have an isomorphism* $\mathrm{Eis}_B^{\mathrm{spec}}(e_*\mathcal{O}_\lambda) \cong \nu_*A_\lambda$ *in* $\mathrm{Coh}(\mathrm{Par}_G)$ *for any λ.*

Proof sketch. The key point is that there is commutative diagram of (derived) Artin stacks

$$\begin{tikzcd}
B\hat{T} \arrow[d, "e"] & \hat{\mathfrak{u}}/\hat{B} \cong \tilde{\mathcal{N}}/\hat{G} \arrow[l, "q^{\mathrm{nil}}"'] \arrow[r, "\pi"] \arrow[d, "f"] & \mathcal{N}/\hat{G} \arrow[d, "\nu"] \\
\mathrm{Par}_T & \mathrm{Par}_B \arrow[l, "q^{\mathrm{spec}}"] \arrow[r, "p^{\mathrm{spec}}"'] & \mathrm{Par}_G
\end{tikzcd}$$

where the vertical maps are closed immersions, and moreover the lefthand square is Cartesian and

$q^{\rm spec}$ is flat in a neighborbood of ime.[11] It is then straightforward to compute

$$\begin{aligned}\mathrm{Eis}_B^{\rm spec}(e_*\mathcal{O}_\lambda) &= p_*^{\rm spec} q^{\rm spec *} e_*\mathcal{O}_\lambda \\ &\simeq p_*^{\rm spec} f_* q^{\rm nil *}\mathcal{O}_\lambda \\ &\simeq \nu_*\pi_* q^{\rm nil *}\mathcal{O}_\lambda \\ &\simeq \nu_* A_\lambda\end{aligned}$$

where the second line follows from flat base change, the third line is trivial, and the fourth line follows from the definition of A_λ plus the simple observation that $q^{\rm nil *}\mathcal{O}_\lambda = \eta^*\mathcal{L}_\lambda$ as line bundles on $\tilde{\mathcal{N}}/\hat{G}$. ☐

Proposition 2.2.4. *Notation as above, we have an isomorphism* $\mathrm{Eis}_B(i_{\lambda !}\mathbf{1}) \simeq i_{b_\lambda !}^{\rm ren}\pi_\lambda$ *for* λ *dominant.*

Proof. This follows from Remark 1.4.8 plus a little thought. See also [Ham22, Proposition 9.4] and the discussion immediately afterwards. ☐

Now we can put things together: since $e_*\mathcal{O}_\lambda$ matches $i_{\lambda !}\mathbf{1}$ under the (known) categorical equivalence for T, compatibility of the categorical equivalence with Eisenstein functors on both sides *forces* us to expect that for any dominant λ, $\nu_*\overline{\nabla}_\lambda \simeq \mathrm{Eis}_B^{\rm spec}(e_*\mathcal{O}_\lambda)$ should match $i_{b_\lambda !}^{\rm ren}\pi_\lambda \simeq \mathrm{Eis}_B(i_{\lambda !}\mathbf{1})$ under the categorical conjecture for G. This gives the first isomorphism in Conjecture 2.2.1. The second isomorphism follows from the expected compatibility with duality. Indeed, one checks directly that $\mathbf{D}_{\rm twGS}(\nu_* A_\lambda) \cong \nu_* A_{w_0(\lambda)}[-\dim T]$ for any λ (this follows, for instance, from the arguments in section 4 of [AH19]), and also that $\mathbf{D}_{\rm BZ}(i_{b_\lambda !}^{\rm ren}\pi_\lambda) \cong i_{b_\lambda \sharp}^{\rm ren}\pi_\lambda[-\dim T]$, which is an easy consequence of Proposition 1.1.4. Conjecture 1.7.5 then implies the desired isomorphism.

We can use Conjecture 2.2.1 to do some actual calculations. Let us illustrate this in the simplest case $G = \mathrm{PGL}_2$. Here we identify $X^*(\hat{T})^+ = \mathbf{Z}_{\geq 0}$ in the usual way, and write $\overline{\nabla}_n$ and $\overline{\Delta}_n$ for the associated perverse coherent sheaves as defined above. Let $b_n \in |\mathrm{Bun}_{\mathrm{PGL}_2}|$ be the point labelled by the highest weight n, so b_n corresponds to the image of $\mathrm{diag}(\varpi^n, 1)$ in $B(\mathrm{PGL}_2)$. Note that one component of $\mathrm{Bun}_{\mathrm{PGL}_2}$ consists of the chain of specializations $b_0 \rightsquigarrow b_2 \rightsquigarrow b_4 \cdots$, while the other component consists of the chain $b_{1/2} \rightsquigarrow b_1 \rightsquigarrow b_3 \cdots$. Here $b_{1/2}$ is the "missing" point, which won't play any role in our discussion (since no sheaf supported at this point can have trivial L-parameter). When $n = 0$, $G_{b_0} = G$ and $\pi_0 = i_B^G(\mathbf{1})$ is an irreducible principal series representation. For $n \geq 1$, $G_{b_n}(E) = E^\times$ and the representation π_n is just the trivial representation. In particular, the sheaf $i_{b_n !}^{\rm ren}\mathbf{1}$ corresponding to $\nu_*\overline{\nabla}_n$ is very simple and explicit, and is supported at one point. But how to calculate the stalks of the sheaf $i_{b_n \sharp}^{\rm ren}\mathbf{1}$? It turns out that the categorical conjecture lets us do this!

For this, recall that (as we already mentioned) there is a unique nonzero map $\overline{\Delta}_\lambda \to \overline{\nabla}_\lambda$ whose image is an irreducible perverse coherent sheaf IC_λ. Translating the diagram $\nu_*\overline{\Delta}_\lambda \to \nu_* IC_\lambda \to \nu_*\overline{\nabla}_\lambda$ to the other side of the categorical conjecture, we now predict the existence of a canonical indecomposable sheaf $\mathscr{F}_\lambda$ on Bun_G admitting maps $i_{b_\lambda \sharp}^{\rm ren}\pi_\lambda \to \mathscr{F}_\lambda \to i_{b_\lambda !}^{\rm ren}\pi_\lambda$ whose composite is the canonical map, and such that $\mathbf{D}_{\rm BZ}\mathscr{F}_\lambda \simeq \mathscr{F}_\lambda[-\dim T]$. In general, these sheaves are hard to calculate. However, for PGL_2, everything can be made very explicit. In particular, the following results are known [Ach15, Section 5]:

a) there are isomorphisms $\overline{\Delta}_0 \cong \overline{\nabla}_0 \cong IC_0 \cong \mathcal{O}_{\mathcal{N}/\hat{G}}$ and $\overline{\Delta}_1 \cong \overline{\nabla}_1 \cong IC_1$

[11] The hard part here is showing that the left-hand square is actually Cartesian, and not just Cartesian on classical truncations, since Par_B is genuinely a derived Artin stack. To verify this, one needs to check that the derived structure of Par_B is trivial in a neighborhood of $(q^{\rm spec})^{-1}(\mathrm{ime})$. See Proposition 2.3.3 for a much more general statement.

b) for $n \geq 2$ there are short exact sequences of perverse coherent sheaves

$$0 \to IC_n \to \overline{\nabla}_n \to \overline{\nabla}_{n-2} \to 0$$

and

$$0 \to \overline{\Delta}_{n-2} \to \overline{\Delta}_n \to IC_n \to 0,$$

and isomorphisms $IC_n \simeq i_{0*}V_{n-2}[-1]$, where $i_0 : B\hat{G} \to \mathcal{N}/\hat{G}$ is the inclusion of the closed orbit.

Note that the information in a) translates into isomorphisms $i^{\mathrm{ren}}_{b_0\sharp}\pi_0 \simeq i^{\mathrm{ren}}_{b_0!}\pi_0$ and $i^{\mathrm{ren}}_{b_1\sharp}\mathbf{1} \simeq i^{\mathrm{ren}}_{b_1!}\mathbf{1} \simeq \mathscr{F}_1$. The first of these is completely tautological, while the second is not tautological but follows a priori from the fact that no sheaf supported at $b_{1/2}$ (which is the unique generization of b_1) can have trivial L-parameter.

Using the first sequence in b) we can now calculate the stalks of $\mathscr{F}_n$ inductively via the distinguished triangles $\mathscr{F}_n \to i^{\mathrm{ren}}_{b_n!}\pi_n \to i^{\mathrm{ren}}_{b_{n-2}!}\pi_{n-2} \to$ which map to this sequence under the categorical equivalence. The result is easy to discern:

Proposition 2.2.5. *For $n \geq 2$, the stalks of $\mathscr{F}_n$ vanish outside the points b_n and b_{n-2}.*
For $n = 2$ we have $i^*_{b_2}\mathscr{F}_2 = \delta^{1/2}[-2]$ and $i^*_{b_0}\mathscr{F}_2 = \pi_0[-1]$.
For $n > 2$ we have $i^*_{b_n}\mathscr{F}_n = \delta^{1/2}[-n]$ and $i^*_{b_{n-2}}\mathscr{F}_n = \delta^{1/2}[1-n]$.

We can now use this information together with the triangles $i^{\mathrm{ren}}_{b_{n-2}\sharp}\pi_{n-2} \to i^{\mathrm{ren}}_{b_n\sharp}\pi_n \to \mathscr{F}_n \to$ corresponding to the second sequence in b) to inductively calculate the stalks of $i^{\mathrm{ren}}_{b_n\sharp}\mathbf{1}$. For even n, the outcome is the following:

Proposition 2.2.6. *Assume $n \geq 2$ is even. The stalks of $i^{\mathrm{ren}}_{b_n\sharp}\mathbf{1}$ vanish outside the points $b_n, b_{n-2}, \ldots, b_2, b_0$. At these points, all stalk cohomologies are zero except the following:*

i. $H^n(i^*_{b_n} i^{\mathrm{ren}}_{b_n\sharp}\mathbf{1}) = \delta^{1/2}$
ii. *For $0 < 2j < n$,* $H^{2j}(i^*_{b_{2j}} i^{\mathrm{ren}}_{b_n\sharp}\mathbf{1}) = H^{2j+1}(i^*_{b_{2j}} i^{\mathrm{ren}}_{b_n\sharp}\mathbf{1}) = \delta^{1/2}$
iii. $H^0(i^*_{b_0} i^{\mathrm{ren}}_{b_n\sharp}\mathbf{1}) = H^1(i^*_{b_0} i^{\mathrm{ren}}_{b_n\sharp}\mathbf{1}) = i^G_B(\mathbf{1})$.

We encourage the reader to check this for themselves, and to formulate and prove a similar result for $n \geq 3$ odd. As a notable consequence of these calculations, we get the following suggestive result.

Proposition 2.2.7. *For all $n \geq 0$ the sheaves $i^{\mathrm{ren}}_{b_n\sharp}\pi_n$, $i^{\mathrm{ren}}_{b_n!}\pi_n$, and $\mathscr{F}_n$ are hadal sheaves, and the distinguished triangles*

$$\mathscr{F}_n \to i^{\mathrm{ren}}_{b_n!}\pi_n \to i^{\mathrm{ren}}_{b_{n-2}!}\pi_{n-2} \to$$

and

$$i^{\mathrm{ren}}_{b_{n-2}\sharp}\pi_{n-2} \to i^{\mathrm{ren}}_{b_n\sharp}\pi_n \to \mathscr{F}_n \to$$

are short exact sequences of hadal sheaves. The hadal sheaf $\mathscr{F}_n$ is irreducible.

Proof. For $n = 0, 1$ it is clear that these sheaves are hadal. For $n \geq 2$, the calculation in the previous proposition shows that $i^{\mathrm{ren}}_{b_n\sharp}\pi_n$ is coconnective for the hadal t-structure, but we also know it is connective by definition. Thus $i^{\mathrm{ren}}_{b_n\sharp}\pi_n$ is a hadal sheaf. Now from the first triangle we get that all $\mathscr{F}_n$'s are coconnective for the hadal t-structure, so then the map $i^{\mathrm{ren}}_{b_{n-2}\sharp}\pi_{n-2} \to i^{\mathrm{ren}}_{b_n\sharp}\pi_n$ in the second triangle is an injection of hadal sheaves, which then implies that $\mathscr{F}_n$ is hadal. Now the first triangle implies that $i^{\mathrm{ren}}_{b_n!}\pi_n$ is hadal by an easy induction on n. □

In [FS21, Remark I.10.3] one finds the suggestion that the categorical conjecture might match the perverse t-structure on Bun_G with some perverse coherent t-structure on $\mathrm{Coh}(\mathrm{Par}_G)$. Since the IC_λ's are perverse coherent, this might naively lead one to guess that the $\mathscr{F}_\lambda$'s should be perverse, but already for PGL_2 this is false. Indeed, we calculated above that $i_{b_0}^*\mathscr{F}_2$ has nonzero cohomology in degree 1, which could not happen if $\mathscr{F}_2$ were perverse. However, this numerology matches the fact that $IC_2 \simeq i_{0*}V_0[-1]$ has a nonzero cohomology sheaf in degree one only. We will argue later in these notes that after localizing over a large (and explicit) open substack of Par_G, the categorical equivalence should be t-exact for the perverse t-structure on Bun_G and the *naive* t-structure on $\mathrm{Coh}(\mathrm{Par}_G)$. On the other hand, we also saw above that $\mathscr{F}_n$ is a hadal sheaf. This suggests that perverse coherent t-structures on $\mathrm{Coh}(\mathrm{Par}_G)$ should be relevant, but that they will be related to the *hadal* t-structure on Bun_G instead.

We now make these ideas precise for the trivial parameter, again in the context of a general split G.

Theorem 2.2.8. *Suppose that for all $\lambda \in X^*(\hat{T})^+$ we have isomorphisms*

$$a_\psi(\nu_*\overline{\Delta}_\lambda) \simeq i^{\mathrm{ren}}_{b_\lambda\sharp}\pi_\lambda \quad \text{and} \quad a_\psi(\nu_*\overline{\nabla}_\lambda) \simeq i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$$

as in Remark 2.2.2. Then the following results are true.

i. *The triangulated functor*

$$a_\psi \circ \nu_* : \mathrm{Coh}(\mathcal{N}/\hat{G}) \to D(\mathrm{Bun}_G)_{\mathrm{fin}}$$

is t-exact with respect to the perverse coherent t-structure on the left and the hadal t-structure on the right. In particular, it induces an exact functor

$$a_\psi \circ \nu_* : \mathrm{PCoh}(\mathcal{N}/\hat{G}) \to \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}}.$$

ii. *The sheaves $i^{\mathrm{ren}}_{b_\lambda\sharp}\pi_\lambda$ and $i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$ are hadal.*

iii. *The image of the (unique up to scalar) nonzero map $\overline{\Delta}_\lambda \to \overline{\nabla}_\lambda$ under the functor $a_\psi \circ \nu_*$ is nonzero, and we have an isomorphism*

$$a_\psi(\nu_* IC_\lambda) \simeq i^{\mathrm{ren}}_{b_\lambda \sharp !}\pi_\lambda.$$

In particular, the exact functor

$$a_\psi \circ \nu_* : \mathrm{PCoh}(\mathcal{N}/\hat{G}) \to \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}}$$

is faithful, and sends irreducible objects to irreducible objects.

iv. *Assume that the pairs (b_λ, π_λ) exhaust all pairs (b, π) for which $i^{\mathrm{ren}}_{b!}\pi$ has trivial Fargues-Scholze parameter. Then the sheaves $i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$ are perverse.*

Note that the sheaves $\mathscr{F}_\lambda$ which we predicted earlier, and calculated by hand for PGL_2, are now revealed more clearly: they are exactly the irreducible hadal sheaves $i^{\mathrm{ren}}_{b_\lambda \sharp !}\pi_\lambda$.

We emphasize again that the *only* assumption in this theorem is a set-theoretic matching of objects, and this matching is *forced* on us by the most basic expectations regarding the categorical local Langlands conjecture. The fact that this matching implies so much more is really some magic, related to the remarkable properties of the hadal and perverse coherent t-structures.

Proof. By [Bez03], the left half, resp. right half of the perverse coherent t-structure is generated under extensions by objects of the form $\overline{\Delta}_\lambda[n]$, $n \geq 0$, resp. objects of the form $\overline{\nabla}_\lambda[n]$, $n \leq 0$. Our assumption now guarantees that $a_\psi \circ \nu_*$ sends the left resp. right half into the left resp. right half of the hadal t-structure. This gives i, and then ii. is an immediate consequence of the fact that $\overline{\Delta}_\lambda$ and $\overline{\nabla}_\lambda$ are perverse coherent.

For iii., let $\gamma_\lambda : \overline{\Delta}_\lambda \to \overline{\nabla}_\lambda$ be a nonzero map. Pick a total ordering $<$ on $X^*(\hat{T})^+$ refining the usual partial ordering. By [Bez03], the cone $\mathcal{K}_\lambda$ of the map γ_λ lies in the triangulated subcategory of $\mathrm{Coh}(\mathcal{N}/\hat{G})$ generated by $\overline{\nabla}_\mu[m]$ for $m \in \mathbf{Z}$ and $\mu < \lambda$. This implies that the $*$-stalks of the sheaf $a_\psi(\nu_* \mathcal{K}_\lambda)$ can only be nonzero at points b_μ with $\mu < \lambda$, and in particular the $*$-stalk at b_λ vanishes. This implies that the first map in the distinguished triangle

$$a_\psi(\nu_* \overline{\Delta}_\lambda) \simeq i^{\mathrm{ren}}_{b_\lambda \sharp} \pi_\lambda \stackrel{(a_\psi \circ \nu_*)(\gamma_\lambda)}{\longrightarrow} a_\psi(\nu_* \overline{\nabla}_\lambda) \simeq i^{\mathrm{ren}}_{b_\lambda !} \pi_\lambda \to a_\psi(\nu_* \mathcal{K}_\lambda) \to$$

cannot be zero, because otherwise we would get an isomorphism

$$a_\psi(\nu_* \mathcal{K}_\lambda) \simeq i^{\mathrm{ren}}_{b_\lambda !} \pi_\lambda \oplus i^{\mathrm{ren}}_{b_\lambda \sharp} \pi_\lambda [1],$$

and the right-hand side here clearly has nonzero $*$-stalk at b_λ. Therefore, the image of γ_λ under $a_\psi \circ \nu_*$ is the (unique up to scalar) nonzero map

$$i^{\mathrm{ren}}_{b_\lambda \sharp} \pi_\lambda \stackrel{\delta_\lambda}{\to} i^{\mathrm{ren}}_{b_\lambda !} \pi_\lambda.$$

Since we have already seen that the source and target of δ_λ are hadal sheaves, it is clear that $\mathrm{im}\delta_\lambda = i^{\mathrm{ren}}_{b_\lambda \sharp !} \pi_\lambda$ by the definition of the latter as in Theorem 1.2.7. On the other hand, $a_\psi \circ \nu_*$ is an exact functor of abelian categories, so it preserves images, and therefore

$$i^{\mathrm{ren}}_{b_\lambda \sharp !} \pi_\lambda = \mathrm{im}\delta_\lambda \simeq a_\psi \circ \nu_*(\mathrm{im}\gamma_\lambda) \simeq a_\psi \circ \nu_*(IC_\lambda).$$

This also implies that $a_\psi \circ \nu_*$ is conservative (since it doesn't send any irreducible object to the zero object), and thus faithful.

For iv., we already know that $i^{\mathrm{ren}}_{b_\lambda !} \pi_\lambda$ lies in the left half of the of the perverse t-structure. To see that it lies in the right half, it is enough (by the exhaustion assumption in iv.) to check that

$$\mathrm{Hom}(i^{\mathrm{ren}}_{b_\mu !} \pi_\mu [n], i^{\mathrm{ren}}_{b_\lambda !} \pi_\lambda) = 0$$

for all μ and all $n > 0$. We already know that $i^{\mathrm{ren}}_{b_\lambda !} \pi_\lambda \in {}^h D^{\geq 0}$ in general, but from ii. we also know that $i^{\mathrm{ren}}_{b_\mu !} \pi_\mu$ is hadal for all μ, and therefore $i^{\mathrm{ren}}_{b_\mu !} \pi_\mu [n] \in {}^h D^{\leq -1}$ for all $n > 0$. The desired Hom-vanishing then follows from basic properties of t-structures. □

Exercise 2.2.9. 1. Show that for λ, μ dominant with $\lambda \neq \mu$, we have $\mathrm{Hom}(i^{\mathrm{ren}}_{b_\lambda \sharp} \pi_\lambda, i^{\mathrm{ren}}_{b_\mu !} \pi_\mu) = 0$. (Hint: Use Proposition 1.1.2.iii.) Match this under the categorical conjecture with Proposition 4.b) from [Bez03].

2. Show that for λ, μ dominant with $\lambda \not\geq \mu$, we have $\mathrm{Hom}(i^{\mathrm{ren}}_{b_\lambda !} \pi_\lambda, i^{\mathrm{ren}}_{b_\mu !} \pi_\mu) = 0$. (Hint: Use Proposition 1.1.2.ii.) Match this under the categorical conjecture with the vanishing portion of [Ach15, Theorem 4.12]. How does the "other part" of [Ach15, Theorem 4.12] translate under the categorical conjecture?

3. Assume that the pairs (b_λ, π_λ) exhaust all pairs (b, π) for which $i^{\mathrm{ren}}_{b!} \pi$ has trivial Fargues-Scholze parameter. Pick a total ordering $<$ on $X^*(\hat{T})^+$ refining the usual partial ordering. Prove

by hand that the collection $\{i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda\}_{\lambda\in X^*(\hat{T})^+}$ generates the triangulated category $D(\mathrm{Bun}_G)_{\mathrm{fin},\phi_{\mathrm{triv}}}$ and defines a dualizable quasi-exceptional set within it, whose associated t-structure is the hadal t-structure, and with dual quasi-exceptional set $\left\{i^{\mathrm{ren}}_{b_\lambda\sharp}\pi_\lambda\right\}_{\lambda\in X^*(\hat{T})^+}$.

4. Assume that $G=\mathrm{GL}_n$. If λ is minuscule, show that the natural map $i^{\mathrm{ren}}_{b_\lambda\sharp}\pi_\lambda \to i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$ is an isomorphism. Observe that this is *predicted* by the categorical conjecture for all G: for λ minuscule, the natural map $\overline{\Delta}_\lambda \to \overline{\nabla}_\lambda$ is always an isomorphism (see e.g. [Ach15, Proof of Prop. 3.8]). (Hint: If b is a proper generization of b_λ, show that G_b is not quasisplit. Deduce that no sheaf supported at b can have trivial L-parameter, using the known compatibility of the FS correspondence with the classical LLC for inner forms of GL_n [HKW22].)

Remark 2.2.10. Let $i_0 : B\hat{G}\to\mathcal{N}/\hat{G}$ be the inclusion of the closed orbit. It is a general fact [Ach15, Proposition 3.9] that for any dominant λ, $IC_{\lambda+2\rho}\simeq i_{0*}V_\lambda[-\dim U]$.[12] In the particular case of $G=\mathrm{PGL}_2$, this reduces to isomorphisms $IC_n\simeq i_{0*}V_{n-2}[-1]$ for all $n\geq 2$, as we've already noted. Writing $\mathcal{O}(\hat{G})\simeq \oplus_{n\geq 2}V_{n-2}^{\oplus n-1}$, we formally see that $\nu_* i_{0*}\mathcal{O}(\hat{G})\simeq \oplus_{n\geq 2}\nu_* IC_n^{\oplus n-1}[1]$, so translating to the other side we get that

$$\begin{aligned} a_\psi(\nu_* i_{0*}\mathcal{O}(\hat{G})) &\simeq \oplus_{n\geq 2} a_\psi(IC_n)^{\oplus n-1}[1] \\ &\simeq \oplus_{n\geq 2}\mathscr{F}_n^{\oplus n-1}[1] \end{aligned}$$

should be a Hecke eigensheaf with trivial eigenvalue. For general G, similar reasoning leads formally to the expectation that $\mathscr{G}=\oplus_{\lambda\in X^*(\hat{T})^+}\mathscr{F}_{\lambda+2\rho}^{\oplus \dim V_\lambda}$ should be a Hecke eigensheaf with trivial eigenvalue.

It seems that much more can be said about the sheaves $\mathscr{F}_\lambda$. We already noted above that there are isomorphisms $\mathbf{D}_{\mathrm{BZ}}\mathscr{F}_\lambda\simeq\mathscr{F}_\lambda[-\dim T]$. However, a heuristic argument with compactified Eisenstein series suggests that for $\lambda\in 2\rho+X^*(\hat{T})^+$, we should also have isomorphisms $\mathbf{D}_{\mathrm{Verd}}\mathscr{F}_\lambda\simeq\mathscr{F}_\lambda[2\dim U]$, and $\mathscr{F}_\lambda[\dim U]$ should be an irreducible perverse sheaf. We will come back to this later in the notes. For now, we note that the special case of PGL_2 can be understood directly using our previous calculations. Indeed, recall from earlier the natural map $i^{\mathrm{ren}}_{b_n !}\pi_n\to i^{\mathrm{ren}}_{b_{n-2}!}\pi_{n-2}$, which is a surjection of hadal sheaves with kernel $\mathscr{F}_n$. However, the source and target of this map are also perverse (by Theorem 2.2.8.iv), and by some general nonsense this same map must be an *injection* of perverse sheaves, with cokernel in the perverse category given by $\mathscr{F}_n[1]$! This immediately implies that $\mathscr{F}_n[1]$ is perverse, and a small additional argument identifies it with the intermediate extension $i^{\mathrm{ren}}_{b_{n-2}!*}\pi_{n-2}$. In other words, we have short exact sequences of perverse sheaves

$$0\to i^{\mathrm{ren}}_{b_{n+2}!}\pi_{n+2}\to i^{\mathrm{ren}}_{b_n !}\pi_n\to i^{\mathrm{ren}}_{b_n !*}\pi_n\to 0$$

for all $n\geq 0$. This implies that the perverse sheaf $i^{\mathrm{ren}}_{b_n !}\pi_n$ is uniserial of infinite length, with Jordan-Holder series

$$\cdots i^{\mathrm{ren}}_{b_{n+6}!*}\pi_{n+6} - i^{\mathrm{ren}}_{b_{n+4}!*}\pi_{n+4} - i^{\mathrm{ren}}_{b_{n+2}!*}\pi_{n+2} - i^{\mathrm{ren}}_{b_n !*}\pi_n.$$

Note also that $i^{\mathrm{ren}}_{b_n !}\pi_n$ has infinite length as a perverse sheaf, despite clearly being an object of $D(\mathrm{Bun}_G)_{\mathrm{fin}}$. Finally, taking the Verdier dual of the previous short exact sequence and using the self-duality of intermediate extensions, we get short exact seqences of perverse sheaves

$$0\to i^{\mathrm{ren}}_{b_n !*}\pi_n\to i^{\mathrm{ren}}_{b_n *}\pi_n\to i^{\mathrm{ren}}_{b_{n+2}*}\pi_{n+2}\to 0$$

for all $n\geq 0$. This immediately implies that $i^{\mathrm{ren}}_{b_n *}\pi_n$ is uniserial of infinite length, with the same perverse Jordan-Holder factors as $i^{\mathrm{ren}}_{b_n !}\pi_n$ but arranged in the opposite order. By a simple induction, this triangle also lets us easily compute the $*$-stalks of the sheaves $i^{\mathrm{ren}}_{b_n *}\pi_n$.

[12]This is more or less the only part of the Lusztig-Vogan bijection which can be explicitly understood.

Proposition 2.2.11. *For all $n \geq 0$ and $j > 0$, we have*

$$i^{*\mathrm{ren}}_{b_{n+2j}} i^{\mathrm{ren}}_{b_n *} \pi_n \simeq \pi_{n+2j} \oplus \pi_{n+2j}[1].$$

2.3 Semisimple generic parameters

In this section we sketch our expectations regarding the maximal common generalizations of the discussion in section 2.1 and 2.2. We again allow any quasisplit G. It will be convenient to adopt the following notation: If H is any linear algebraic group over $\overline{\mathbf{Q}_\ell}$, we have the associated Artin stack $\mathcal{N}_H = \mathrm{Lie}(H)^{\mathrm{nil}}/H$ where $\mathrm{Lie}(H)^{\mathrm{nil}}$ is the nilpotent cone. If the connected component H° is reductive, the obvious map $\mathcal{N}_{H^\circ} \to \mathcal{N}_H$ is finite etale, and it is not hard to see that that we can define a perverse coherent t-structure on $\mathrm{Coh}(\mathcal{N}_H)$ whose heart is exactly the objects in $\mathrm{Coh}(\mathcal{N}_H)$ whose pullback to $\mathrm{Coh}(\mathcal{N}_{H^\circ})$ is perverse coherent.

Let ϕ be a semisimple parameter, $x_\phi \in X_G^{\mathrm{spec}}$ the associated closed point. To begin, observe that there is a canonical closed immersion $\nu_\phi : \mathcal{N}_{S_\phi} \to \mathrm{Par}_G$ factoring over the closed substack $q^{-1}(x_\phi)$, which parametrizes exactly the L-parameters with open kernel whose Frobenius-semisimplification is ϕ. We omit the proof, which is an easy exercise with the Jordan-Chevalley decomposition.

Proposition 2.3.1. *The following conditions on a semisimple parameter ϕ are equivalent.*

1) The morphism ν_ϕ factors through an isomorphism $\mathcal{N}_{S_\phi} \xrightarrow{\sim} q^{-1}(x_\phi)$.

2) The parameter ϕ is generic, *i.e. the adjoint L-function $L(s, \mathrm{ad} \circ \phi)$ is regular at $s = 1$.*

3) The composite map $BS_\phi \to \mathcal{N}_{S_\phi} \xrightarrow{\nu_\phi} \mathrm{Par}_G$ defines a smooth point of Par_G.

4) The stack Par_G is smooth in a Zariski neighborhood of $q^{-1}(x_\phi)$.

Proof. Omitted. We note that condition 2) may seem to depend a priori on a choice of isomorphism $\mathbf{C} \simeq \overline{\mathbf{Q}_\ell}$, and we encourage the reader to convince themselves that it is actually independent of any such choice. ☐

Example 2.3.2. Suppose $G = \mathrm{GL}_n$. Then an L-parameter ϕ is semisimple generic iff it is a direct sum $\phi \simeq \phi_1 \oplus \cdots \oplus \phi_d$ where the ϕ_i's are supercuspidal parameters with $\sum_i \dim \phi_i = n$, and with $\phi_i \not\simeq \phi_j(1)$ for any $i \neq j$. However, unlike in the generous case, we allow the ϕ_i's to occur with arbitrary multiplicities. Note that a semisimple L-parameter ϕ for GL_n is generic *if and only if* the associated $\mathrm{GL}_n(E)$-representation π_ϕ is an irreducible parabolic induction of a supercuspidal representation on a Levi subgroup.

It is not hard to see that the locus $X_G^{\mathrm{spec,gen}} \subset X_G^{\mathrm{spec}}$ parametrizing semisimple generic parameters is open and dense, and contains all generous parameters. Here is another key simplifying feature associated with these parameters.

Proposition 2.3.3. *Choose any parabolic $P = MU \subset G$, so we have the natural composite map $\mathrm{Par}_P \xrightarrow{p^{\mathrm{spec}}} \mathrm{Par}_G \xrightarrow{q} X_G^{\mathrm{spec}}$. Let*

$$\mathrm{Par}_P^{G-\mathrm{gen}} = \mathrm{Par}_P \times_{X_G^{\mathrm{spec}}} X_G^{\mathrm{spec,gen}} \subset \mathrm{Par}_P$$

be the evident open substack. Then the derived structure on $\mathrm{Par}_P^{G-\mathrm{gen}}$ is trivial.

Proof. By the calculations in [Zhu21, Section 2.3], we know that $q^{\mathrm{spec}} : \mathrm{Par}_P \to \mathrm{Par}_M$ is a quasismooth morphism of quasismooth derived stacks with classical target, and where the source and target both have virtual (equi)dimension zero. By Proposition B.0.1, it is then enough to prove the following: for every closed point $x_\phi \in X_M^{\mathrm{spec}}$ whose image in X_G^{spec} lies in $X_G^{\mathrm{spec,gen}}$, with associated closed immersion $\nu_\phi^M : \mathcal{N}_{S_\phi} \simeq (q^M)^{-1}(x_\phi) \hookrightarrow \mathrm{Par}_M$, the natural map on classical truncations $\mathrm{Par}_P^{\mathrm{cl}} \times_{\mathrm{Par}_M, \nu_\phi^M} \mathcal{N}_{S_\phi} \to \mathcal{N}_{S_\phi}$ is smooth of relative dimension zero. (Here $S_\phi = \mathrm{Cent}_{\hat{M}}(\phi)$ is computed relative to the Levi.) This follows in turn by explicitly identifying this fiber product with $\mathcal{N}_{S_\phi^P}$ where $S_\phi^P = \mathrm{Cent}_{\hat{P}}(\phi)$, and then using the fact that for any surjection $H \to G$ of linear algebraic groups with unipotent kernel, the associated map $\mathcal{N}_H \to \mathcal{N}_G$ is smooth of relative dimension zero. $\square$

Remark 2.3.4. It is probably true that for most groups, $X_G^{\mathrm{spec,gen}}$ is the maximal open subvariety of X_G^{spec} whose preimage in Par_P has trivial derived structure for all parabolics $P \subset G$.

Let ϕ be a semisimple generic parameter, so we have maps $BS_\phi \overset{i_0}{\to} \mathcal{N}_{S_\phi} \overset{\nu_\phi}{\to} \mathrm{Par}_G$. Here we write i_0 for the inclusion of the zero orbit, so $i_\phi = \nu_\phi \circ i_0 : BS_\phi \to \mathrm{Par}_G$ is the closed orbit of the semisimple parameter ϕ just as in the discussion of generous parameters. For any $\rho \in \mathrm{Irr}(S_\phi)$, we set $\delta_\rho = i_{0*}\rho \in \mathrm{Coh}(\mathcal{N}_{S_\phi})$, so the pushforward $\nu_{\phi*}\delta_\rho = i_{\phi*}\rho$ is still a perfect complex on Par_G. However, there should be several other sheaves in the picture, reflecting the more complicated geometry of $q^{-1}(x_\phi)$ and our experience with the trivial L-parameter. More precisely, we expect that there is a clean and explicit recipe which assigns to any $\rho \in \mathrm{Irr}(S_\phi)$ three canonical perverse-coherent objects $\overline{\Delta}_\rho, IC_\rho, \overline{\nabla}_\rho \in \mathrm{PCoh}(\mathcal{N}_{S_\phi})$ together with maps

$$\overline{\Delta}_\rho \twoheadrightarrow IC_\rho \hookrightarrow \overline{\nabla}_\rho$$

realizing IC_ρ as the socle of $\overline{\nabla}_\rho$ and the cosocle of $\overline{\Delta}_\rho$. Moreover, IC_ρ should be irreducible, and $\overline{\nabla}_\rho$ should be a genuine coherent sheaf. These sheaves should interpolate the following properties.

1. When ϕ is generous, $\mathcal{N}_{S_\phi} \cong BS_\phi$ and $\overline{\Delta}_\rho = IC_\rho = \overline{\nabla}_\rho = \delta_\rho = \rho$.

2. When S_ϕ is connected, $\mathrm{Irr}(S_\phi)$ is parametrized by highest weights and $\overline{\Delta}_\rho, IC_\rho, \overline{\nabla}_\rho$ should be the standard, resp. irreducible, resp. costandard perverse-coherent objects as in the discussion preceding Conjecture 2.2.1.

3. When ρ factors over the quotient $S_\phi \to S_\phi^\natural$, $\overline{\Delta}_\rho = IC_\rho = \overline{\nabla}_\rho$ should be the pullback of ρ along the tautological map $\mathcal{N}_{S_\phi} \to BS_\phi$.

4. For any ρ, there should be a uniquely determined ρ' such that $\delta_\rho \simeq IC_{\rho'}[d]$, where $d = \dim U_{S_\phi^\circ}$ for $U_{S_\phi^\circ}$ the unipotent radical of a(ny) Borel subgroup of S_ϕ°.

5. There should be a surjection of coherent sheaves $\overline{\nabla}_\rho \twoheadrightarrow \delta_\rho$.

Finally, these sheaves should have the property that if (b, π) is the pair matching (ϕ, ρ) under the BM-O bijection, then the categorical equivalence induces the following matchings of sheaves (c_ψ in one direction, a_ψ in the other)

$$\begin{aligned} \nu_{\phi*}\overline{\Delta}_\rho &\longleftrightarrow i_{b\sharp}^{\mathrm{ren}}\pi \\ \nu_{\phi*}IC_\rho &\longleftrightarrow i_{b\sharp!}^{\mathrm{ren}}\pi \\ \nu_{\phi*}\overline{\nabla}_\rho &\longleftrightarrow i_{b!}^{\mathrm{ren}}\pi \\ \nu_{\phi*}\delta_\rho = i_{\phi*}\rho &\longleftrightarrow i_{b!*}^{\mathrm{ren}}\pi \end{aligned}$$

compatible with the evident maps. The first three sheaves on the right should be hadal, and the last two should be perverse. The surjection $\nu_{\phi*}\overline{\nabla}_\rho \twoheadrightarrow \nu_{\phi*}\delta_\rho$ from 5. above should correspond to the natural surjection $i_{b!}^{\mathrm{ren}}\pi \twoheadrightarrow i_{b!*}^{\mathrm{ren}}\pi$ of perverse sheaves. The isomorphism $\delta_\rho \simeq IC_{\rho'}[d]$ from 4. should correspond to an isomorphism $i_{b!*}^{\mathrm{ren}}\pi \simeq i_{b'\sharp!}^{\mathrm{ren}}\pi'[d]$, where (b', π') is the pair matching (ϕ, ρ') under the BM-O bijection. In particular, we see that irreducible perverse sheaves over a semisimple generic parameter should be finite sheaves.

If we believe in this matching, it is also not hard to see that Hecke operators should be t-exact for the hadal t-structure on $D(\mathrm{Bun}_G)_{\mathrm{fin},\phi}$. Indeed, $T_V i_{b\sharp}^{\mathrm{ren}}\pi$ matches with $\nu_{\phi *}(V \otimes \overline{\Delta}_\rho)$, and $V \otimes \overline{\Delta}_\rho$ is still perverse coherent, so by general nonsense about t-structures associated with quasiexceptional collections it will have a finite filtration with graded pieces of the form $\overline{\Delta}_{\rho_i}[n_i]$ for some $n_i \geq 0$. Passing back to the other side, we see that $T_V i_{b\sharp}^{\mathrm{ren}}\pi$ should have a finite filtration with graded pieces of the form $i_{b_i\sharp}^{\mathrm{ren}}\pi_i[n_i]$, which are connective for the hadal t-structure. This shows that T_V is right t-exact, and a similar argument gives left t-exactness.

Remark 2.3.5. It is natural to wonder how Bernstein-Zelevinsky duality interacts with the hadal t-structure. Let us say that a semisimple L-parameter ϕ is *cohomologically inert* if there is a fixed (nonnegative) integer d_ϕ such that for all $b \in B(G)$ and all $\pi \in \Pi(G_b)$ with Fargues-Scholze parameter ϕ, $\mathbf{D}_{\mathrm{coh}}(\pi) \simeq \mathrm{Zel}(\pi)[-d_\phi]$. Here $\mathrm{Zel}(-)$ denotes the Aubert-Zelevinsky involution on $\Pi(G_b)$ as in Remark 1.1.5. It is easy to see that if ϕ is cohomologically inert, then also $\phi^\vee$ is cohomologically inert with $d_\phi = d_{\phi^\vee}$, and one can show in this case that $\mathbf{D}_{\mathrm{BZ}}(-)[d_\phi]$ restricts to an exact anti-equivalence of abelian categories

$$\begin{aligned} \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin},\phi} &\xrightarrow{\sim} \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin},\phi^\vee} \\ A &\mapsto \mathbf{D}_{\mathrm{BZ}}(A)[d_\phi] \end{aligned}$$

sending the irreducible hadal sheaf $\mathscr{G}_{b,\pi}$ to the irreducible hadal sheaf $\mathscr{G}_{b,\mathrm{Zel}(\pi)}$. One can also show that this result is best possible: if ϕ is not cohomologically inert, then no fixed shift of $\mathbf{D}_{\mathrm{BZ}}(-)$ can induce such an equivalence. Finally, all evidence and examples point to the speculation that a semisimple parameter is cohomologically inert if and only if it is generic.[13]

2.4 Perverse t-exactness of Hecke operators

In the previous section we pointed out that Hecke operators should be hadal t-exact on ϕ-local finite sheaves when ϕ is semisimple generic. It is also natural to wonder how the Hecke action interacts with the perverse t-structure. Our main goal in this section is to justify the following conjecture.

Conjecture 2.4.1. *If ϕ is a semisimple generic parameter, the Hecke action of* $\mathrm{Rep}(\hat{G})$ *on* $D(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}}$ *is perverse t-exact.*

When ϕ is supercuspidal, this conjecture was essentially formulated in Fargues-Scholze. For generous parameters, this conjecture is a straightforward consequence of Conjecture 2.1.8, as discussed in the proof of Proposition 2.1.14. For general semisimple generic parameters, however, there is no easy evidence in its favor. Nevertheless, we will rigorously show that when ϕ is the trivial parameter, Conjecture 2.2.1 actually implies Conjecture 2.4.1! A very similar argument will apply to all semisimple generic parameters, once the ideas sketched in section 2.3 are more thoroughly developed.

The essential point on the spectral side is the following result. Here we return to the notation and setup of section 2.2; in particular we assume G is split.

Proposition 2.4.2. *For any $\lambda \in X^*(\hat{T})^+$ and any $V \in \mathrm{Rep}(\hat{G})$, the sheaf $V \otimes A_\lambda \in \mathrm{Coh}(\mathcal{N}/\hat{G})$ admits a finite filtration with graded pieces of the form $A_\nu[n]$ for some $\nu \in X^*(\hat{T})^+$ and $n \geq 0$.*

[13] Note added October 25: I recently realized that for some groups of low rank, this is not true. In particular, for $G = \mathrm{SL}_2$, every semisimple parameter is cohomologically inert.

Despite its innocent nature, I was only able to prove this by using the full power of the Arkhipov-Bezrukavnikov equivalence.[14] To prepare for the argument, recall that we have a diagram of functors between triangulated categories

$$\begin{tikzcd}
\mathrm{Coh}(\tilde{\mathcal{N}}/\hat{G}) \arrow[d, "\pi_*"] & D^b\mathrm{ExCoh}(\tilde{\mathcal{N}}/\hat{G}) \arrow[l, "\sim"'] \arrow[r, "F_{\mathcal{IW}}", "\sim"'] \arrow[d, "\pi_*"] & D^b\mathrm{Perv}_{\mathcal{IW}}(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell}) \arrow[r, "\sim"] & D^b_{\mathcal{IW}}(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell}) \\
\mathrm{Coh}(\mathcal{N}/\hat{G}) & D^b\mathrm{PCoh}(\mathcal{N}/\hat{G}) \arrow[l, "\sim"'] & D^b\mathrm{Perv}_I(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell}) \arrow[r] \arrow[u, "\mathrm{Av}_{\mathcal{IW}}"] & D^b_I(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell}) \arrow[u, "\mathrm{Av}_{\mathcal{IW}}"]
\end{tikzcd}$$

where all but one horizontal arrow is an equivalence of categories. Here $\mathbf{G}/\overline{\mathbf{F}_p}$ is a split reductive group equipped with a fixed identification of dual groups $\hat{\mathbf{G}} \cong \hat{G}$ over $\overline{\mathbf{Q}_\ell}$, and all our remaining notation essentially follows the book [AR23], which we will refer to heavily. Aside from notation we have already seen in section 2.2, we recall that $\mathrm{ExCoh}(\tilde{\mathcal{N}}/\hat{G})$ denotes the heart of the "exotic" t-structure on equivariant coherent sheaves on the Springer resolution, and $F_{\mathcal{IW}}$ denotes the Arkhipov-Bezrukavnikov equivalence. All unlabelled horizontal arrows are induced by the usual realization functors, and we will elide them in our notation; this should cause no confusion. We also recall that $F_{\mathcal{IW}}$ is t-exact for the exotic and perverse t-structures, and defines an exact equivalence of abelian categories between ExCoh and $\mathrm{Perv}_{\mathcal{IW}}$. Moreover, π_* is t-exact for the exotic and perverse coherent t-structures. Finally, $\mathrm{Av}_{\mathcal{IW}}$ is t-exact for the evident perverse t-structures.

Each of the abelian categories PCoh, ExCoh, $\mathrm{Perv}_{\mathcal{IW}}$, Perv_I is equipped with a canonical collection of costandard objects. For PCoh these are exactly the sheaves $A_\lambda = \overline{\nabla}_\lambda$, $\lambda \in X^*(\hat{T})^+$, which we have seen in section 2.2. For ExCoh and $\mathrm{Perv}_{\mathcal{IW}}$ the indexing set for the costandard objects is the set of all $\mu \in X^*(\hat{T})$, and we write $\nabla^{\mathrm{ex}}_\mu \in \mathrm{ExCoh}$ resp. $\nabla^{\mathcal{IW}}_\mu \in \mathrm{Perv}_{\mathcal{IW}}$ for the associated costandard objects. For Perv_I the indexing set for costandard objects is the extended affine Weyl group $\widetilde{W} = W \ltimes X^*(\hat{T})$, and we write ∇^I_w for the costandard object corresponding to an element $w \in \widetilde{W}$.

For $\mathcal{C}$ any one of these four abelian categories, we write $D^b_+\mathcal{C}$ for the full (but not triangulated!) subcategory of $D^b\mathcal{C}$ spanned by objects which admit a finite filtration with graded pieces of the form $A[n]$ where A is costandard and $n \geq 0$. We also write $D^b_{\mathcal{IW}}(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell})_+$ and $D^b_I(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell})_+$ for the full subcategories spanned by objects admitting finite filtrations whose graded pieces are nonnegative shifts of costandard objects.

Proposition 2.4.3. *The functors π_*, $F_{\mathcal{IW}}$, and $\mathrm{Av}_{\mathcal{IW}}$, together with the realization functors, induce a commutative diagram of functors*

$$\begin{tikzcd}
D^b_+\mathrm{ExCoh}(\tilde{\mathcal{N}}/\hat{G}) \arrow[r, "F_{\mathcal{IW}}", "\sim"'] \arrow[d, "\pi_*"] & D^b_+\mathrm{Perv}_{\mathcal{IW}}(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell}) \arrow[r, "\sim"] & D^b_{\mathcal{IW}}(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell})_+ \\
D^b_+\mathrm{PCoh}(\mathcal{N}/\hat{G}) & D^b_+\mathrm{Perv}_I(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell}) \arrow[r] \arrow[u, "\mathrm{Av}_{\mathcal{IW}}"] & D^b_I(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell})_+ \arrow[u, "\mathrm{Av}_{\mathcal{IW}}"]
\end{tikzcd}$$

where both upper horizontal arrows are equivalences.

Proof. Recall that all three functors in the left half of the diagram come from exact functors on the evident abelian categories. For π_* the claim follows from the fact that $\pi_*\nabla^{\mathrm{ex}}_\mu \simeq \overline{\nabla}_{\mathrm{dom}(\mu)}$ for all μ,

[14]It is quite striking that in order to prove something about the Hecke action on $D(\mathrm{Bun}_G)$, we will first go to the spectral side via the categorical equivalence, and then pass through the Langlands mirror *again* via the AB equivalence.

where $\mathrm{dom}(\mu)$ is the unique dominant weight in the W-orbit of μ [AR23, Lemma 7.3.10]. For $F_{\mathcal{IW}}$ the claim follows from the fact that $F_{\mathcal{IW}}(\nabla^{\mathrm{ex}}_\mu) \simeq \nabla^{\mathcal{IW}}_\mu$ for all μ [AR23, Proposition 7.1.5]. Finally, for $\mathrm{Av}_{\mathcal{IW}}$ the claim follows from the fact that $\mathrm{Av}_{\mathcal{IW}}(\nabla^I_w) \simeq \nabla^{\mathcal{IW}}_\mu$, where $\mu \in X^*(\hat{T})$ is the unique element with $W \cdot (1,\mu) = W \cdot w$ [AR23, Lemma 6.4.5]. □

We now return to the task of proving Proposition 2.4.2. Note that in the "+" notation introduced above, we are simply trying to prove that for all $\lambda \in X^*(\hat{T})^+$ and $V \in \mathrm{Rep}(\hat{G})$, $\overline{\nabla}_\lambda \otimes V$ lies in $D^b_+\mathrm{PCoh}(\mathcal{N}/\hat{G})$. A trivial projection formula gives an isomorphism $\overline{\nabla}_\lambda \otimes V \simeq \pi_*(\nabla^{\mathrm{ex}}_\lambda \otimes V)$, so by Proposition 2.4.3 it's enough to prove that $\nabla^{\mathrm{ex}}_\lambda \otimes V$ lies in $D^b_+\mathrm{ExCoh}(\tilde{\mathcal{N}}/\hat{G})$. Going to the other side of the Arkhipov-Bezrukavnikov equivalence and using Proposition 2.4.3 again, it's enough in turn to prove that $F_{\mathcal{IW}}(\nabla^{\mathrm{ex}}_\lambda \otimes V)$ lies in $D^b_+\mathrm{Perv}_{\mathcal{IW}}(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell})$.

Now the magic happens. Recall that $D^b_{\mathcal{IW}}$ is a right module over D^b_I via the convolution action of D^b_I on itself, compatibly with the functor $\mathrm{Av}_{\mathcal{IW}}$. It is then true that

$$F_{\mathcal{IW}}(\mathcal{G} \otimes V) \simeq F_{\mathcal{IW}}(\mathcal{G}) \star^I \mathscr{Z}(V)$$

for any $\mathcal{G} \in D^b\mathrm{ExCoh}$ and any $V \in \mathrm{Rep}(\hat{G})$, where $\mathscr{Z}(V)$ denotes the central sheaf in Perv_I associated with V. Applying this property with $\mathcal{G} = \nabla^{\mathrm{ex}}_\lambda$, using the identification of costandard objects under $F_{\mathcal{IW}}$ and $\mathrm{Av}_{\mathcal{IW}}$ stated previously, and rearranging using the centrality of $\mathscr{Z}(V)$, we get that

$$\begin{aligned} F_{\mathcal{IW}}(\nabla^{\mathrm{ex}}_\lambda \otimes V) &\simeq F_{\mathcal{IW}}(\nabla^{\mathrm{ex}}_\lambda) \star^I \mathscr{Z}(V) \\ &\simeq \nabla^{\mathcal{IW}}_\lambda \star^I \mathscr{Z}(V) \\ &\simeq \mathrm{Av}_{\mathcal{IW}}\left(\nabla^I_{w_\lambda} \star^I \mathscr{Z}(V)\right) \\ &\simeq \mathrm{Av}_{\mathcal{IW}}\left(\mathscr{Z}(V) \star^I \nabla^I_{w_\lambda}\right) \\ &\simeq \mathrm{Av}_{\mathcal{IW}}(\mathscr{Z}(V)) \star^I \nabla^I_{w_\lambda} \end{aligned}$$

where w_λ is the evident lift. By [AR23, Theorem 6.5.2], $\mathrm{Av}_{\mathcal{IW}}(\mathscr{Z}(V))$ admits a finite filtration in $\mathrm{Perv}_{\mathcal{IW}}$ with costandard graded pieces. This immediately reduces us to proving that any complex of the form $\nabla^{\mathcal{IW}}_\nu \star^I \nabla^I_w$ lies in $D^b_+\mathrm{Perv}_{\mathcal{IW}}$. But

$$\nabla^{\mathcal{IW}}_\nu \star^I \nabla^I_w \simeq \mathrm{Av}_{\mathcal{IW}}(\nabla^I_{w_\nu} \star^I \nabla^I_w),$$

so using Proposition 2.4.3 one more time, we're now reduced to showing that any complex of the form $\nabla^I_{w'} \star^I \nabla^I_w$ lies in $D^b_I(\mathrm{Fl}_{\mathbf{G}}, \overline{\mathbf{Q}_\ell})_+$. But this is exactly the second half of [AR23, Lemma 6.5.8].

Theorem 2.4.4. *Assume Conjecture 2.2.1, and also that the pairs (b_λ, π_λ) exhaust all pairs (b,π) for which $i^{\mathrm{ren}}_{b!}\pi$ has trivial Fargues-Scholze parameter. Then the Hecke action of* $\mathrm{Rep}(\hat{G})$ *on* $D(\mathrm{Bun}_G)^{\mathrm{ULA}}_{\phi_{\mathrm{triv}}}$ *is perverse t-exact.*

The exhaustion hypothesis is known unconditionally for many groups, including GL_n, SL_n, and GSp_4.

Proof. It suffices to prove that any Hecke operator T_V acting on $D(\mathrm{Bun}_G)^{\mathrm{ULA}}_{\phi_{\mathrm{triv}}}$ is perverse right t-exact. Indeed, since the right adjoint of T_V is $T_{V^\vee}$, this automatically implies that $T_{V^\vee}$ is perverse left t-exact. Varying over all V, we get the claimed reduction.

Next, using the exhaustion hypothesis, one checks that $D(\mathrm{Bun}_G)^{\mathrm{ULA}}_{\phi_{\mathrm{triv}}} \cap {}^pD^{\leq 0}$ is generated under extensions and colimits by objects of the form $i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda[n]$ for $\lambda \in X^*(\hat{T})^+$ and $n \geq 0$. This reduces us to checking that any sheaf of the form $T_V i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$ is perverse connective. Under the categorical equivalence, Conjecture 2.2.1, $T_V i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$ corresponds to the coherent complex $\nu_*(V\otimes A_\lambda)$. By Proposition 2.4.2, this has a finite filtration with graded pieces of the form $\nu_* A_\mu[n]$ for some $\mu \in X^*(\hat{T})^+$ and $n \geq 0$. Translating back to the other side, we deduce that $T_V i^{\mathrm{ren}}_{b_\lambda !}\pi_\lambda$ has a finite filtration with graded pieces of the form $i^{\mathrm{ren}}_{b_\mu !}\pi_\mu[n]$ for some $\mu \in X^*(\hat{T})^+$ and $n \geq 0$. Since these graded pieces are perverse connective, we get the desired result. □

2.5 Two t-exactness conjectures

At this point, we are ready to confidently pose some precise t-exactness conjectures for the categorical equivalence with restricted variation, localized over semisimple generic parameters.

Set $\mathrm{Par}_G^{\mathrm{gen}} = \mathrm{Par}_G \times_{X_G^{\mathrm{spec}}} X_G^{\mathrm{spec,gen}}$, so $\mathrm{Par}_G^{\mathrm{gen}}$ is a smooth algebraic stack. In fact, $X_G^{\mathrm{spec,gen}}$ is the *maximal* open subscheme of X_G^{spec} with the property that its preimage in Par_G is a smooth algebraic stack.

Let $D(\mathrm{Bun}_G)^{\mathrm{gen}}_{\mathrm{fin}}$ be the full subcategory of finite sheaves A whose ϕ-local summand A_ϕ vanishes for every semisimple parameter ϕ which is not semisimple generic. It is clear that Conjecture 1.7.8 localizes to a conjectural equivalence

$$c_\psi : D(\mathrm{Bun}_G)^{\mathrm{gen}}_{\mathrm{fin}} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G^{\mathrm{gen}})_{\mathrm{fin}}.$$

Conjecture 2.5.1. i. *The category $D(\mathrm{Bun}_G)^{\mathrm{gen}}_{\mathrm{fin}}$ is stable under the perverse truncation functors.*

ii. *The equivalence*

$$D(\mathrm{Bun}_G)^{\mathrm{gen}}_{\mathrm{fin}} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G^{\mathrm{gen}})_{\mathrm{fin}}$$

is t-exact with respect to the perverse *t-structure on the left-hand side and the* standard *t-structure on the right-hand side.*

Note that ii. is not well-posed unless one assumes i. is true. However, we warn the reader that i. definitely fails before passing to the localization around semisimple generic parameters. For a concrete example, take $G = \mathrm{SL}_2$ and let $i_1 : [*/\underline{\mathrm{SL}_2(E)}] \to \mathrm{Bun}_G$ be the inclusion of the open stratum, with closed complement $h : Z \to \mathrm{Bun}_G$. Considering the distinguished triangle

$$i_{1!}\overline{\mathbf{Q}_\ell} \to \overline{\mathbf{Q}_\ell} \to h_*\overline{\mathbf{Q}_\ell} \xrightarrow{[1]},$$

it is easy to see that the constant sheaf $\overline{\mathbf{Q}_\ell}$ is perverse, and (by consideration of $*$-stalks) that $h_*\overline{\mathbf{Q}_\ell}$ can only have nonvanishing perverse cohomology sheaves in degrees ≤ -2. Therefore applying ${}^pH^0(-)$ to the first map of this triangle induces an isomorphism

$${}^pH^0(i_{1!}\overline{\mathbf{Q}_\ell}) \xrightarrow{\sim} \overline{\mathbf{Q}_\ell},$$

where of course $i_{1!}\overline{\mathbf{Q}_\ell}$ is finite but $\overline{\mathbf{Q}_\ell}$ is not. However, all of these sheaves are ϕ-local for the unramified parameter $\phi : W_E \to \mathrm{PGL}_2(\overline{\mathbf{Q}_\ell})$ sending Frobenius to $\begin{pmatrix} 1 & \\ & q \end{pmatrix}$, and this parameter is certainly not semisimple generic.

On the other hand, part ii. of Conjecture 2.5.1 is not a wild guess. Aside from the analogy with the t-exactness results in [FR22, Section 1.6.2], we have the following result.

Proposition 2.5.2. *If Conjectures 1.7.8 and 2.4.1 are true, then*

$$c_\psi : D(\mathrm{Bun}_G)^{\mathrm{gen}}_{\mathrm{fin}} \to \mathrm{Coh}(\mathrm{Par}_G^{\mathrm{gen}})_{\mathrm{fin}}$$

satisfies the t-exactness property of Conjecture 2.5.1.

Sketch. If $A \in D(\mathrm{Bun}_G)^{\mathrm{gen}}_{\mathrm{fin}}$ is perverse, then $T_V A$ is perverse for all $V \in \mathrm{Rep}(\hat{G})$ by Conjecture 2.4.1, so $i_1^* T_V A$ is concentrated in degree zero and then

$$R\mathrm{Hom}(i_{1!}W_\psi, T_V A) \simeq R\mathrm{Hom}(W_\psi, i_1^* T_V A)$$

is also concentrated in degree zero by the projectivity of W_ψ (see the discussion around Theorem A.0.1). On the other side of Conjecture 1.7.8, this translates into the knowledge that

$$R\Gamma(\mathrm{Par}_G, V \otimes c_\psi(A)) \simeq R\mathrm{Hom}(i_{1!}W_\psi, T_V A)$$

is concentrated in degree zero for all $V \in \mathrm{Rep}(\hat{G})$. But this implies that $c_\psi(A)$ is concentrated in degree zero for the standard t-structure. □

Finally we formulate a t-exactness conjecture for the hadal t-structure. This is somewhat implicit in the discussion from section 2.3, but for completeness we spell it out fully. Here we freely reuse the notations introduced in section 2.3. For a fixed semisimple generic parameter ϕ, define ${}^{pcoh}D^{\leq 0}$ resp. ${}^{pcoh}D^{\geq 0}$ inside $\mathrm{Coh}(\mathrm{Par}_G)_\phi$ as the full subcategory generated under extensions by $\nu_{\phi *}\overline{\Delta}_\rho[n]$ for $n \geq 0$, resp. by $\nu_{\phi *}\overline{\nabla}_\rho[n]$ for $n \leq 0$.

Conjecture 2.5.3. i. *The pair* $({}^{pcoh}D^{\leq 0}, {}^{pcoh}D^{\geq 0})$ *defines a perverse coherent t-structure on* $\mathrm{Coh}(\mathrm{Par}_G)_\phi$. *Writing* $\mathrm{PCoh}(\mathrm{Par}_G)_\phi$ *for the heart, the functor* $\nu_{\phi *} : \mathrm{Coh}(\mathcal{N}_{S_\phi}) \to \mathrm{Coh}(\mathrm{Par}_G)_\phi$ *should induce a faithful exact functor*

$$\mathrm{PCoh}(\mathcal{N}_{S_\phi}) \to \mathrm{PCoh}(\mathrm{Par}_G)_\phi$$

which is bijective on isomorphism classes of irreducible objects.

ii. *The equivalence*

$$c_\psi : D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G)_\phi$$

induced by ϕ-localizing the equivalence of Conjecture 1.7.8 should be t-exact with respect to the hadal *t-structure on the left-hand side and the* perverse coherent *t-structure defined above on the right-hand side. In particular, it should restrict to an exact equivalence of abelian categories*

$$\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \xrightarrow{\sim} \mathrm{PCoh}(\mathrm{Par}_G)_\phi.$$

Part i. of this conjecture should be quite easy to verify, conditionally on working out the speculations from section 2.3.

3 Additional conjectures

3.1 ULA sheaves and generic parameters

The reader may have noticed that general (i.e. non-finite) ULA sheaves have been largely absent from our discussion. This is due to a psychological complication, which we have tried to avoid confronting until now: when translating general ULA sheaves to the spectral side, we are forced unavoidably to reckon with IndCoh. More precisely, we have the following definition.

Definition 3.1.1. An object $A \in \mathrm{IndCoh}(\mathrm{Par}_G)$ is *admissible* if for all $B \in \mathrm{Coh}(\mathrm{Par}_G)$, $R\mathrm{Hom}(B, A)$ lies in $\mathrm{Perf}(\overline{\mathbf{Q}_\ell})$. We write $\mathrm{Adm}(\mathrm{Par}_G) \subset \mathrm{IndCoh}(\mathrm{Par}_G)$ for the evident stable ∞-category of admissible ind-coherent sheaves.

If we believe the categorical conjecture, then these are exactly the sheaves on the spectral side which should match with ULA sheaves on the automorphic side.

Proposition 3.1.2. *If Conjecture 1.7.3 is true, then the ind-extension of c_ψ to an equivalence $D(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Par}_G)$ restricts to an equivalence of categories*

$$c_\psi^{\mathrm{ULA}} : D(\mathrm{Bun}_G)^{\mathrm{ULA}} \xrightarrow{\sim} \mathrm{Adm}(\mathrm{Par}_G)$$

linear over the $\otimes$-action of $\mathrm{Perf}(\mathrm{Par}_G)$.

Proof. This is immediate from the fact that ULA sheaves A on Bun_G are characterized by the condition that $R\mathrm{Hom}(B, A)$ lies in $\mathrm{Perf}(\overline{\mathbf{Q}_\ell})$ for all $B \in D(\mathrm{Bun}_G)^\omega$, which follows from [FS21, Prop. VII.7.4 and Prop. VII.7.9]. □

Again, we emphasize that after restricting to $D(\mathrm{Bun}_G)_{\mathrm{fin}} \subset D(\mathrm{Bun}_G)^{\mathrm{ULA}}$ the functors c_ψ^{ULA} and c_ψ coincide, but they do not agree on all ULA sheaves, as discussed in the warning before Conjecture 1.7.5.

Exercise 3.1.3. 1. Show that there is an inclusion $\mathrm{Adm}(\mathrm{Par}_G) \cap \mathrm{Coh}(\mathrm{Par}_G) \subset \mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}$.
2. Show that there is an inclusion $\mathrm{Perf}(\mathrm{Par}_G)_{\mathrm{fin}} \subset \mathrm{Adm}(\mathrm{Par}_G) \cap \mathrm{Coh}(\mathrm{Par}_G)$.
3. Show that if the categorical conjecture is true, then the inclusion in 1. must be an equality.

We warn the reader that most admissible sheaves are not coherent. For instance, let $x \in \mathrm{Par}_G(\overline{\mathbf{Q}_\ell})$ be any point in the smooth locus of the stack Par_G, with associated residual gerbe BS_x, and let $i_x : BS_x \to \mathrm{Par}_G$ be the evident immersion. If ρ is any irreducible algebraic representation of S_x, one can show that $i_{x*}^{\mathrm{IndCoh}}\rho$ is always an admissible sheaf (we will prove a more general result below). However, if x is not a closed point, this sheaf will not be coherent.

Next we formulate a duality conjecture. If X is any quasismooth algebraic stack over $\mathrm{Spec}\,\overline{\mathbf{Q}_\ell}$ with structure map f_X, and $A \in \mathrm{IndCoh}(X)$ is a given object, then we get a contravariant functor

$$\begin{aligned} \mathrm{IndCoh}(X) &\to \mathrm{IndCoh}(\mathrm{Spec}\,\overline{\mathbf{Q}_\ell}) = \mathrm{Vect}_{\overline{\mathbf{Q}_\ell}} \\ B &\mapsto R\mathrm{Hom}(f_{X*}^{\mathrm{IndCoh}}(A \otimes^! B), \overline{\mathbf{Q}_\ell}) \end{aligned}$$

where the notation for pushforward and tensor product of ind-coherent sheaves follows Gaitsgory-Rozenblyum's book. By some general nonsense with the ∞-categorical adjoint functor theorem, this functor is representable by $R\mathrm{Hom}(-, \mathbf{D}_{\mathrm{adm}}A)$ for a uniquely determined object $\mathbf{D}_{\mathrm{adm}}A \in \mathrm{IndCoh}(X)$. The association $A \mapsto \mathbf{D}_{\mathrm{adm}}A$ is a contravariant endofunctor of $\mathrm{IndCoh}(X)$. For general stacks and general sheaves, this functor of "admissible dual" will not be well-behaved. However, for the stack of L-parameters, we expect the following.[15]

Conjecture 3.1.4. i. *The functor $\mathbf{D}_{\mathrm{adm}}(-)$ defines an involutive anti-equivalence from* $\mathrm{Adm}(\mathrm{Par}_G)$ *to itself.*

ii. *The equivalence $c_\psi^{\mathrm{ULA}} : D(\mathrm{Bun}_G)^{\mathrm{ULA}} \xrightarrow{\sim} \mathrm{Adm}(\mathrm{Par}_G)$ conjectured in Proposition 3.1.2 should satisfy the duality compatibility*

$$\mathbf{D}_{\mathrm{tw.adm}} \circ c_\psi^{\mathrm{ULA}} \simeq c_{\psi^{-1}}^{\mathrm{ULA}} \circ \mathbf{D}_{\mathrm{Verd}},$$

where $\mathbf{D}_{\mathrm{tw.adm}} = c^*\mathbf{D}_{\mathrm{adm}}$ is the composition of admissible duality with pullback along the Chevalley involution.

[15] A proof of Conjecture 3.1.4.i and substantial evidence for Conjecture 3.1.4.ii will be given in future joint work with Lucas Mann [HM23].

This suggests another perspective on the "best hope" discussed at the beginning of section 2. Specifically, if we believe in Proposition 3.1.2 and Conjecture 3.1.4, we are forced to believe in the existence of an exotic t-structure on $\mathrm{Adm}(\mathrm{Par}_G)$ matching the perverse t-structure on $D(\mathrm{Bun}_G)^{\mathrm{ULA}}$, whose left and right halves are swapped by $\mathbf{D}_{\mathrm{tw.adm}}$. Over the smooth locus of Par_G, this should just be the standard t-structure, but it seems extremely subtle to extend the desired t-structure over the singularities of the stack of L-parameters. For instance, one can show that for most $V \in \mathrm{Rep}(\hat{G})$, the endofunctor $V \otimes -$ corresponding to the Hecke operator T_V is neither left nor right t-exact for this t-structure, and its failure of t-exactness around a given point x seems to correlate with "how singular" the stack is at x. This seems to suggest that any direct definition of this t-structure will need to use specific features of the stack of singularities of Par_G. We will discuss these ideas in more detail elsewhere.

Next we formulate some conjectures attaching ULA Hecke eigensheaves on Bun_G to generic L-parameters. In some sense this brings us back to the very origins of the entire subject in Fargues's 2014 MSRI lecture [Far14]. To simplify the discussion, we will consider L-parameters satisfying the following condition.

(†) ϕ is Frobenius-semisimple and generic, and S_ϕ° is reductive.

We emphasize that in contrast to most of the discussion in section 2, we are no longer requiring ϕ to be semisimple. In fact, the condition (†) is very mild. For instance, it holds for all discrete parameters, all parameters which are ι-essentially tempered for some isomorphism $\iota : \overline{\mathbf{Q}_\ell} \overset{\sim}{\to} \mathbf{C}$, all semisimple generic parameters, and all generic parameters such that $\phi(\mathrm{Fr})$ is regular semisimple. The last two claims here are easy, and the first two (which require some work) are proved in [BMIY22, Section 3]. Given any such L-parameter, we get a canonical immersion $i_\phi : BS_\phi \to \mathrm{Par}_G$ which factors through the smooth locus in the stack of L-parameters by our genericity assumption.

Proposition 3.1.5. *Let $M \in \mathrm{IndCoh}(BS_\phi)$ be any object with the property that for each $\rho \in \mathrm{Irr}(S_\phi)$, the total multiplicity $\sum_n \dim \mathrm{Hom}_{S_\phi}(\rho, H^n(M))$ is finite. Then $i_{\phi*}^{\mathrm{IndCoh}} M$ is an admissible sheaf.*

Proof. If B lies in $\mathrm{Coh}(\mathrm{Par}_G)$, then $i_\phi^{*\mathrm{IndCoh}} B$ lies in $\mathrm{Coh}(BS_\phi)$, using that i_ϕ has finite tor-dimension since it is a regular immersion of stacks. We then get that

$$R\mathrm{Hom}(B, i_{\phi*}^{\mathrm{IndCoh}} M) \simeq R\mathrm{Hom}(i_\phi^{*\mathrm{IndCoh}} B, M)$$

by adjunction, and this lies in $\mathrm{Perf}(\overline{\mathbf{Q}_\ell})$ by the coherence of $i_\phi^{*\mathrm{IndCoh}} B$, the reductivity of S_ϕ°, and our assumption on M. □

Note that the condition in this proposition is trivially satisfied if M is an irreducible algebraic S_ϕ-representation, but it is also satisfied if $M = \mathcal{O}(S_\phi)$ is the regular representation, even though the latter is not coherent! In particular, we get an admissible sheaf $i_{\phi*}^{\mathrm{IndCoh}} \mathcal{O}(S_\phi)$ which (by our reductivity assumption) admits a canonical direct sum decomposition with admissible pieces

$$i_{\phi*}^{\mathrm{IndCoh}} \mathcal{O}(S_\phi) \simeq \oplus_{\rho \in \mathrm{Irr}(S_\phi)} i_{\phi*}^{\mathrm{IndCoh}} \rho^{\oplus \dim \rho}.$$

Moreover, by [AG15, Lemma 8.3.2], we see that $i_{\phi*}^{\mathrm{IndCoh}} \rho \simeq i_{\phi*}\rho$ lies in $\mathrm{QCoh} \subset \mathrm{IndCoh}$, so we are actually in the regime where the inverse to the equivalence c_ψ^{ULA} should be realized by the spectral action functor a_ψ. In particular, we obtain a canonical sheaf $\mathscr{F}_\phi = a_\psi(i_{\phi*}\mathcal{O}(S_\phi))$ admitting a direct sum decomposition

$$\mathscr{F}_\phi \simeq \oplus_{\rho \in \mathrm{Irr}(S_\phi)} \mathscr{F}_{\phi,\rho}^{\oplus \dim \rho}$$

where of course we set $\mathscr{F}_{\phi,\rho} = a_\psi(i_{\phi*}\rho)$, which should match the above direct sum decomposition of ind-coherent sheaves under the functor c_ψ^{ULA}. However, we stress that the definition of $\mathscr{F}_{\phi,\rho}$

is totally unconditional. Moreover, we expect that these sheaves should enjoy many remarkable properties.

1. By the usual formal argument, $\mathscr{F}_\phi$ is a Hecke eigensheaf with eigenvalue ϕ. A more refined argument shows that for any $V \in \mathrm{Rep}({}^L G)$, there is an isomorphism

$$T_V \mathscr{F}_{\phi,\rho} \simeq \bigoplus_{\rho' \in \mathrm{Irr}(S_\phi)} \mathscr{F}_{\phi,\rho'} \boxtimes \mathrm{Hom}_{S_\phi}(\rho^\vee \otimes \rho', V \circ \phi)$$

as objects of $D(\mathrm{Bun}_G)^{BW_E}$.

2. We expect that $\mathscr{F}_\phi$ is ULA and perverse, and each $\mathscr{F}_{\phi,\rho}$ is ULA, perverse and indecomposable. Moreover, the "easy part" of the BM-O algorithm begins by (unconditionally!) attaching to the pair (ϕ, ρ) an element $b = b_{\phi,\rho} \in B(G)$, and we expect that $\mathrm{supp}\mathscr{F}_{\phi,\rho} \subseteq \overline{\{b\}}$. Additionally, we expect that $i_b^{*\mathrm{ren}} \mathscr{F}_{\phi,\rho}$ is a single irreducible $G_b(E)$-representation π concentrated in degree zero, and that this is *the* $G_b(E)$-representation attached to (ϕ, ρ) by the $B(G)$ local Langlands correspondence. In other words, we expect that this stalk realizes the true $B(G)$ local Langlands correspondence at L-parameters satisfying (†)! Since the condition (†) includes all tempered parameters, and there is a standard procedure to pass from tempered local Langlands to full local Langlands, we conclude that *the spectral action should completely pin down the long-sought set-theoretic local Langlands correspondence in all generality!*

We also expect that the induced map

$$i_{b!}^{\mathrm{ren}} \pi \to \mathscr{F}_{\phi,\rho}$$

induces a surjection (in the perverse category) from ${}^p H^0(i_{b!}^{\mathrm{ren}}\pi)$ onto the socle of $\mathscr{F}_{\phi,\rho}$. In general, $\mathscr{F}_{\phi,\rho}$ should *not* be the intermediate extension $i_{b!*}^{\mathrm{ren}}\pi$ - we believe that this happens exactly when ϕ is semisimple generic. When ϕ is discrete, $\mathscr{F}_{\phi,\rho}$ seems to be a *tilting* perverse sheaf [BBM04].

3. We expect that $\mathbf{D}_{\mathrm{Verd}}(\mathscr{F}_{\phi,\rho}) \simeq \mathscr{F}_{\phi^\vee, c\circ\rho^\vee}$, where $\mathscr{F}_{\phi^\vee, c\circ\rho^\vee} = a_{\psi^{-1}}(i_{\phi^\vee *} c \circ \rho^\vee)$ is the sheaf (conjecturally) matching $i_{\phi^\vee *} c \circ \rho^\vee$ under the functor $c_{\psi^{-1}}^{\mathrm{ULA}}$. Indeed, one can prove unconditionally that $\mathbf{D}_{\mathrm{tw.adm}}$ exchanges $i_{\phi *}^{\mathrm{IndCoh}} \rho$ and $i_{\phi^\vee *}^{\mathrm{IndCoh}} c \circ \rho^\vee$, so this expectation is forced upon us by Conjecture 3.1.4.ii.

4. We expect that suitable linear combinations (over varying ρ) of the virtual stalks $[i_b^{*\mathrm{ren}} \mathscr{F}_{\phi,\rho}]$ can be described explicitly. For instance, if ϕ is discrete and b has trivial Kottwitz invariant, we expect that for every elliptic endoscopic datum $\mathcal{H} = (H, s, \eta)$ such that ϕ admits a factorization $\phi = {}^L\eta \circ \phi^H$ there should be an equality of the form

$$\sum_{\rho \in \mathrm{Irr}(\pi_0(\overline{S}_\phi))} \mathrm{tr}\rho(s) \cdot [i_b^{*\mathrm{ren}} \mathscr{F}_{\phi,\rho}] = \mathrm{Red}_b^{\mathcal{H}}(S\Theta_{\phi^H}).$$

Here $\mathrm{Red}_b^{\mathcal{H}}$ is the map from stable virtual representations of $H(E)$ towards virtual representations of $G_b(E)$ defined in [BM21, Definition 5.6], and $S\Theta_{\phi^H}$ is the stable virtual representation of $H(E)$ attached to ϕ^H. In general, the stalks of the individual sheaves $\mathscr{F}_{\phi,\rho}$ seem extremely hard to describe, even as virtual representations.

3.2 Generalized coherent Springer sheaves

Some of the phenomena predicted in section 3.1 would be neatly explained by yet another conjecture, with many consequences.[16] To formulate this, note that for any b and any open compact $K \subset G_b(E)$,

[16]Conjecture 3.2.1 and Proposition 3.2.2 below were independently discovered by Koshikawa, who also independently noticed the utility of the condition (†). I would also like to acknowledge that the tilting property for Hecke

$i_{b!}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}$ is compact, so Conjecture 1.7.3 predicts that

$$\mathscr{C}_{b,K} \overset{\mathrm{def}}{=} c_\psi\left(i_{b!}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}\right)$$

is a bounded coherent complex with quasicompact support. (Strictly speaking, $\mathscr{C}_{b,K}$ depends also on the Whittaker datum, but we suppress this from the notation.)

Conjecture 3.2.1. *For all b and K as above (and all Whittaker data), $\mathscr{C}_{b,K}$ is a genuine coherent sheaf, i.e. it lies in the heart $\mathrm{Coh}(\mathrm{Par}_G)^\heartsuit$ of the standard t-structure on* Coh.

When $b=1$, G is unramified, and $K=I$ is an Iwahori, $\mathscr{C}_{1,I}$ should coincide with the "coherent Springer sheaf" studied by Ben–Zvi-Chen-Helm-Nadler, Hellmann and Zhu. For non-basic b, Conjecture 3.2.1 is a natural Bun_G variant of the conjecture formulated in [Zhu21, Remark 4.6.5].[17] This conjecture turns out to be a magic wand, both for predicting qualitative properties of the categorical equivalence, and for generating new conjectures purely on the automorphic side.

Proposition 3.2.2. *Assume Conjectures 1.7.3 and 1.7.5, and also assume that Conjecture 3.2.1 is true. Then the following hold.*

i. *For all b and K, the coherent complex $c_\psi\left(i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}\right)$ is concentrated in nonnegative degrees.*

ii. *The functor c_ψ is perverse right t-exact, i.e. it carries ${}^pD^{\leq 0}(\mathrm{Bun}_G)$ into $\mathrm{QCoh}^{\leq 0}(\mathrm{Par}_G)$. In particular, for any b and any $\pi\in\Pi(G_b)$, $c_\psi(i_{b!}^{\mathrm{ren}}\pi)$ is concentrated in nonpositive cohomological degrees.*

iii. *For any L-parameter ϕ satisfying (†) as in Section 3.1 and any $\rho\in\mathrm{Irr}(S_\phi)$, the sheaf $\mathscr{F}_{\phi,\rho}$ defined in Section 3.1 is perverse. Moreover, if ϕ is a discrete parameter then $\mathscr{F}_{\phi,\rho}$ is a* tilting *perverse sheaf: for all b, $i_b^{*\mathrm{ren}}\mathscr{F}_{\phi,\rho}$ and $i_b^{!\mathrm{ren}}\mathscr{F}_{\phi,\rho}$ are concentrated in degree zero.*

iv. *For any local shtuka datum (G,μ,b) and any open compact subgroup $K\subset G(E)$, the compactly supported intersection cohomology*

$$R\Gamma_c(\mathrm{Sht}(G,\mu,b)_K, IC_\mu)$$

of the local shtuka space $\mathrm{Sht}(G,\mu,b)_K$ at level K is concentrated in degrees $[\langle 2\rho_G,\nu_b\rangle, \langle 2\rho_G,\mu\rangle]$.

v. *For all b and $K\subset G_b(E)$ as above, the sheaf $i_{b!}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}$ is perverse, and the sheaf $i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}$ is hadal.*

Here $R\Gamma_c(\mathrm{Sht}(G,\mu,b)_K, IC_\mu)\in D(G_b(E),\overline{\mathbf{Q}_\ell})^\omega$ is defined as in [HKW22, Lemma 6.4.4], taking $\Lambda=\overline{\mathbf{Q}_\ell}$ in the notation there. A priori, the complex $R\Gamma_c(\mathrm{Sht}(G,\mu,b)_K, IC_\mu)$ lives in degrees $[-\langle 2\rho_G,\mu\rangle, \langle 2\rho_G,\mu\rangle]$. When μ is minuscule, the Stein property formulated in [Han23a, Conjecture 1.10] would imply concentration in degrees $[0,\langle 2\rho_G,\mu\rangle]$. Thus for non-basic b, the vanishing result in iv. above goes strictly beyond what is predicted by geometry. We also observe that i. is best possible: it is certainly not true in general that $c_\psi\left(i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}\right)$ is concentrated in degree zero, or in any single degree.

Proof. For i., we have a duality isomorphism

$$\mathbf{D}_{\mathrm{BZ}}\left(i_{b!}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}\right)\simeq i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}.$$

eigensheaves at discrete parameters was suggested by a beautiful example explained to me by Koshikawa. I discovered Conjecture 3.2.1 while trying to explain this tilting property.

[17] I do not believe Zhu's formulation is quite correct.

Combining this with the duality compatibility gives

$$c_{\psi^{-1}}\left(i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}\right) \simeq \mathbf{D}_{\mathrm{twGS}}\left(\mathscr{C}_{b,K}\right),$$

and the right-hand side is clearly concentrated in nonnegative degrees. For ii., the claim follows by observing that ${}^pD^{\leq 0}(\mathrm{Bun}_G)$ is generated under extensions and colimits by objects of the form $i_{b!}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}[n]$ for arbitrary b, K and arbitrary $n \geq 0$.

For iii. it is enough to prove that $\mathscr{F}_{\phi,\rho}$ is perverse coconnective, and that $i_b^{!\mathrm{ren}}\mathscr{F}_{\phi,\rho}$ is concentrated in degree zero when ϕ is discrete. These reductions follow from the Verdier duality property of $\mathscr{F}_{\phi,\rho}$ implied by Conjecture 3.1.4.ii as in the discussion of section 3.1. In turn, Conjecture 3.1.4.ii is an *unconditional* consequence of Conjectures 1.7.3 and 1.7.5; the argument for this will appear in [HM23]. Now by design, Conjecture 1.7.3 implies that

$$R\mathrm{Hom}(\mathscr{C}_{b,K}, c_\psi^{\mathrm{ULA}}(-)) : D(\mathrm{Bun}_G)^{\mathrm{ULA}} \to D(\overline{\mathbf{Q}_\ell})$$

is exactly the functor $A \mapsto (i_b^{!\mathrm{ren}}A)^K$, so we compute that

$$\begin{aligned}(i_b^{!\mathrm{ren}}\mathscr{F}_{\phi,\rho})^K &\simeq R\mathrm{Hom}(\mathscr{C}_{b,K}, c_\psi^{\mathrm{ULA}}(\mathscr{F}_{\phi,\rho})) \\ &\simeq R\mathrm{Hom}(\mathscr{C}_{b,K}, i_{\phi*}^{\mathrm{IndCoh}}\rho) \\ &\simeq R\mathrm{Hom}(i_\phi^{*\mathrm{IndCoh}}\mathscr{C}_{b,K}, \rho).\end{aligned}$$

Since $\mathscr{C}_{b,K}$ is in degree zero by assumption, $i_\phi^{*\mathrm{IndCoh}}\mathscr{C}_{b,K}$ is concentrated in nonpositive degrees, so we immediately get that the above expression is concentrated in nonnegative degrees. Moreover, if ϕ is discrete, one can check that $i_\phi^{*\mathrm{IndCoh}}\mathscr{C}_{b,K}$ is necessarily concentrated in degree zero, which implies that the above expression is concentrated also in degree zero. If G is semisimple the concentration of $i_\phi^{*\mathrm{IndCoh}}\mathscr{C}_{b,K}$ in degree zero is obvious since for a discrete parameter ϕ the morphism i_ϕ is an open immersion.[18] For general groups one needs a small extra argument, using that the orbit of any discrete parameter ϕ under unramified twisting is open in Par_G together with the equivariance of $\mathscr{C}_{b,K}$ under unramified twisting.

For iv., concentration in degrees $\leq \langle 2\rho_G, \mu\rangle$ follows from the general étale cohomology formalism. For the bound in the other direction, the key point is the formula

$$R\Gamma_c(\mathrm{Sht}(G,\mu,b)_K, IC_\mu)^{K'} \simeq R\mathrm{Hom}\left(\mathscr{C}_{1,K}\otimes V_\mu, \mathscr{C}_{b,K'}\right)\left[-\langle 2\rho_G, \nu_b\rangle\right]$$

where $K' \subset G_b(E)$ is any open compact subgroup.[19] This formula is an unconditional consequence of Conjecture 1.7.3. Indeed, [HKW22, Lemma 6.4.4] gives an isomorphism

$$R\Gamma_c(\mathrm{Sht}(G,\mu,b)_K, IC_\mu) \simeq i_b^* T_{V_\mu} i_{1!}\mathrm{ind}_K^{G(E)}\overline{\mathbf{Q}_\ell},$$

which easily implies an isomorphism

$$R\Gamma_c(\mathrm{Sht}(G,\mu,b)_K, IC_\mu)^{K'} \simeq R\mathrm{Hom}(i_{b\sharp}\mathrm{ind}_{K'}^{G_b(E)}\overline{\mathbf{Q}_\ell}, T_{V_\mu} i_{1!}\mathrm{ind}_K^{G(E)}\overline{\mathbf{Q}_\ell}).$$

Applying BZ duality on the right-hand side, we can rewrite it as

$$R\mathrm{Hom}(T_{V_\mu} i_{1!}\mathrm{ind}_K^{G(E)}\overline{\mathbf{Q}_\ell}, i_{b!}\mathrm{ind}_{K'}^{G_b(E)}\overline{\mathbf{Q}_\ell}[-2\langle 2\rho_G, \nu_b\rangle]).$$

[18] For semisimple groups, the morphism i_ϕ is an open immersion exactly when ϕ is a discrete parameter.

[19] This is a variant of [Zhu21, Conjecture 4.7.18], but again I do not think Zhu's formulation is correct.

Now we have isomorphisms

$$c_\psi \left(T_{V_\mu} i_{1!} \mathrm{ind}_K^{G(E)} \overline{\mathbf{Q}_\ell} \right) \simeq V_\mu \otimes \mathscr{C}_{1,K}$$

and

$$c_\psi \left(i_{b!} \mathrm{ind}_{K'}^{G_b(E)} \overline{\mathbf{Q}_\ell}[-2 \langle 2\rho_G, \nu_b \rangle] \right) \simeq \mathscr{C}_{b,K'}[- \langle 2\rho_G, \nu_b \rangle]$$

by unwinding the definitions and using the linearity of c_ψ over the spectral action, so passing to the other side via Conjecture 1.7.3 we arrive at the desired formula. Then by Conjecture 3.2.1, $R\mathrm{Hom}\,(\mathscr{C}_{1,K} \otimes V_\mu, \mathscr{C}_{b,K'})$ is concentrated in nonnegative degrees, so accounting for the shift and shrinking K' arbitrarily gives the claim.

For v., we prove the second claim, the first being similar (and easier). It is clear that $i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell}$ is connective for the hadal t-structure. To show it is coconnective is equivalent to showing that all stalks

$$i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell}$$

are concentrated in nonnegative degrees. But now for any open compact $K' \subset G_{b'}(E)$ we have isomorphisms

$$\begin{aligned}
\left(i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell} \right)^{K'} &\simeq R\mathrm{Hom}\left(i_{b'\sharp}^{\mathrm{ren}} \mathrm{ind}_{K'}^{G_{b'}(E)} \overline{\mathbf{Q}_\ell}, i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell} \right) \\
&\simeq R\mathrm{Hom}\left(i_{b!}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell}, i_{b'!}^{\mathrm{ren}} \mathrm{ind}_{K'}^{G_{b'}(E)} \overline{\mathbf{Q}_\ell} \right) \\
&\simeq R\mathrm{Hom}(\mathscr{C}_{b,K}, \mathscr{C}_{b,K'})
\end{aligned}$$

where the first line follows from the obvious adjunctions, the second line follows from BZ duality, and the third line follows from Conjecture 1.7.3 and the definition of $\mathscr{C}_{b,K}$. Then $R\mathrm{Hom}(\mathscr{C}_{b,K}, \mathscr{C}_{b,K'})$ is concentrated in nonnegative degrees by Conjecture 3.2.1, so shrinking K' arbitrarily gives the desired coconnectivity. □

Remark 3.2.3. In fact, the sheaves $i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell}$ are hadal for all b, K if and only if the sheaves $i_{b!}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell}$ are perverse for all b, K. Since both are clearly connective in the relevant t-structure, this follows from the chain of identities

$$\begin{aligned}
\left(i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell} \right)^{K'} &\simeq R\mathrm{Hom}\left(i_{b'\sharp}^{\mathrm{ren}} \mathrm{ind}_{K'}^{G_{b'}(E)} \overline{\mathbf{Q}_\ell}, i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell} \right) \\
&\simeq R\mathrm{Hom}\left(i_{b!}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell}, i_{b'!}^{\mathrm{ren}} \mathrm{ind}_{K'}^{G_{b'}(E)} \overline{\mathbf{Q}_\ell} \right) \\
&\simeq \left(i_b^{!\mathrm{ren}} i_{b'!}^{\mathrm{ren}} \mathrm{ind}_{K'}^{G_{b'}(E)} \overline{\mathbf{Q}_\ell} \right)^K
\end{aligned}$$

arguing as in the proof of 3.2.2.v.

The conjectural hadal property of the sheaves $i_{b\sharp}^{\mathrm{ren}} \mathrm{ind}_K^{G_b(E)} \overline{\mathbf{Q}_\ell}$ admits the following very concrete reinterpretation. Let $\widetilde{\mathcal{M}}_b \to \mathcal{M}_b$ be the canonical $G_b(E)$-torsor over the local chart $\mathcal{M}_b$ as constructed in Fargues-Scholze. For any $b' \preceq b$, pick a complete algebraically closed field $C/\mathbf{F}_p$ and a map $\mathrm{Spd} C \to \mathrm{Bun}_G$ covering the stratum $\mathrm{Bun}_G^{b'}$, and define $X_{b,b'}$ by the pullback diagram

$$\begin{tikzcd}
X_{b,b'} \arrow[r] \arrow[d] & \widetilde{\mathcal{M}}_b \arrow[d] \\
\mathrm{Spd} C \arrow[r] & \mathrm{Bun}_G
\end{tikzcd}$$

of small v-stacks. Then $X_{b,b'}$ is a partially proper locally spatial diamond over $\mathrm{Spd}C$ of ℓ-cohomological dimension $\langle 2\rho_G, \nu_b\rangle$, and a priori $R\Gamma_c(X_{b,b'}, \mathbf{F}_\ell)$ is concentrated in degrees $[0, 2\langle 2\rho_G, \nu_b\rangle]$.

Conjecture 3.2.4. *For all $b' \preceq b$, $R\Gamma_c(X_{b,b'}, \mathbf{F}_\ell)$ is concentrated in degrees $[\langle 2\rho_G, \nu_b + \nu_{b'}\rangle, 2\langle 2\rho_G, \nu_b\rangle]$.*

If this conjecture holds for b fixed and all $b' \preceq b$, then $i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}$ is hadal for all $K \subset G_b(E)$.[20] The essential point here is the formula

$$\mathrm{colim}_{K\to\{1\}} i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell} \cong \left(R\Gamma_c(X_{b,b'}, \mathbf{Z}_\ell)[\langle 2\rho_G, \nu_b + \nu_{b'}\rangle]^{G_{b'}(E)\times G_b(E)-\mathrm{sm}}\right) \otimes_{\mathbf{Z}_\ell} \overline{\mathbf{Q}_\ell}$$

where the superscript $(-)^{G_{b'}(E)\times G_b(E)-\mathrm{sm}}$ indicates the (exact) functor of $G_{b'}(E) \times G_b(E)$-smooth vectors. This conjecture already has nontrivial content in the degenerate case $b = b'$, where it reduces to the fact that the $\mathbf{F}_\ell$-étale cohomology of $\tilde{G}_{b,C}^{>0}$ (in the notation of [FS21, Proposition III.5.1]) is entirely concentrated in degree $2\langle 2\rho_G, \nu_b\rangle$. When $G = \mathrm{GL}_2$, $\mathcal{E}_b = \mathcal{O}(1) \oplus \mathcal{O}(-1)$, and $\mathcal{E}_{b'} = \mathcal{O}^2$, Conjecture 3.2.4 has been checked by Miles.

We can squeeze quite a bit more out of Conjecture 3.2.1.

Conjecture 3.2.5. *Assume that b is basic and $\pi \in \Pi(G_b)$ is supercuspidal, with Fargues-Scholze parameter ϕ. Let $\nu_\phi : V = q^{-1}(x_\phi) \to \mathrm{Par}_G$ be the natural closed immersion of the fiber over x_ϕ. Then $c_\psi(i_{b!}\pi)$ is a genuine coherent sheaf, which is moreover of the form $\nu_{\phi*}\mathcal{F}$ for some $\mathcal{F} \in \mathrm{Coh}(V)^\heartsuit$.*

Assuming moreover that the fiber V is smooth, then V is $-\mathrm{dim}Z(\hat{G})^\Gamma$-dimensional and $\mathcal{F}$ is a locally free sheaf.

Note that $-\mathrm{dim}Z(\hat{G})^\Gamma$ is the maximal dimension of any fiber of q, and a fiber achieves this dimension if and only if it contains a discrete L-parameter, which we expect is automatic under the assumptions of this conjecture. Indeed, the fiber V should contain the true L-parameter of π, which we expect is always a discrete parameter. However, we do not always expect the fiber V to be smooth under the assumptions of this conjecture, even when G_b is split.[21]

Here is a heuristic argument in support of this conjecture. For simplicity, we assume G is semisimple, so $-\mathrm{dim}Z(\hat{G})^\Gamma = 0$ and $\mathbf{D}_{\mathrm{coh}}(\pi) \simeq \pi^\vee$. By some general nonsense (using e.g. [Bus90, Theorem 2]), any supercuspidal π will occur as a summand of $\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}$ for any sufficiently small K, so $c_\psi(i_{b!}\pi)$ is a summand of the corresponding $\mathscr{C}_{b,K}$, which is concentrated in degree zero by Conjecture 3.2.1. Thus $c_\psi(i_{b!}\pi)$ is a genuine coherent sheaf. By the compatibility of c_ψ with the action of the spectral Bernstein center, $c_\psi(i_{b!}\pi)$ is also killed by the maximal ideal $\mathfrak{m}_\phi \subset \mathcal{O}(X_G^{\mathrm{spec}})$ whose preimage under q cuts out V, so it is supported scheme-theoretically on V. This gives the first part. For the second part, Conjecture 1.7.5 predicts that $\mathbf{D}_{\mathrm{twGS}}c_\psi(i_{b!}\pi) \simeq c_{\psi^{-1}}(i_{b!}\pi^\vee)$ is also concentrated in degree zero, so the (untwisted) Grothendieck-Serre dual of $\nu_{\phi*}\mathcal{F}$ is also concentrated in degree zero. Since Grothendieck-Serre duality commutes with proper pushforward, this implies that $R\mathscr{H}\mathrm{om}(\mathcal{F}, \omega_V)$ is also concentrated in degree zero. Note that V is closed in the zero-dimensional stack Par_G, so it is necessarily of dimension $-d \leq 0$. Since it is moreover smooth

[20] It is plausible that for some small primes ℓ, Conjecture 3.2.4 is not true as stated. However, for the intended application, it would be enough to prove the weaker conjecture that for any fixed $i < \langle 2\rho_G, \nu_b + \nu_{b'}\rangle$, the group $H_c^i(X_{b,b'}, \mathbf{Z}/\ell^n\mathbf{Z})$ is killed by an integer independent of n.

[21] I thank Wee Teck Gan for showing me a counterexample on G_2. In brief, the (Weil-Deligne incarnation of the) L-parameter $\phi : W_E \times \mathrm{SL}_2 \to G_2$ which is trivial on W_E and embeds $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2$ as a subregular unipotent element is discrete, and the packet $\Pi_\phi(G_2)$ contains a supercuspidal representation π_3^ε, but the fiber of q over $x_{\phi^{\mathrm{ss}}}$ is not smooth. It would be very interesting to give a conjectural description of the coherent sheaf $c_\psi(i_{1!}\pi_3^\varepsilon)$.

by assumption, the dualizing complex is of the form $\omega_V \simeq \mathcal{L}[-d]$ for some line bundle $\mathcal{L}$ on V. But then $R\mathscr{H}\mathrm{om}(\mathcal{F}, \omega_V) \simeq R\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{L})[-d]$ is automatically concentrated in degrees $\geq d$, so we must have $d = 0$. We then see that $\mathcal{F}$ is maximal Cohen-Macaulay on the smooth zero-dimensional algebraic stack V, and this implies that $\mathcal{F}$ is locally free.

A similar heuristic also leads to the following expectation.

Conjecture 3.2.6. *Assume that b is basic and $\pi \in \Pi(G_b)$ is supercuspidal. Then $i_{b!}\pi$ and $i_{b*}\pi$ are perverse.*

Indeed, assume G is semisimple for simplicity. Then $i_{b!}\pi$ is a summand of $i_{b!}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}$ for any sufficiently small K, and Proposition 3.2.2.v suggests that $i_{b!}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}$ is always perverse, so also $i_{b!}\pi$ should be perverse. Then $i_{b*}\pi \simeq \mathbf{D}_{\mathrm{Verd}} i_{b!}\pi^\vee$ should also be perverse, using the Verdier self-duality of the perverse t-structure.

We emphasize that !-extensions of irreducible representations from basic strata need not be perverse in general. For instance, if $G = \mathrm{SL}_2$, the discussion after Conjecture 2.5.1 implies that $i_{1!}\overline{\mathbf{Q}_\ell}$ has a nonvanishing perverse cohomology sheaf in some degree ≤ -1.

Next, we sketch some consequences of Conjecture 3.2.1 for the categorical conjecture over the semisimple generic locus; the full details of these arguments will appear in [HM23]. To set things up, note that the equivalence predicted in Proposition 3.1.2 localizes to an equivalence

$$c_\psi^{\mathrm{ULA,gen}} : D(\mathrm{Bun}_G)^{\mathrm{ULA,gen}} \xrightarrow{\sim} \mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}})$$

where both categories appearing here are the evident localizations of the analogous category appearing in Proposition 3.1.2. Note that $\mathrm{Par}_G^{\mathrm{gen}}$ is smooth, so $\mathrm{IndCoh}(\mathrm{Par}_G^{\mathrm{gen}}) \cong \mathrm{QCoh}(\mathrm{Par}_G^{\mathrm{gen}})$.

Lemma 3.2.7. *The standard t-structure on $\mathrm{QCoh}(\mathrm{Par}_G^{\mathrm{gen}})$ restricts to a ("standard") t-structure on $\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}})$, whose left and right halves are exchanged by $\mathbf{D}_{\mathrm{tw.adm}}$.*

This is a general feature of admissible ind-coherent sheaves on algebraic stacks of the form $[X/H]$ where X is a smooth affine variety and H is a reductive group. It turns out that every connected component of $\mathrm{Par}_G^{\mathrm{gen}}$ has this structure.

Proposition 3.2.8. *Assume Conjectures 1.7.3 and 1.7.5, and also assume that Conjecture 3.2.1 is true. Then the equivalence $c_\psi^{\mathrm{ULA,gen}}$ is t-exact with respect to the perverse t-structure on $D(\mathrm{Bun}_G)^{\mathrm{ULA,gen}}$ and the standard t-structure on $\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}})$.*

We remark that the equivalence appearing here restricts further to the equivalence $D(\mathrm{Bun}_G)^{\mathrm{gen}}_{\mathrm{fin}} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G^{\mathrm{gen}})_{\mathrm{fin}}$ discussed in section 2.5, and the standard t-structure on $\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}})$ clearly restricts to the standard t-structure on $\mathrm{Coh}(\mathrm{Par}_G^{\mathrm{gen}})_{\mathrm{fin}}$. From here, it is easy to see that the assumptions in the previous proposition actually imply Conjecture 2.5.1. Since tensoring with a vector bundle is t-exact for the standard t-structure on $\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}})$, we also see that Conjecture 2.4.1 is implied by the same set of assumptions.

Sketch. By our assumptions, Proposition 3.2.2.ii implies that $c_\psi^{\mathrm{ULA,gen}}$ carries ${}^pD^{\leq 0}(\mathrm{Bun}_G)^{\mathrm{ULA,gen}}$ fully faithfully into $\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}})$ with image contained inside $\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}}) \cap \mathrm{QCoh}^{\leq 0}$. Passing to right orthogonals in $D(\mathrm{Bun}_G)^{\mathrm{ULA,gen}}$ resp. in $\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}})$ and swapping the Whittaker data, we get a containment

$$\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}}) \cap \mathrm{QCoh}^{\geq 0} \subseteq c_{\psi^{-1}}^{\mathrm{ULA,gen}}\left({}^pD^{\geq 0}(\mathrm{Bun}_G)^{\mathrm{ULA,gen}}\right).$$

Applying $\mathbf{D}_{\mathrm{tw.adm}}$ on both sides and using the previous lemma together with the duality compatibility of $c_{\psi^{-1}}^{\mathrm{ULA,gen}}$ (which follows from our assumptions), we get a containment

$$\mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}}) \cap \mathrm{QCoh}^{\leq 0} \subseteq c_{\psi}^{\mathrm{ULA,gen}}\left({}^{p}D^{\leq 0}(\mathrm{Bun}_G)^{\mathrm{ULA,gen}}\right).$$

But we already know that the left side contains the right, so putting things together we see that $c_{\psi}^{\mathrm{ULA,gen}}$ induces an equivalence

$${}^{p}D^{\leq 0}(\mathrm{Bun}_G)^{\mathrm{ULA,gen}} \overset{\sim}{\to} \mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}}) \cap \mathrm{QCoh}^{\leq 0}.$$

Passing to right orthogonals again we see that it also induces an equivalence

$${}^{p}D^{\geq 0}(\mathrm{Bun}_G)^{\mathrm{ULA,gen}} \overset{\sim}{\to} \mathrm{Adm}(\mathrm{Par}_G^{\mathrm{gen}}) \cap \mathrm{QCoh}^{\geq 0}.$$

Therefore the t-structures are compatible, as desired. □

Pushing these ideas in a slightly different direction, we can also say something about the long-neglected functor a_ψ, which also yields a substantial upgrade to Proposition 3.2.2.ii.[22]

Proposition 3.2.9. *Assume Conjectures 1.7.3 and 1.7.5, and also assume that Conjecture 3.2.1 is true. Then* $a_\psi : \mathrm{QCoh}(\mathrm{Par}_G) \to D(\mathrm{Bun}_G)$ *is perverse right t-exact, i.e. it carries* $\mathrm{QCoh}^{\leq 0}$ *into* ${}^{p}D^{\leq 0}$.

Moreover, $c_\psi : D(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{Par}_G)$ *is t-exact for the perverse t-structure on the source and the standard t-structure on the target.*

From here it is easy to give yet another conditional justification for Conjectures 2.4.1 and 2.5.1.

Proof. It suffices to prove the first part. Indeed, we already know from Proposition 3.2.2.ii that c_ψ is perverse right t-exact under the same set of assumptions. But if a_ψ is also perverse right t-exact, then its right adjoint c_ψ is automatically perverse left t-exact as well.

For the first part, we easily reduce to the claim that a_ψ carries any $\mathcal{F} \in \mathrm{Perf}^{\mathrm{qc}} \cap \mathrm{QCoh}^{\leq 0}$ into ${}^{p}D^{\leq 0}$. Fix such an $\mathcal{F}$. Then Conjecture 1.7.3 implies that a_ψ is fully faithful and that $a_\psi(\mathcal{F})$ is compact, so the natural adjunction gives an isomorphism $\mathcal{F} \simeq c_\psi(a_\psi(\mathcal{F}))$. Now fix b and $K \subset G_b(E)$, and set $\mathscr{C}'_{b,K} = c_{\psi^{-1}}(i_{b!}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell})$; note the change in Whittaker datum. Then Conjectures 1.7.3 and 1.7.5 together imply that

$$\begin{aligned}(i_b^{*\mathrm{ren}}A)^K &\simeq R\mathrm{Hom}\left(i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}, A\right)\\ &\simeq R\mathrm{Hom}\left(c_\psi\left(i_{b\sharp}^{\mathrm{ren}}\mathrm{ind}_K^{G_b(E)}\overline{\mathbf{Q}_\ell}\right), c_\psi(A)\right)\\ &\simeq R\mathrm{Hom}(\mathbf{D}_{\mathrm{twGS}}(\mathscr{C}'_{b,K}), c_\psi(A))\end{aligned}$$

for all compact sheaves $A \in D(\mathrm{Bun}_G)$. Applying this with $A = a_\psi(\mathcal{F})$ and using the full faithfulness of a_ψ and the compactness of A, we get

$$\begin{aligned}(i_b^{*\mathrm{ren}}a_\psi(\mathcal{F}))^K &\simeq R\mathrm{Hom}(\mathbf{D}_{\mathrm{twGS}}(\mathscr{C}'_{b,K}), \mathcal{F})\\ &\simeq R\Gamma\left(\mathrm{Par}_G, R\mathscr{H}\mathrm{om}(\mathbf{D}_{\mathrm{twGS}}(\mathscr{C}'_{b,K}), \mathcal{F})\right).\end{aligned}$$

[22] I again thank Koshikawa for very interesting discussions on these matters, and for suggesting the key trick in the next proof.

Now the key trick, which was suggested to me by Koshikawa, is that because $\mathcal{F}$ is perfect by assumption, we can rewrite the internal hom appearing here as

$$R\mathscr{H}\mathrm{om}(\mathbf{D}_{\mathrm{twGS}}(\mathscr{C}'_{b,K}), \mathcal{F}) \simeq R\mathscr{H}\mathrm{om}(\mathbf{D}_{\mathrm{twGS}}(\mathscr{C}'_{b,K}), \mathcal{O}_{\mathrm{Par}_G}) \otimes \mathcal{F}.$$

Moreover, by the involutivity of Grothendieck-Serre duality and the definition of $\mathbf{D}_{\mathrm{twGS}}$, we easily get an isomorphism $R\mathscr{H}\mathrm{om}(\mathbf{D}_{\mathrm{twGS}}(\mathscr{C}'_{b,K}), \mathcal{O}_{\mathrm{Par}_G}) \simeq c^*\mathscr{C}'_{b,K}$. Combining all of our observations so far, we get an isomorphism

$$(i_b^{*\mathrm{ren}} a_\psi(\mathcal{F}))^K \simeq R\Gamma(\mathrm{Par}_G, c^*\mathscr{C}'_{b,K} \otimes \mathcal{F}).$$

Now we win: $\mathcal{F}$ is a perfect complex concentrated in nonpositive degrees by assumption, and $c^*\mathscr{C}'_{b,K}$ is concentrated in degree zero by Conjecture 3.2.1, so $c^*\mathscr{C}'_{b,K} \otimes \mathcal{F}$ is a bounded coherent complex with quasicompact support concentrated in nonpositive degrees. Since each connected component of Par_G is a quotient of an affine variety by a reductive group, the functor $R\Gamma(\mathrm{Par}_G, -) : \mathrm{Coh}(\mathrm{Par}_G) \to D(\overline{\mathbf{Q}_\ell})$ is t-exact, so we conclude that

$$(i_b^{*\mathrm{ren}} a_\psi(\mathcal{F}))^K \simeq R\Gamma(\mathrm{Par}_G, c^*\mathscr{C}'_{b,K} \otimes \mathcal{F})$$

is concentrated in nonpositive degrees. Varying b and K arbitrarily, we conclude that $a_\psi(\mathcal{F})$ is perverse connective as desired. $\square$

Remark 3.2.10. Proposition 3.2.9 is in perfect accord with the t-exactness results proved in [FR22]. More precisely, we expect that there is an intrinsically defined subcategory $D(\mathrm{Bun}_G)^{\mathrm{temp}} \subset D(\mathrm{Bun}_G)$ stable under colimits and Hecke operators, and coinciding with the image of $\mathrm{QCoh}(\mathrm{Par}_G)$ under a_ψ. Since a_ψ supposedly matches with the fully faithful embedding $\Xi : \mathrm{QCoh} \to \mathrm{IndCoh}$ under the categorical equivalence, passing to right adjoints implies there should be a commutative diagram

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}"] \arrow[dr, "c_\psi"] & \mathrm{IndCoh}(\mathrm{Par}_G) \arrow[d, "\Psi"] \\ D(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[r, "\sim"] & \mathrm{QCoh}(\mathrm{Par}_G) \end{tikzcd}$$

where $\mathrm{pr} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)^{\mathrm{temp}}$ is the right adjoint of the evident inclusion and the lower horizontal arrow is the essential inverse of a_ψ. Transporting the t-exactness result of the previous proposition across the lower horizontal equivalence, we see that there should be an intrinsic t-structure on $D(\mathrm{Bun}_G)^{\mathrm{temp}}$ compatible with the perverse t-structure on $D(\mathrm{Bun}_G)$ via pr. In the setting of classical geometric Langlands, such a t-structure is constructed in [FR22].

We end this section by noting that while the coherence of $\mathscr{C}_{b,K}$ seems to lie very deep, there is a reasonably explicit criterion for these gadgets to be genuine quasicoherent sheaves.

Proposition 3.2.11. *Fix G and $b \in B(G)$. Then the following are equivalent.*
i. *For all $K \subset G_b(E)$ open compact, $\mathscr{C}_{b,K}$ lies in the heart $\mathrm{QCoh}^\heartsuit$ of the standard t-structure on $\mathrm{QCoh}(\mathrm{Par}_G)$.*
ii. *For all $V \in \mathrm{Rep}\,\hat{G}$, $i_b^* T_V i_{1!} W_{\psi^{-1}}$ is zero outside degree $\langle 2\rho_G, \nu_b \rangle$.*

We leave the proof as an exercise for the interested reader. The essential point is that the defining property of c_ψ leads to the (unconditional!) formula

$$R\Gamma(\mathrm{Par}_G, V \otimes \mathscr{C}_{b,K}) \cong \left(i_b^{*\mathrm{ren}} T_{V^\vee} i_{1!} W_{\psi^{-1}}\right)^K$$

for any $V \in \operatorname{Rep} \hat{G}$. One then uses the fact that the functors

$$R\Gamma(\mathrm{Par}_G, V \otimes -) : \mathrm{QCoh}^{\mathrm{qc}}(\mathrm{Par}_G) \to D(\overline{\mathbf{Q}_\ell})$$

form a *t-exact* conservative family as one varies over all $V \in \operatorname{Rep} \hat{G}$.

Exercise 3.2.12. Assume Conjectures 1.7.3 and 3.2.1. Deduce that $T_V i_{1!} W_\psi$ is perverse for all $V \in \operatorname{Rep} \hat{G}$.

3.3 More examples

In this section we collect some miscellaneous remarks and examples. We also try to illustrate how complicated the situation becomes once we allow L-parameters with nontrivial monodromy into the discussion.

Example 3.3.1. It is natural to expect that $c_\psi \circ i_{1!}$ should coincide with Hellmann's functor R_ψ. Taking this for granted, we can use the calculations in [Hel23] as a source of examples. In particular, take $G = \mathrm{GL}_2$, and let St be the Steinberg representation, so we have a short exact sequence

$$0 \to \mathrm{St} \to \mathrm{Ind}_B^G \delta_B \to \mathbf{1} \to 0.$$

Using [Hel23, Theorem 4.34 and Remark 4.43] it is not difficult to see that R_ψ should send this sequence to the distinguished triangle

$$i_{Z*}\mathcal{O}_Z \to i_{y*}\mathcal{O}_y \to i_{Z*}\mathcal{L}[1] \to .$$

Here $Z \simeq \mathbf{A}^1/\mathbf{G}_m^2 \subset \mathrm{Par}_G$ is the closure of the orbit of the Steinberg parameter, $y \simeq B(\mathbf{G}_m^2) \in Z$ is the unique closed orbit, i_Z and i_y are the evident closed immersions, and $\mathcal{L}$ is the line bundle on Z of functions which vanish at y. Note that this triangle is a rotation of the more natural triangle

$$i_{Z*}\mathcal{L} \to i_{Z*}\mathcal{O}_Z \to i_{y*}\mathcal{O}_y \to$$

which is an honest short exact sequence in $\mathrm{Coh}^\heartsuit$.

It is very instructive to see how this sequence interacts with duality. On the Bun_G side, it is easy to see that $\mathbf{D}_{\mathrm{coh}}$ sends $\mathbf{1}$ to $\mathrm{St}[-2]$ and St to $\mathbf{1}[-2]$. It is also well-known that $\mathbf{D}_{\mathrm{coh}}$ intertwines Ind_B^G and $\mathrm{Ind}_{\overline{B}}^G$. Putting these observations together appropriately, we see that $\mathbf{D}_{\mathrm{coh}}$ sends our initial short exact sequence to the distinguished triangle

$$\mathrm{St}[-2] \to \mathrm{Ind}_{\overline{B}}^G \delta_{\overline{B}}[-2] \to \mathbf{1}[-2] \to,$$

which is isomorphic to the sequence we began with shifted by -2. In order for this to be compatible with duality on the L-parameter side, we must have $\mathbf{D}_{\mathrm{twGS}} i_{Z*}\mathcal{O}_Z \simeq i_{Z*}\mathcal{L}[-1]$, or equivalently we must have

$$\omega_Z = i_Z^! \mathcal{O}_{\mathrm{Par}_G} \simeq \mathcal{L}[-1].$$

This is indeed true, and follows from an explicit calculation via the isomorphism $Z \simeq \mathbf{A}^1/\mathbf{G}_m^2$ noted above.

Examination of this example and various other examples in [Hel23], together with some optimism, leads to the following speculation. Let G be quasisplit with a fixed Whittaker datum as usual. Let $\pi \in \mathrm{Rep}(G(E))$ be irreducible and ψ-generic, with Fargues-Scholze parameter ϕ. Assume that $\phi(\mathrm{Fr})$

is *regular semisimple*. Write $Z = q^{-1}(x_\phi)$, so this is an Artin stack which comes with a tautological closed immersion $i : Z \hookrightarrow \mathrm{Par}_G$. Our assumption on ϕ guarantees that Z is extremely nice, and is nothing more than the Vogan variety for the infinitesimal character ϕ. Explicitly, there is a presentation

$$Z \cong \left\{ N \in \mathfrak{g}^{\mathrm{ad}\phi(I_E)} \mid \mathrm{ad}\phi(\mathrm{Fr}) \cdot N = q^{-1}N \right\} / S_\phi.$$

Question. Is it true that $c_\psi(i_{1!}\pi) \simeq i_*\mathcal{O}_Z$?

Example 3.3.2. In this example we illustrate how complicated the functor $i^{\mathrm{ren}}_{b\sharp}$ can be in general. Take $E = \mathbf{Q}_p$ with $p > 2$, and $G = \mathrm{GSp}_4$, so we have an exceptional isomorphism $\hat{G} \simeq \mathrm{GSp}_4 \subset \mathrm{GL}_4$. Let b' be the element such that $\mathcal{E}_{b'} \simeq \mathcal{O}(\frac{1}{2}) \oplus \mathcal{O}(-\frac{1}{2})$. Note that b' is an immediate specialization of the point $b = 1$. There is a natural identification $G_{b'}(\mathbf{Q}_p) \simeq D^\times \times \mathbf{Q}_p^\times$, where $D/\mathbf{Q}_p$ is the usual quaternion algebra. Let ρ be an irreducible $D^\times$-representation of dimension > 1 with trivial central character, with L-parameter $\phi_\rho : W_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{Q}_\ell})$. There is then a unique supercuspidal representation π of $\mathrm{GSp}_4(\mathbf{Q}_p)$ with semisimple L-parameter $\phi_\rho \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}}$. Note that this parameter is *not* semisimple generic in the sense of section 2.3: it is the semisimplification of a discrete parameter with nontrivial monodromy, which is the true L-parameter of π.

Claim. For $n \in \{0,1\}$, $H^n(i_1^* i^{\mathrm{ren}}_{b'\sharp}(\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}}))$ contains π with multiplicity one. Moreover, this is the entire supercuspidal part of $H^*(i_1^* i^{\mathrm{ren}}_{b'\sharp}(\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}}))$.

The proof of this is very indirect. The idea is to compute

$$A \overset{\mathrm{def}}{=} i_1^* T_V i_{b!} i_{P_b}^{G_b}(\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}})$$

in two different ways. Here $P = MU \subset G$ is the Siegel parabolic, $b \in B(G)$ is (a representative in $M(\mathbf{Q}_p)$ of) the basic element such that $\mathcal{E}_b \simeq \mathcal{O}(\frac{1}{2})^2$, $P_b = M_b U_b \subset G_b$ is the evident twist, and V is the dual standard representation of $\hat{G}$. Note that $M_b \cong G_{b'}$ as inner forms of $M \simeq \mathrm{GL}_2 \times \mathbf{G}_m$. On one hand, there is a short exact sequence

$$0 \to \mathrm{St}(\rho, 1) \to i_{P_b}^{G_b}(\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}}) \to \mathrm{Sp}(\rho, 1) \to 0$$

(cf. Proposition 5.3.(i) of [GT14]), and one can compute the supercuspidal part of $i_1^* T_V i_{b!}(-)$ applied to the outer terms of this sequence using a result of Ito-Mieda (Theorem 3.1 of [Mie21]; note that $\mathrm{St}(\rho,1) = \rho_{\mathrm{disc}}$ and $\mathrm{Sp}(\rho,1) = \rho_{\mathrm{nt}}$ in Mieda's notation). This yields a two-term filtration on the supercuspidal part of A with comprehensible graded pieces.

On the other hand, we can rewrite A as $i_1^* T_V \mathrm{Eis}_P^G i_{b!}^M(\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}})$. Using the filtered commutation of Eis with Hecke operators (Conjecture 1.5.2), the latter expression acquires a three-term filtration with graded pieces $i_1^* \mathrm{Eis}_P^G T_{W_k} i_{b!}^M(\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}})$. Here the W_k are the irreducible algebraic $\hat{M}$-representations whose inflations to the Klingen parabolic[23] $\hat{P}$ yield the three irreducible subquotients of $V|\hat{P}$. These graded pieces can be computed explicitly, and after passing to the supercuspidal part only one of them survives and yields exactly the supercuspidal part of $i_1^* i^{\mathrm{ren}}_{b'\sharp}(\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}})$. A careful comparison of these two calculations now yields the claim.

Note that by adjunction, the claim easily implies that $i_{b'}^{*\mathrm{ren}} i_{1*}\pi$ is nonzero, with $\rho|\mathrm{Nm}|^{-\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}}$ occurring as a subquotient. In the early days of the Fargues-Scholze program, many people (including the present author) hoped the functors $i_b^* i_{1*}$ would have a reasonably simple explicit description in terms of Jacquet modules and other concrete representation-theoretic operations. Examples of this sort suggest that such hopes are woefully misguided.

[23] We remind the reader that under the exceptional identification of GSp_4 with its own dual group, the Klingen and Siegel parabolics correspond.

Exercise 3.3.3. Give a similar analysis of the supercuspidal part of $H^n(i_1^* i_{b'\sharp}^{\mathrm{ren}}(\rho|\mathrm{Nm}|^{\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}}))$ - note the change in the twisting exponent on ρ. (Hint: There is a relevant variant of the short exact sequence used above where Sp and St are swapped.)

3.4 Grothendieck groups and vanishing conjectures

Since $D(\mathrm{Bun}_G)$ is cocomplete, its K_0 is zero by the usual trick. On the other hand, $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ is much smaller, and its K_0 is very interesting.

For any b, $i_b^{*\mathrm{ren}}$ induces a map $[i_b^{*\mathrm{ren}}] : K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}} \to K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}}$, as well as maps $[i_{b!}^{\mathrm{ren}}], [i_{b\sharp}^{\mathrm{ren}}] : K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}} \to K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}}$. Note that $[i_{b\sharp}^{\mathrm{ren}}]$ is well-defined, since we know from the proof of Theorem 1.6.3 that $i_{b\sharp}^{\mathrm{ren}}\pi$ is a finite sheaf for π of finite length. These clearly assemble into maps

$$\gamma_! : \bigoplus_{b \in B(G)} K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}} \overset{\sum_b [i_{b!}^{\mathrm{ren}}]}{\longrightarrow} K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}},$$

$$\gamma_\sharp : \bigoplus_{b \in B(G)} K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}} \overset{\sum_b [i_{b\sharp}^{\mathrm{ren}}]}{\longrightarrow} K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}}$$

and

$$\gamma^* : K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}} \overset{\{[i_b^{*\mathrm{ren}}]\}_b}{\longrightarrow} \bigoplus_{b \in B(G)} K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}}.$$

Proposition 3.4.1. *The maps $\gamma_!$, $\gamma_\sharp$ and γ^* are isomorphisms of abelian groups. The map γ^* is a left inverse of $\gamma_!$.*

Proof. The last part is clear, since $i_b^{*\mathrm{ren}} i_{b!}^{\mathrm{ren}} = \mathrm{id}$ and $i_{b'}^{*\mathrm{ren}} i_{b!}^{\mathrm{ren}} = 0$ for all $b' \neq b$. The first part is an easy consequence of the semiorthogonal decomposition together with the fact that $i_{b\sharp}^{\mathrm{ren}}$ preserves finite sheaves as in the proof of Theorem 1.6.3. □

Of course, the unrenormalized variants of these statements are also true.

By Theorem 1.6.3, Bernstein-Zelevinsky duality induces a well-defined involution on $K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}}$, which we denote $[\mathbf{D}_{\mathrm{BZ}}]$. Similarly, cohomological duality induces an involution on $K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}}$ for each b; taking the direct sum over b, we get an involution $[\mathbf{D}_{\mathrm{coh}}]$ on $\bigoplus_{b \in B(G)} K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}}$.

Proposition 3.4.2. *The following statements are equivalent.*

1) *The maps $\gamma_!$ and $\gamma_\sharp$ are equal.*
2) *For all $b' \neq b$, the map $[i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}}] : K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}} \to K_0\mathrm{Rep}(G_{b'}(E))_{\mathrm{fin}}$ is zero.*
2') *For all $b' \neq b$, the map $[i_{b'}^* i_{b\sharp}] : K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin}} \to K_0\mathrm{Rep}(G_{b'}(E))_{\mathrm{fin}}$ is zero.*
3) *There is an equality of maps $\gamma_! \circ [\mathbf{D}_{\mathrm{coh}}] = [\mathbf{D}_{\mathrm{BZ}}] \circ \gamma_!$.*
4) *There is an equality of maps $\gamma^* \circ [\mathbf{D}_{\mathrm{BZ}}] = [\mathbf{D}_{\mathrm{coh}}] \circ \gamma^*$.*
5) *The maps γ^* and $\gamma_!$ are mutually inverse.*

Proof. The equivalence of 1) and 2) is immediate from the definitions together with the fact that γ^* is a left inverse of $\gamma_!$. The equivalence of 2) and 2') is trivial. The equivalence of 1) and 3) follows from the equality $\gamma_\sharp \circ [\mathbf{D}_{\mathrm{coh}}] = [\mathbf{D}_{\mathrm{BZ}}] \circ \gamma_!$ together with the fact that both dualities are isomorphisms on the relevant K_0's. The equivalence of 3) and 4) follows along similar lines. Finally, we leave the implications 3) + 4) $\Rightarrow$5) $\Rightarrow$1) as exercises. □

We now have the following key vanishing conjecture.

Conjecture 3.4.3. *The equivalent statements in Proposition 3.4.2 are true.*

This conjecture seems to lie very deep, and it has significant consequences for the cohomology of local Shimura varieties.

It is fruitful to study this conjecture one L-parameter at a time. More precisely, passing to ϕ-localizations, one can introduce an additional grading on the source and target of the γ's, indexed by semisimple L-parameters. Conjecture 3.4.3 is then equivalent, for instance, to the statement that for all semisimple L-parameters ϕ and all $b' \neq b$, the map

$$[i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}}] : K_0\mathrm{Rep}(G_b(E))_{\mathrm{fin},\phi} \to K_0\mathrm{Rep}(G_{b'}(E))_{\mathrm{fin},\phi}$$

is zero. Note that if ϕ is generous, the final part of Conjecture 2.1.8 predicts that

$$i_{b'}^{*\mathrm{ren}} i_{b\sharp}^{\mathrm{ren}} : D(G_b(E), \overline{\mathbf{Q}}_\ell)_{\mathrm{fin},\phi} \to D(G_{b'}(E), \overline{\mathbf{Q}}_\ell)_{\mathrm{fin},\phi}$$

is *already* identically zero before passing to K_0! However, in general this map will not vanish before passing to K_0.

A good example of this phenomenon is given by the trivial L-parameter. Here, to see that $\gamma_!$ and $\gamma_\sharp$ are equal on the summand indexed by the trivial parameter, we need to see that $[i_{b_\lambda !}^{\mathrm{ren}} \pi_\lambda] = [i_{b_\lambda \sharp}^{\mathrm{ren}} \pi_\lambda]$ for all dominant λ. On the other side of the categorical conjecture, this corresponds to the expectation that $[\nu_* A_\lambda] = [\nu_* A_{w_0(\lambda)}]$ in $K_0\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}$. But this is true! In fact, the equality $[A_\lambda] = [A_{w_0(\lambda)}]$ already holds in $K_0\mathrm{Coh}(\mathcal{N}/\hat{G})$, which follows from the results in section 4 of [AH19]. Indeed, Achar-Hardesty prove that twisted Grothendieck-Serre duality induces the *identity* on $K_0\mathrm{Coh}(\mathcal{N}/\hat{G})$, but on the other hand it is easy to check that $\mathbf{D}_{\mathrm{twGS}} A_\lambda \simeq A_{w_0(\lambda)}$.

There is another, closely related conjecture. Fix a Levi subgroup $M \subset G$ and a parabolic $P = MU$ containing it. Recall from our discussion of Eisenstein series that Eis_P^G is expected to preserve compact objects, and also ULA objects with quasicompact support. With $\overline{\mathbf{Q}}_\ell$-coefficients, we thus expect that it will preserve finite sheaves, and in particular it will induce a functor

$$[\mathrm{Eis}_P^G] : K_0 D(\mathrm{Bun}_M)_{\mathrm{fin}} \to K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}}.$$

Conjecture 3.4.4. *The functor* $[\mathrm{Eis}_P^G]$ *depends only on the Levi* M.

This is a geometric analogue of the classical fact that for any parabolic $P = MU \subset G$, the map $[i_P^G] : K_0\mathrm{Rep}(M(E))_{\mathrm{fin}} \to K_0\mathrm{Rep}(G(E))_{\mathrm{fin}}$ depends only on the Levi M, which is an easy consequence of van Dijk's formula for the Harish-Chandra character of a parabolic induction [vD72].

Proposition 3.4.5. *Conjecture 3.4.3 and Conjecture 3.4.4 are equivalent.*

Sketch. One implication follows immediately from Remark 1.4.8. The other implication is much less obvious, and a detailed proof will appear in [HHS]. □

It turns out that Conjecture 1.5.2 and Conjecture 3.4.4 together imply a vast generalization of the "Harris-Viehmann conjecture" describing the K_0-class of the cohomology of non-basic local Shimura varieties in terms of parabolic inductions. A detailed discussion will appear in [HHS].

Exercise 3.4.6. Return to the notation and setup of Section 2.2.

1) (Difficult.) For any $\lambda \in X^*(\hat{T})$ and any $w \in W$, prove that $[A_\lambda] = [A_{w(\lambda)}]$ in $K_0\mathrm{Coh}(\mathcal{N}/\hat{G})$.

2) Show that 1) is consistent with Conjecture 3.4.4 and the categorical conjecture. Hint: Generalize the arguments in Section 2.2 to show that for λ dominant and w arbitrary, $\mathrm{Eis}_B(i_{w(\lambda)!}\mathbf{1}) \simeq \mathrm{Eis}_{{}^wB}(i_{\lambda !}\mathbf{1})$ should match with $A_{w(\lambda)}$ under the categorical conjecture.

3) (Difficult.) Assume Conjecture 2.2.1. Prove that for any $\lambda \in X^*(\hat{T})^+$,

$$[\mathscr{F}_{\lambda+2\rho}] = \sum_{w \in W} (-1)^{\ell(w)} [i^{\mathrm{ren}}_{b_{\mathrm{dom}(\lambda+\rho+w\rho)}!} \pi_{\mathrm{dom}(\lambda+\rho+w\rho)}]$$

in $K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}}$. Here $\mathrm{dom}(\mu)$ denotes the unique dominant element in the W-orbit of μ. For $G = \mathrm{PGL}_3$ and $\lambda = 0$, compute the right-hand side explicitly.

Remark 3.4.7. According to Conjecture 3.4.3, the elements $[i^{\mathrm{ren}}_{b!}\pi] = [i^{\mathrm{ren}}_{b\sharp}\pi]$ should coincide, and should give a canonical $\mathbf{Z}$-basis for $K_0 D(\mathrm{Bun}_G)_{\mathrm{fin}}$ parametrized by pairs (b, π). There is a second canonical $\mathbf{Z}$-basis parametrized by the same set, given by the elements $[\mathscr{G}_{b,\pi}]$ where $\mathscr{G}_{b,\pi}$ is the irreducible hadal sheaf constructed in Theorem 1.2.7. In particular, we can write any $[\mathscr{G}_{b,\pi}]$ uniquely as a finite $\mathbf{Z}$-linear combination of $[i^{\mathrm{ren}}_{b'!}\pi']$, and vice versa. Experiments (including the previous exercise) suggest that the coefficients in the expansion $[\mathscr{G}_{b,\pi}] = \sum_{(b',\pi')} c_{b',\pi'}[i^{\mathrm{ren}}_{b'!}\pi']$ are somewhat complicated, with unpredictable signs. However, in every example I have worked out, the coefficients in the expansion

$$[i^{\mathrm{ren}}_{b!}\pi] = \sum_{(b',\pi')} \alpha_{b',\pi'}[\mathscr{G}_{b',\pi'}]$$

are *nonnegative* integers. At semisimple generic parameters, this is predicted by the discussion in section 2.3, but this is also true in some nontrivial cases around the Steinberg parameter for GL_2. Does this positivity phenomenon persist in general?

A Homological properties of Whittaker functions

Fix $E/\mathbf{Q}_p$ finite, G (the E-points of) a quasisplit reductive group, $B = TU \subset G$ as usual. If R is any commutative ring, we write $\mathrm{Mod}_R(G)$ for the category of smooth $R[G]$-modules.

Let Λ be any $\mathbf{Z}[\frac{1}{p}, \zeta_{p^\infty}]$ algebra, and let $\psi : U \to \Lambda^\times$ be any nondegenerate additive character. We are interested in the space

$$W_\psi = \mathrm{ind}_U^G(\psi)$$

of compactly supported Whittaker functions with coefficients in Λ. In other words, $W_\psi \subset \mathcal{C}(G, \Lambda)$ is the space of functions such that $f(ug) = \psi(u)f(g)$ for all $u \in U$ and $g \in G$, f is right-invariant by some open compact subgroup of G, and the support of f has compact image in $U\backslash G$.

In the most classical situation where $\Lambda = \mathbf{C}$, Chan-Savin proved that W_ψ is a projective object in $\mathrm{Mod}_{\mathbf{C}}(G)$ [CS19], and Bushnell-Henniart proved that the summand $W_{\psi,\mathfrak{s}}$ corresponding to an individual Bernstein component $\mathfrak{s}$ is finitely generated [BH03]. The following theorem is a generalization of these results.

Theorem A.0.1. i. *For any Λ, W_ψ is a projective object in $\mathrm{Mod}_\Lambda(G)$.*

ii. *Let $e_r : \mathrm{Mod}_\Lambda(G) \to \mathrm{Mod}_\Lambda(G)_r$ be the projector onto the depth $\leq r$ factor category. Then $e_r W_\psi \in \mathrm{Mod}_\Lambda(G)$ is finitely generated projective, and $\mathbf{D}_{\mathrm{coh}}(e_r W_\psi) \simeq e_r W_{\psi^{-1}}$.*

iii. *Assume that $\Lambda = \mathbf{C}$. Then $\mathbf{D}_{\mathrm{coh}}(W_{\psi,\mathfrak{s}}) \simeq W_{\psi^{-1},\mathfrak{s}^\vee}$ for any Bernstein component $\mathfrak{s}$.*

Here as before, $\mathbf{D}_{\mathrm{coh}}(-)$ denotes the cohomological duality functor $R\mathrm{Hom}(-, \mathcal{C}_c^\infty(G, \Lambda))$. We note that a different proof of i. and of the finite generation statement in ii. was also discovered by Dat-Helm-Kurinczuk-Moss.

The main tools in our analysis of W_ψ are Bushnell-Henniart's finiteness results over $\mathbf{C}$, together with a beautiful approximation technique due to Rodier [Rod75], which seems perhaps under-appreciated. To explain this, fix Λ-valued Haar measures dg on G and du on U. We may assume that the volume of any pro-p open subgroup is a unit in Λ. Following Rodier, we may choose a sequence of pairs $(K_n, \psi_n)_{n\geq 1}$, where $K_n \subset G$ is an open compact pro-p subgroup and $\psi_n : K_n \to \Lambda^\times$ is a smooth character, with the following remarkable properties.

P1 $K_n \cap U$ is an increasing sequence of groups, with $U = \bigcup_n K_n \cap U$.

P2 $K_n \cap \overline{B}$ is a decreasing sequence of groups, with $\{e\} = \bigcap_n K_n \cap \overline{B}$.

P3 For all n, $K_n = (K_n \cap U) \cdot (K_n \cap \overline{B}) = (K_n \cap \overline{B}) \cdot (K_n \cap U)$.

P4 $\psi_n|_{K_n \cap \overline{B}} = 1$ and $\psi_n|_{K_n\cap U} = \psi|_{K_n \cap U}$.

P5 Let $\varphi_n \in \mathcal{C}_c^\infty(G, \Lambda)$ be the function on G obtained from ψ_n via extension by zero. Then there is an explicit integer B such that for all $n \geq B$,

$$\varphi_n * \varphi_{n+1} * \varphi_n = \mathrm{vol}(K_n)\mathrm{vol}(K_{n+1} \cap K_n)\varphi_n$$

where $*$ denotes the usual convolution structure on $\mathcal{C}_c^\infty(G, \Lambda)$ relative to the chosen Haar measure.

The existence of a sequence of pairs satisfying P1-P4 is not particularly hard, but P5 lies significantly deeper. Note that Rodier's paper only treated the case of G split and p sufficiently large, in which case P5 corresponds to the key "Lemme 5", whose proof occupies the entirety of [Rod75, Section 5]. In the present generality, the existence of a system satisfying P1-P5 follows from Varma's paper [Var14].[24] (Note that we write K_n where Varma writes G'_n.) The essential point is that P5 follows from Lemma 9 of [Var14], upon observing that in the present situation, the element Y is principal nilpotent and contained in $\overline{\mathfrak{b}}$, so $G(Y) \subset \overline{B}$ and then (by P2) $G'_{n+1} \cap G(Y) \subset G'_{n+1} \cap \overline{B} \subset G'_n$, so we may take $\mathcal{Y}_n = \{e\}$ in Lemma 9 of [Var14]. Note that [Rod75] and [Var14] are written in the usual setting where the coefficient ring is $\mathbf{C}$, but it is immediate that the arguments work for any Λ as above.

With these preparations in hand, consider the compact inductions $\mathrm{ind}_{K_n}^G(\psi_n)$. Then for all positive integers m, n, we have the G-equivariant map

$$\begin{aligned} A_n^m : \mathrm{ind}_{K_n}^G(\psi_n) &\to \mathrm{ind}_{K_m}^G(\psi_m) \\ f &\mapsto \frac{1}{\mathrm{vol}(K_m)} \int_{K_m} \psi_m(k)^{-1} f(kg)dk. \end{aligned}$$

Note that A_n^m is well-defined independently of the choice of Haar measure dk on K_m. Note also that $A_m^\ell \circ A_n^m = A_n^\ell$ for all $\ell \geq m \geq n$, so the representations $\mathrm{ind}_{K_n}^G(\psi_n)$ form an inductive system with transition maps given by the A_n^m's. It is also easy to check that for $m \geq n$, the formula

$$A_n^m(f) = \frac{1}{\mathrm{vol}(K_m \cap U)} \int_{K_m \cap U} \psi(u)^{-1} f(ug)du$$

is true. On the other hand, for any $n \geq 1$, we have a G-equivariant map

$$\begin{aligned} \phi_n : \mathrm{ind}_{K_n}^G(\psi_n) &\to W_\psi \\ f &\mapsto \int_U \psi(u)^{-1} f(ug)du. \end{aligned}$$

[24] I am very grateful to Sandeep Varma for helpful communications regarding his paper [Var14].

Using the preceding formula for A_n^m, it is easy to check that $\phi_n = \phi_m \circ A_n^m$ for all $m \geq n$, so passing to the colimit we get a G-equivariant map

$$\phi_\infty : \mathrm{colim}_n \mathrm{ind}_{K_n}^G(\psi_n) \to W_\psi.$$

It is then formal (Proposition 3 in Rodier's paper) that ϕ_∞ is an isomorphism. Note that all the arguments so far used only P1-P4.

The key remaining observation is that for all $m \geq n \geq B$, A_n^m is injective and A_m^n is surjective. This follows immediately from the claim that $A_m^n \circ A_n^m = \frac{\mathrm{vol}(K_m \cap K_n)}{\mathrm{vol}(K_m)}\mathrm{id}$ for all $m \geq n \geq B$ (note that the fraction here is a power of p, hence a unit in Λ). To prove this claim, first note (using P1-P3) that $\frac{\mathrm{vol}(K_m \cap K_n)}{\mathrm{vol}(K_m)} = \frac{\mathrm{vol}(K_n \cap U)}{\mathrm{vol}(K_m \cap U)}$, whence the claim reduces by induction to the special case $m = n+1$. The case $m = n+1$ in turn is a direct consequence of P5 above, upon noting that $\mathrm{ind}_{K_n}^G(\psi_n) \subset \mathcal{C}_c^\infty(G, \Lambda)$ is exactly the image of the map

$$\begin{aligned} \mathcal{C}_c^\infty(G,\Lambda) &\to \mathcal{C}_c^\infty(G,\Lambda) \\ f &\mapsto \varphi_n * f, \end{aligned}$$

and that the map $A_{n+1}^n \circ A_n^{n+1}$ coincides with further left convolution by $\varphi_n * \varphi_{n+1}$ up to an explicit scaling. For details, see the proof of [Rod75, Proposition 4].

The equality $A_m^n \circ A_n^m = \frac{\mathrm{vol}(K_m \cap K_n)}{\mathrm{vol}(K_m)}\mathrm{id}$ immediately implies that for all $m \geq n \geq B$, A_n^m is a split injection, whence also $\phi_n = \phi_\infty \circ \mathrm{colim}_m A_n^m$ is a split injection.

Proof of Theorem A.0.1. First we prove i. With the above preparations on Rodier approximation at hand, this result proves itself. The only thing to observe is that each $\mathrm{ind}_{K_n}^G(\psi_n)$ is projective, so Rodier approximation presents W_ψ as the colimit of a directed system of projective objects along split injective transition maps. But the colimit of any such directed system is itself a projective object.

Next we prove iii. Note first that in the discussion of Rodier approximation, we may replace ψ resp. ψ_n with ψ^{-1} resp. ψ_n^{-1} everywhere without changing the validity of any statements. In particular, we may simultaneously write $W_\psi \simeq \mathrm{colim}_n \mathrm{ind}_{K_n}^G(\psi_n)$ and $W_{\psi^{-1}} \simeq \mathrm{colim}_n \mathrm{ind}_{K_n}^G(\psi_n^{-1})$, where for all sufficiently large n the transition maps in both systems are (split) injections. Now pick any Bernstein component $\mathfrak{s}$. Again, we have $W_{\psi,\mathfrak{s}} \simeq \mathrm{colim}_n \mathrm{ind}_{K_n}^G(\psi_n)_{\mathfrak{s}}$ and $W_{\psi^{-1},\mathfrak{s}^\vee} \simeq \mathrm{colim}_n \mathrm{ind}_{K_n}^G(\psi_n^{-1})_{\mathfrak{s}^\vee}$, where for all sufficiently large n the transition maps are injections. Since $W_{\psi,\mathfrak{s}}$ is finitely generated, this immediately implies that the map $\mathrm{ind}_{K_n}^G(\psi_n)_{\mathfrak{s}} \to W_{\psi,\mathfrak{s}}$ is an isomorphism for all sufficiently large n. Repeating the same argument, we also get that the map $\mathrm{ind}_{K_n}^G(\psi_n^{-1})_{\mathfrak{s}^\vee} \to W_{\psi^{-1},\mathfrak{s}^\vee}$ is an isomorphism for all sufficiently large n. But $\mathbf{D}_{\mathrm{coh}}(\mathrm{ind}_{K_n}^G(\psi_n)_{\mathfrak{s}}) \simeq \mathrm{ind}_{K_n}^G(\psi_n^{-1})_{\mathfrak{s}^\vee}$, which gives the result.

Finally we prove ii. Observe first that the formation of the representations W_ψ and $\mathrm{ind}_{K_n}^G(\psi_n)$ and the maps A_n^m, ϕ_n, ϕ_∞ are compatible with extension of scalars along any ring map $\Lambda \to \Lambda'$. Since BZ duality and the depth $\leq r$ projector e_r are also compatible with any extension of scalars, we are immediately reduced to the universal case $\Lambda = \mathbf{Z}[\frac{1}{p}, \zeta_{p^\infty}]$. Pick an embedding $\Lambda \to \mathbf{C}$; we will write $(-)_{\mathbf{C}}$ for objects obtained by the evident extension of scalars along this map.

Since $\mathrm{Mod}_{\mathbf{C}}(G)_r$ is a product of finitely many Bernstein components, the proof of iii. shows that the maps $e_r\phi_{n,\mathbf{C}} : e_r\mathrm{ind}_{K_n}^G(\psi_n)_{\mathbf{C}} \to e_r W_{\psi,\mathbf{C}}$ and $e_r\mathrm{ind}_{K_n}^G(\psi_n^{-1})_{\mathbf{C}} \to e_r W_{\psi^{-1},\mathbf{C}}$ are isomorphisms for sufficiently large n. Now consider the maps $e_r\phi_n : e_r\mathrm{ind}_{K_n}^G(\psi_n) \to e_r W_\psi$. Since ϕ_n is a split injection for all sufficiently large n, we get also that $e_r\phi_n$ is a split injection for all sufficiently large

n. Moreover, the source and target of ϕ_n are projective Λ-modules,[25] so also the source and target of $e_r\phi_n$ are projective Λ-modules. In particular, $\operatorname{coker} e_r\phi_n$ is a projective Λ-module for sufficiently large n. But we've already established that

$$(\operatorname{coker} e_r\phi_n) \otimes_\Lambda \mathbf{C} \simeq \operatorname{coker} e_r\phi_{n,\mathbf{C}} = 0$$

for all sufficiently large n. This implies that $\operatorname{coker} e_r\phi_n = 0$ for all sufficiently large n, so $e_r\phi_n : e_r\mathrm{ind}_{K_n}^G(\psi_n) \to e_rW_\psi$ is an isomorphism for all sufficiently large n. Since $\mathrm{ind}_{K_n}^G(\psi_n)$ and its summand $e_r\mathrm{ind}_{K_n}^G(\psi_n)$ are finitely generated, we deduce that e_rW_ψ is finitely generated.

Repeating the same argument with inverses on all characters, we also get that $e_r\mathrm{ind}_{K_n}^G(\psi_n^{-1}) \to e_rW_{\psi^{-1}}$ is an isomorphism for all sufficiently large n. Since $\mathbf{D}_{\mathrm{coh}}(\mathrm{ind}_{K_n}^G(\psi_n)) = \mathrm{ind}_{K_n}^G(\psi_n^{-1})$ and e_r commutes with $\mathbf{D}_{\mathrm{coh}}$, we compute that

$$\begin{aligned}\mathbf{D}_{\mathrm{coh}}(e_rW_\psi) &\simeq \mathbf{D}_{\mathrm{coh}}(e_r\mathrm{ind}_{K_n}^G(\psi_n))\\ &= e_r\mathbf{D}_{\mathrm{coh}}(\mathrm{ind}_{K_n}^G(\psi_n))\\ &= e_r\mathrm{ind}_{K_n}^G(\psi_n^{-1})\\ &\simeq e_rW_{\psi^{-1}}.\end{aligned}$$

This concludes the proof. □

Corollary A.0.2. *Suppose Λ is an algebraically closed field of characteristic $\neq p$ and $\pi \in \mathrm{Mod}_\Lambda(G)$ is irreducible. Then π is ψ-generic if and only if $\pi^\vee$ is ψ^{-1}-generic. More generally, for any π of finite length, there is an isomorphism*

$$\mathrm{Hom}(W_\psi, \pi)^* \simeq \mathrm{Hom}(W_{\psi^{-1}}, \pi^\vee).$$

When $\Lambda = \mathbf{C}$, the first part of this corollary was previously proved by Prasad [Pra19, Lemma 2].

Proof. Choose r large so π is of depth $\leq r$, and then choose n large enough so that $e_rW_\psi \simeq e_r\mathrm{ind}_{K_n}^G(\psi_n)$. Then arguing as in the previous proof we get isomorphisms

$$\begin{aligned}\mathrm{Hom}(W_\psi, \pi) &\simeq \mathrm{Hom}(e_rW_\psi, \pi)\\ &\simeq \mathrm{Hom}(e_r\mathrm{ind}_{K_n}^G(\psi_n), \pi)\\ &\simeq \mathrm{Hom}(\mathrm{ind}_{K_n}^G(\psi_n), \pi)\\ &\simeq (\pi|_{K_n} \otimes \psi_n^{-1})^{K_n}\end{aligned}$$

and similarly $\mathrm{Hom}(W_{\psi^{-1}}, \pi^\vee) \simeq (\pi^\vee|_{K_n} \otimes \psi_n)^{K_n}$. We conclude by the easy fact that $(\pi|_{K_n} \otimes \psi_n^{-1})^{K_n}$ and $(\pi^\vee|_{K_n} \otimes \psi_n)^{K_n}$ are canonically dual to each other as Λ-vector spaces. □

A.1 A reasonable condition

In this section we again take our coefficients to be $\Lambda = \overline{\mathbf{Q}_\ell}$. Let G/E be any connected reductive group.

[25] For $\mathrm{ind}_{K_n}^G(\psi_n)$ this is clear, and for W_ψ it follows from the proof of i. Is it clear from first principles that W_ψ is projective as a Λ-module?

Definition A.1.1. i. The group G is *reasonable* if the Fargues-Scholze map $\Psi_G^{\mathrm{geom}} : X_G \to X_G^{\mathrm{spec}}$ is finite, or equivalently (by Lemma 1.6.4) if the associated map $\pi_0 X_G \to \pi_0 X_G^{\mathrm{spec}}$ has finite fibers.

ii. The group G is *very reasonable* if G_b is reasonable for all $b \in B(G)$.

It is clear that if G is very reasonable, then G_b is very reasonable for all $b \in B(G)$.

Exercise A.1.2. Prove that if G is reasonable, then any Levi subgroup $M \subset G$ is reasonable.

Our main reason for considering these conditions is the following basic finiteness result, which we leave to the reader as an exercise. (The key ingredient for i. is Bushnell-Henniart's finiteness theorem cited above.)

Proposition A.1.3. i. *If* $\mathrm{char} E = 0$ *and* G *is quasisplit and reasonable, the functor* $\mathcal{F} \mapsto \mathcal{F} * i_{1!} W_\psi$ *sends* $\mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_G)$ *into* $D(\mathrm{Bun}_G)^\omega$.

ii. *If* G *is very reasonable, then for every* b *and every semisimple parameter* $\phi : W_E \to {}^L G(\overline{\mathbf{Q}_\ell})$, *the set of irreducible smooth representations* $\pi \in \Pi(G_b)$ *with Fargues-Scholze parameter* ϕ *is finite.*

iii. *If* G *is very reasonable, then every ULA sheaf in* $D(\mathrm{Bun}_G)$ *is a filtered colimit of finite sheaves.*

Note that the condition in Proposition A.1.3.ii is actually equivalent to very reasonableness. This follows from finiteness of the map $X_{G_b}^{\mathrm{spec}} \to X_G^{\mathrm{spec}}$, Theorem 1.4.2, and basic structure theory a la Bernstein.

It's also quite plausible that Proposition A.1.3 iii. could be proved unconditionally for any G, but I didn't try very hard to check this. If G is very reasonable, however, it is nearly trivial: if A is ULA, then for any semisimple parameter ϕ and any quasicompact open substack $j : U \to \mathrm{Bun}_G$, the sheaf $j_! j^* A_\phi$ is finite by the very reasonableness condition, and the natural map

$$\mathrm{colim}_{U,S} \oplus_{\phi \in S} j_! j^* A_\phi \to A$$

is an isomorphism (using the decomposition $A \cong \oplus_\phi A_\phi$ proved in [Han23b]), where S runs over (the filtered collection of) finite sets of semisimple L-parameters.

Of course we expect that every group is very reasonable. Right now, we know that GL_n is very reasonable [FS21, HKW22], as well as GSp_4 with $E/\mathbf{Q}_p$ unramified and $p > 2$ [Ham21], unramified (G)$\mathrm{U}_{2n+1}/\mathbf{Q}_p$ [BMHN22], SO_{2n+1} with $E/\mathbf{Q}_p$ unramified and $p > 2$ (H., unpublished), and groups obtained from these by passing to derived groups, products, central isogenies, twisted Levis, etc. To my knowledge, there is no group which is known to be reasonable but not known to be very reasonable.

Exercise A.1.4. Show that if G and G' have isomorphic adjoint groups and G is reasonable, then G' is reasonable. Can you formulate a similar statement for "very reasonableness"?

B A dimensional classicality criterion for derived stacks, by Adeel Khan

We define (co)dimension of derived schemes and stacks on classical truncations (see [Sta21, Tag 04N3] or [GD71, 0_IV, 14.1.2, 14.2.4] for schemes, and [Sta21, Tags 0AFL and 0DRL] for stacks).

Proposition B.0.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a quasi-smooth morphism of derived* 1*-Artin stacks where* $\mathcal{Y}_{\mathrm{cl}}$ *is Cohen-Macaulay.*[26] *If* $x \in |\mathcal{X}|$ *is a point at which the relative dimension of* f *is equal to the relative virtual dimension of* f, *then* $\mathcal{X} \times^{\mathbf{R}}_{\mathcal{Y}} \mathcal{Y}_{\mathrm{cl}}$ *is classical in a Zariski neighbourhood of* x.

[26] Equivalently, $\mathcal{Y}$ admits a smooth surjection $Y \twoheadrightarrow \mathcal{Y}$ where Y_{cl} is a Cohen–Macaulay scheme.

We will make use of the following lemma from [KR18, 2.3.12].

Lemma B.0.2. *Let $Z \to X$ be a quasi-smooth closed immersion of derived schemes where X_{cl} is Cohen–Macaulay. Then we have $-\operatorname{vdim}(Z/X) \geqslant \operatorname{codim}(Z, X)$, with equality if and only if $Z \times^{\mathbf{R}}_X X_{\mathrm{cl}}$ is classical in a Zariski neighbourhood of x.*

Proof of Proposition B.0.1. The statement is invariant under replacing $\mathcal{Y}$ by $\mathcal{Y}_{\mathrm{cl}}$ and $\mathcal{X}$ by $\mathcal{X} \times^{\mathbf{R}}_{\mathcal{Y}} \mathcal{Y}_{\mathrm{cl}}$, so we may assume $\mathcal{Y}$ classical.

Suppose first that $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are schemes. Since $f : X \to Y$ is quasi-smooth, there exists for every $x \in |X|$ over y a Zariski neighbourhood $U \subseteq X$ of x, a derived scheme M which is smooth over Y, and a quasi-smooth closed immersion $U \hookrightarrow M$ over Y (see [KR18, Prop. 2.3.14]). We have

$$\operatorname{vdim}_x(U_y/M_y) = \operatorname{vdim}_x(U/M),$$
$$\operatorname{codim}_x(U_y, M_y) \leqslant \operatorname{codim}_x(U, M).$$

Since $\operatorname{vdim}_x(U_y/\kappa(y)) = \dim_x(U_y)$ by assumption, we also have

$$-\operatorname{vdim}_x(U_y/M_y) = \dim_x(M_y) - \dim_x(U_y) = \operatorname{codim}_x(U_y, M_y)$$

where the last equality holds because M_y is catenary (see [GD71, 0_IV Cor.16.5.12]; [GD71, IV_2 Prop. 5.1.9]). We conclude that

$$-\operatorname{vdim}_x(U/M) \leqslant \operatorname{codim}_x(U, M).$$

Now Lemma B.0.2 implies that U is classical in a Zariski neighbourhood of x.

Next suppose that $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are algebraic spaces. Choose an étale surjection $X_0 \twoheadrightarrow X$ where X_0 is a derived scheme, and let $x_0 \in |X_0|$ be a lift of the given point $x \in |X|$. Choose also an étale surjection $Y_0 \twoheadrightarrow Y$ where Y_0 is a Cohen–Macaulay scheme and a lift $y_0 \in |Y_0|$ of y. Since Y has schematic diagonal, $X_0 \times_Y Y_0$ is a derived scheme. Applying the case above to the morphism $X_0 \times_Y Y_0 \to Y_0$, we obtain a Zariski neighbourhood of $(x_0, y_0) \in X_0 \times_Y Y_0$ which is classical. Its image along the étale morphism $X_0 \times_Y Y_0 \twoheadrightarrow X_0 \twoheadrightarrow X$ is then a Zariski neighbourhood of $x \in X$ which is classical.

Finally we consider the general case. Choose a smooth surjection $X \twoheadrightarrow \mathcal{X}$ where X is a derived scheme, a lift $x_0 \in |X_0|$ of the given point $x \in |X|$, a smooth surjection $Y_0 \twoheadrightarrow Y$ where Y_0 is a Cohen–Macaulay scheme, and a lift $y_0 \in |Y_0|$ of y. Since Y has representable diagonal, $X \times_{\mathcal{Y}} Y$ is a derived algebraic space. Hence the previous case applied to the morphism $X \times_{\mathcal{Y}} Y \to Y$ yields a Zariski neighbourhood of $(x_0, y_0) \in X \times_{\mathcal{Y}} Y$ which is classical. Its image along the smooth morphism $X \times_{\mathcal{Y}} Y \twoheadrightarrow X \twoheadrightarrow \mathcal{X}$ is then a Zariski neighbourhood of $x \in \mathcal{X}$ which is classical. ☐

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